Totally positive spaces: topology and applications

by

Pavel Galashin

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Abstract

This thesis studies topological spaces arising in total positivity. Examples include the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$, Lusztig's totally nonnegative part $(G/P)_{\geq 0}$ of a partial flag variety, Lam's compactification of the space of electrical networks, and the space of (boundary correlation matrices of) planar Ising networks. We show that all these spaces are homeomorphic to closed balls. In addition, we confirm conjectures of Postnikov and Williams that the CW complexes $\mathrm{Gr}_{\geq 0}(k,n)$ and $(G/P)_{\geq 0}$ are regular. This implies that the closure of each positroid cell inside $\mathrm{Gr}_{\geq 0}(k,n)$ is homeomorphic to a closed ball. We discuss the close relationship between the above spaces and the physics of scattering amplitudes, which has served as a motivation for most of our results.

In the second part of the thesis, we investigate the space of planar Ising networks. We give a simple stratification-preserving homeomorphism between this space and the totally nonnegative orthogonal Grassmannian, describing boundary correlation matrices of the planar Ising model by inequalities. Under our correspondence, Kramers–Wannier's high/low temperature duality transforms into the cyclic symmetry of $\mathrm{Gr}_{\geq 0}(k,n)$.

Thesis Supervisor: Alexander Postnikov
Title: Professor of Applied Mathematics

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Introduction

Постепенно человек утрачивает свою форму и становится шаром.

И став шаром, человек утрачивает все свои желания.

– Даниил Хармс

We investigate various topological spaces arising in the theory of total positivity. Each space comes equipped with a natural decomposition into cells, and is conjectured to be a regular CW complex homeomorphic to a closed ball. A *regular CW complex* is a topological space subdivided into cells, such that the closure of each cell is homeomorphic to a ball, and the boundary of each cell is homeomorphic to a sphere. A prototypical example of a regular CW complex is a convex polytope.

We show that each of the totally nonnegative spaces introduced in Sections 1.1–1.6 is homeomorphic to a closed ball. Additionally, we show that the spaces from Sections 1.1–1.3 are regular CW complexes. As we discuss in Section 1.7, the spaces that we consider are surprisingly closely related to each other and to the physics of scattering amplitudes.

This thesis is based on papers [GKL17, GKL19, GP18].

1.1 The totally nonnegative Grassmannian

Let $\mathrm{Gr}(k,n)$ denote the Grassmannian of k-planes in $\mathbb{R}^n$. Postnikov [Pos07] defined its totally nonnegative part $\mathrm{Gr}_{\geq 0}(k,n)$ as the set of $X \in \mathrm{Gr}(k,n)$ whose Plücker coordinates are all nonnegative. Postnikov conjectured that $\mathrm{Gr}_{\geq 0}(k,n)$ is a regular CW complex homeomorphic to a closed ball. He gave a decomposition of $\mathrm{Gr}_{\geq 0}(k,n)$ into *positroid cells*, where each cell is specified by requiring some subset of the Plücker coordinates to be strictly positive, and requiring the rest to equal zero.

Over the past decade, much work has been done towards Postnikov's conjecture. The face poset of the positroid cell decomposition (described in [Rie98, Rie06, Pos07]) was shown to be thin and shellable by Williams [Wil07]. Combined with Björner's results [Bjö84], this implies that there exists *some* regular CW complex with the same face poset. Postnikov, Speyer, and Williams [PSW09] showed that the cell decomposition is a CW complex, and Rietsch and Williams [RW10] showed that the closure of each cell is contractible. We start by giving a simple proof of the following result.

Theorem 1.1.1. $\mathrm{Gr}_{\geq 0}(k,n)$ *is homeomorphic to a* $k(n-k)$*-dimensional closed ball.*

This proves a special case of Postnikov's conjecture: there is one top-dimensional positroid cell in $\mathrm{Gr}_{\geq 0}(k,n)$ whose closure is the whole $\mathrm{Gr}_{\geq 0}(k,n)$, and thus Theorem 1.1.1 shows that the closure of this cell is homeomorphic to a ball. Our proof of Theorem 1.1.1 employs a certain *cyclic shift vector field* τ on $\mathrm{Gr}_{\geq 0}(k,n)$. The flow defined by τ contracts all of $\mathrm{Gr}_{\geq 0}(k,n)$ to the unique *cyclically symmetric point* $X_0 \in \mathrm{Gr}_{\geq 0}(k,n)$. We construct a homeomorphism from $\mathrm{Gr}_{\geq 0}(k,n)$ to a closed ball $B \subset \mathrm{Gr}_{\geq 0}(k,n)$ centered at X_0, by mapping each trajectory in $\mathrm{Gr}_{\geq 0}(k,n)$ to its intersection with B. A feature of our construction is that we do not rely on any cell decomposition of $\mathrm{Gr}_{\geq 0}(k,n)$.

We also prove Postnikov's conjecture in full generality, although in this case the proof is much more involved, see Sections 3.1 and 3.2.

Theorem 1.1.2. $\mathrm{Gr}_{\geq 0}(k,n)$ *is a regular CW complex homeomorphic to a closed ball.*

1.2 The totally nonnegative part of a partial flag variety

Let G be a simple and simply connected algebraic group, split over $\mathbb{R}$, and let $P \subset G$ be a parabolic subgroup. Lusztig [Lus94] introduced the totally nonnegative part of the partial flag variety G/P, denoted $(G/P)_{\geq 0}$. He called $(G/P)_{\geq 0}$ a "remarkable polyhedral subspace", and conjectured that $(G/P)_{\geq 0}$ has a decomposition into cells $\Pi_g^{>0} \cong \mathbb{R}^{\dim(g)}$, which was proved by Rietsch [Rie99]. The following result was conjectured by Williams [Wil07].

Theorem 1.2.1. *$(G/P)_{\geq 0}$ is a regular CW complex homeomorphic to a closed ball.*

For a specific choice of G and P, $(G/P)_{\geq 0}$ becomes the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$, thus Theorems 1.1.1 and 1.1.2 are special cases of Theorem 1.2.1. Similarly to the case of $\mathrm{Gr}_{\geq 0}(k,n)$, Williams [Wil07] proved that the face poset of $(G/P)_{\geq 0}$ is thin and shellable, and then Rietsch–Williams [RW08, RW10] showed that $(G/P)_{\geq 0}$ is a CW complex such that the closure of every cell is contractible.

1.3 The totally nonnegative part of the unipotent radical

The theory of total positivity originated in the 1930's, and concerns real matrices whose minors are all nonnegative [Sch30, GK50, Whi52]. Later, Lusztig [Lus94] was motivated by a question of Kostant to consider connections between totally nonnegative matrices and his theory of canonical bases for quantum groups [Lus90]. This led him to introduce the totally nonnegative part $G_{\geq 0}$ of a split semisimple G.

Fomin and Shapiro [FS00] realized that Lusztig's work may be used to address a long-standing problem in poset topology. Namely, the Bruhat order of the Weyl group W of G had been shown to be shellable by Björner and Wachs [BW82], and by general results of Björner [Bjö84] it follows that there exists a "synthetic" regular CW complex whose face poset coincides with $(W, \leq)$. The motivation of [FS00] was to answer a natural question due to Bernstein and Björner of whether such a regular CW complex exists "in nature". Let $U \subset G$ be the unipotent radical of the standard Borel subgroup, and let $U_{\geq 0} := U \cap G_{\geq 0}$ be its totally nonnegative part. For $G = \mathrm{SL}_n$, $U_{\geq 0}$ is the semigroup of upper-triangular unipotent matrices with all minors nonnegative. The work of Lusztig [Lus94] implies that $U_{\geq 0}$ has a cell decomposition whose face poset is $(W, \leq)$. The space $U_{\geq 0}$ is not compact, but Fomin and Shapiro [FS00] conjectured that taking the link of the identity element in $U_{\geq 0}$, which also has $(W, \leq)$ as its face poset, gives the desired regular CW complex. Their conjecture was confirmed by Hersh [Her14]. Alternatively, it follows as a corollary to our proof of Theorem 1.2.1, see Remark 3.3.12. Thus we provide an independent proof of the Fomin–Shapiro conjecture.

Corollary 1.3.1 ([Her14])**.** *The link of the identity in $U_{\geq 0}$ is a regular CW complex.*

1.4 The cyclically symmetric amplituhedron

A robust connection between the totally nonnegative Grassmannian and the physics of scattering amplitudes was developed in [AHBC$^+$16], which led Arkani-Hamed and Trnka [AHT14] to define topological spaces called *amplituhedra*.

Let k, m, n be nonnegative integers with $k+m \leq n$, and Z be a $(k+m)\times n$ matrix whose $(k+m)\times(k+m)$ minors are all positive. We regard Z as a linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$, which induces a map Z_{Gr} on $\mathrm{Gr}(k,n)$ taking the subspace X to the subspace $\{Z(v) : v \in X\}$. The *(tree) amplituhedron* $\mathcal{A}_{n,k,m}(Z)$ is the image of $\mathrm{Gr}_{\geq 0}(k,n)$ in $\mathrm{Gr}(k,k+m)$ under the map Z_{Gr} [AHT14, Section 4]. When $k=1$, the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(1,n)$ is a simplex in $\mathbb{P}^{n-1}$, and the amplituhedron $\mathcal{A}_{n,1,m}(Z)$ is a *cyclic polytope* in $\mathbb{P}^m$ [Stu88].

We now take m to be even, and $Z = Z_0$ such that the rows of Z_0 span the unique element of $\mathrm{Gr}_{\geq 0}(k+m,n)$ invariant under $\mathbb{Z}/n\mathbb{Z}$-cyclic action (cf. [Kar18]). We call $\mathcal{A}_{n,k,m}(Z_0)$ the *cyclically symmetric amplituhedron.* When $k=1$ and $m=2$, $\mathcal{A}_{n,1,2}(Z_0)$ is a regular n-gon in the plane. More generally, $\mathcal{A}_{n,1,m}(Z_0)$ is a polytope whose vertices are n regularly spaced points on the *trigonometric moment curve* in $\mathbb{P}^m$.

Theorem 1.4.1. *The cyclically symmetric amplituhedron $\mathcal{A}_{n,k,m}(Z_0)$ is homeomorphic to a km-dimensional closed ball.*

It is expected that every amplituhedron is homeomorphic to a closed ball. In some special cases, this is indeed true, see [KW19, BGPZ19].

1.5 The space of planar electrical networks

Let Γ be an electrical network consisting only of resistors, modeled as an undirected graph whose edge weights (conductances) are positive real numbers. The electrical properties of

Γ are encoded by the response matrix $\Lambda(\Gamma) : \mathbb{R}^n \to \mathbb{R}^n$, sending a vector of voltages at n distinguished boundary vertices to the vector of currents induced at the same vertices. The response matrix can be computed using (only) Kirchhoff's law and Ohm's law. Following Curtis, Ingerman, and Morrow [CIM98] and Colin de Verdière, Gitler, and Vertigan [CdVGV96], we consider the space Ω_n of response matrices of planar electrical networks: those Γ embedded into a disk, with boundary vertices on the boundary of the disk. This space is not compact; a compactification E_n was defined by Lam [Lam18]. It comes equipped with a natural embedding $\iota : E_n \hookrightarrow \mathrm{Gr}_{\geq 0}(n-1, 2n)$. We exploit this embedding to establish the following result.

Theorem 1.5.1. *The space E_n is homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

A cell decomposition of E_n was defined in [Lam18], extending earlier work in [CIM98, CdVGV96]. The face poset of this cell decomposition had been defined and studied by Kenyon [Ken12, Section 4.5.2]. Theorem 1.5.1 says that the closure of the unique cell of top dimension in E_n is homeomorphic to a closed ball. In [Lam15], Lam showed that the face poset of the cell decomposition of E_n is Eulerian, and conjectured that it is shellable. Hersh and Kenyon recently proved this conjecture [HK18]. Björner's results [Bjö84] therefore imply that this poset is the face poset of some regular CW complex homeomorphic to a ball. We expect that E_n forms such a CW complex, so that the closure of every cell of E_n is homeomorphic to a closed ball. Proving this remains an open problem.

1.6 The space of planar Ising networks

The Ising model, introduced by Lenz in 1920 as a model for ferromagnetism and solved by Ising [Isi25] in dimension 1, plays a central role in statistical mechanics and conformal field theory. One of the main features of this model is that it undergoes a phase transition in dimensions larger than 1, and in particular its critical temperature $\frac{1}{2}\log(\sqrt{2}+1)$ in dimension 2 has been computed by Kramers and Wannier [KW41], who found a duality transformation exchanging subcritical and supercritical temperatures. The free energy of the model was computed by Onsager [Ons44] and Yang [Yan52], and since then it became a subject of active mathematical and physical research. Conformal invariance of the scaling limit was conjectured in [BPZ84b, BPZ84a] in relation to conformal field theory, and proven more recently as a part of a series of groundbreaking results by Smirnov, Chelkak, Hongler, Izyurov, and others [Smi10, CHI15, CS12, HS13, CDCH$^+$14].

Among the most important quantities associated with the Ising model are two-point correlation functions. It was shown in [Gri67] and later generalized in [KS68] that these correlation functions satisfy natural inequalities, and in particular, the question of characterizing correlation functions coming from the Ising model was raised in the appendix of [KS68].

A starting point for our results was recent insightful work of Lis [Lis17], where he discovered a deep connection between the planar Ising model and total positivity, and used it to prove new inequalities on boundary two-point correlation functions in the planar case.

Despite the enormous amount of research on the planar Ising model, some basic questions seem to have remained unanswered. Let us denote by $\mathcal{X}_n \subset \mathrm{Mat}_n(\mathbb{R})$ the space of all boundary correlation matrices of planar Ising networks with n boundary nodes embedded in a disk. This is a subspace of the space $\mathrm{Mat}_n(\mathbb{R})$ of $n \times n$ matrices with real entries. Every matrix in $\mathcal{X}_n$ is symmetric and has diagonal entries equal to 1, but $\mathcal{X}_n$ is neither a closed nor an open subset of the space of such matrices. Let $\overline{\mathcal{X}}_n$ denote the closure of $\mathcal{X}_n$ inside $\mathrm{Mat}_n(\mathbb{R})$, i.e., $\overline{\mathcal{X}}_n$ is the space of boundary correlation matrices of a slightly more general class of planar Ising networks, as discussed in Section 4.5.

Two fundamental questions about $\overline{\mathcal{X}}_n$ that we answer (see Theorem 4.1.3) are:

- Describe $\overline{\mathcal{X}}_n$ by equalities and inequalities inside $\mathrm{Mat}_n(\mathbb{R})$.

- Describe the topology of $\overline{\mathcal{X}}_n$.

Using a construction similar to the one in [Lis17], we give a simple embedding ϕ of the space $\overline{\mathcal{X}}_n$ as a subset of $\mathrm{Gr}_{\geq 0}(n, 2n)$ which turns out to be precisely the *totally nonnegative orthogonal Grassmannian* $\mathrm{OG}_{\geq 0}(n, 2n)$, introduced in [HW13, HWX14] in the study of ABJM scattering amplitudes. This gives a solution to the first question. Next, we show that $\overline{\mathcal{X}}_n$ is homeomorphic to an $\binom{n}{2}$-dimensional closed ball using the same cyclic shift vector field τ that was used in the proof of Theorem 1.1.1. Surprisingly, we find (Theorem 4.2.4) that the

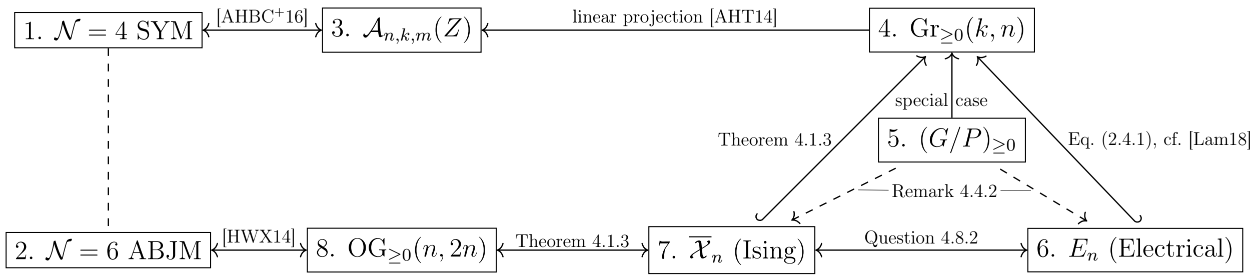


Figure 1-1: Connections between various totally positive spaces, see Section 1.7.

celebrated Kramers–Wannier's duality [KW41] is translated by our map ϕ into the cyclic shift on $\mathrm{Gr}_{\geq 0}(k, n)$.

1.7 Connections

Totally positive spaces have attracted a lot of interest due to their appearances in other contexts such as cluster algebras [FZ02] and the physics of scattering amplitudes [AHBC+16]. In particular, our original motivation came from studying the amplituhedron of [AHT14] and the more general Grassmann polytopes of [Lam16]. The faces of a Grassmann polytope are linear projections of closures of positroid cells, which is why it is essential to understand the topology of these closures in order to develop a theory of Grassmann polytopes.

The relationship between the following objects is shown schematically in Figure 1-1:

1. the planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory;
2. the three-dimensional $\mathcal{N} = 6$ supersymmetric Chern–Simons matter theory (also known as *ABJM theory*);
3. the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ from Section 1.4;
4. the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$ from Section 1.1;
5. the totally nonnegative part $(G/P)_{\geq 0}$ of a partial flag variety from Section 1.2;
6. the compactification E_n of the space of electrical networks from Section 1.5;
7. the space $\overline{\mathcal{X}}_n$ of planar Ising networks from Section 1.6;

8. the totally nonnegative orthogonal Grassmannian $\mathrm{OG}_{\geq 0}(n, 2n)$ from Section 1.6.

The scattering amplitudes in the planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory can be computed by formally integrating a certain differential form over the amplituhedron $\mathcal{A}_{n,k,m}(Z)$, cf. [AHBC+16, AHT14]. By definition, $\mathcal{A}_{n,k,m}(Z)$ is a linear projection of $\mathrm{Gr}_{\geq 0}(k, n)$, see Section 1.4. As we explained in Section 1.2, Postnikov's $\mathrm{Gr}_{\geq 0}(k, n)$ is a special case of Lusztig's $(G/P)_{\geq 0}$. Next, Lam [Lam18] constructed an embedding of his compactification E_n of the space of electrical networks into $\mathrm{Gr}_{\geq 0}(n-1, 2n)$, see (2.4.1). Similarly, the space $\overline{\mathcal{X}}_n$ of planar Ising networks from Section 1.6 is identified via Theorem 4.1.3 with the totally nonnegative orthogonal Grassmannian $\mathrm{OG}_{\geq 0}(n, 2n)$, which is a subset of $\mathrm{Gr}_{\geq 0}(n, 2n)$ by definition. The spaces E_n and $\overline{\mathcal{X}}_n$ share a lot of common properties (cf. Section 4.8), for example, their face posets are isomorphic. However, the precise relationship between them remains completely mysterious to us, see Question 4.8.2. By Remark 4.4.2, each of these spaces can be realized as a subset of G/P for a suitable choice of G and P, but this subset does not coincide with the totally nonnegative part $(G/P)_{\geq 0}$. Finally, the totally nonnegative orthogonal Grassmannian $\mathrm{OG}_{\geq 0}(n, 2n)$ (which is homeomorphic to $\overline{\mathcal{X}}_n$) was first introduced in [HW13, HWX14] in connection with the $\mathcal{N} = 6$ ABJM theory, analogously to the relationship [AHBC+16] between $\mathrm{Gr}_{\geq 0}(k, n)$ and the $\mathcal{N} = 4$ SYM theory.

1.8 Outline

Chapters 2, 3, 4 correspond to papers [GKL17, GKL19, GP18]. Theorems 1.1.1, 1.4.1, and 1.5.1 are proved in Chapter 2 using the cyclic shift vector field τ. Theorem 1.2.1 and its special case Theorem 1.1.2 are proved in Chapter 3. Chapter 4 develops the theory of planar Ising networks, see Section 4.1 for the precise statement of the main results.

2

The cyclic shift vector field

In this chapter, give a simple unified approach to show that the spaces $\mathrm{Gr}_{\geq 0}(k,n)$, $\mathcal{A}_{n,k,m}(Z_0)$, and E_n are homeomorphic to closed balls, thus proving Theorems 1.1.1, 1.4.1, and 1.5.1. We note that the same approach can be used to show that the spaces $U_{\geq 0}$ (when $G = \mathrm{SL}_n$) and $(G/P)_{\geq 0}$ (when G and P are arbitrary) are homeomorphic to closed balls as well, see [GKL17, §4] and [GKL18].

In Section 2.1, we introduce *contractive flows*, and then we use them in Sections 2.2, 2.3, and 2.4 to prove Theorems 1.1.1, 1.4.1, and 1.5.1, respectively.

2.1 Contractive flows

In this section we prove Lemma 2.1.3, which we will repeatedly use in establishing our theorems. Consider a real normed vector space $(\mathbb{R}^N, \|\cdot\|)$. Thus for each $r > 0$, the closed ball $B_r^N := \{p \in \mathbb{R}^N : \|p\| \leq r\}$ of radius r is a compact convex body in $\mathbb{R}^N$ whose interior contains the origin. We denote its boundary by ∂B_r^N, which is the sphere of radius r.

Definition 2.1.1. We say that a map $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a *contractive flow* if the following conditions are satisfied:

(1) the map f is continuous;

(2) for all $p \in \mathbb{R}^N$ and $t_1, t_2 \in \mathbb{R}$, we have $f(0,p) = p$ and $f(t_1 + t_2, p) = f(t_1, f(t_2, p))$; and

(3) for all $p \neq 0$ and $t > 0$, we have $\|f(t,p)\| < \|p\|$.

The condition (2) says that f induces a group action of $(\mathbb{R},+)$ on $\mathbb{R}^N$. In particular, $f(t,p) = q$ is equivalent to $f(-t,q) = p$, so (3) implies that if $t \neq 0$ and $f(t,p) = p$, then $p = 0$. The converse to this statement is given below in Lemma 2.1.2(i).

For $K \subset \mathbb{R}^N$ and $t \in \mathbb{R}$, we let $f(t,K)$ denote $\{f(t,p) : p \in K\}$.

Lemma 2.1.2. Let $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be a contractive flow.

(i) We have $f(t,0) = 0$ for all $t \in \mathbb{R}$.

(ii) Let $p \neq 0$. Then the function $t \mapsto \|f(t,p)\|$ is strictly decreasing on $(-\infty,\infty)$.

(iii) Let $p \neq 0$. Then $\lim_{t\to\infty} \|f(t,p)\| = 0$ and $\lim_{t\to-\infty} \|f(t,p)\| = \infty$.

Proof. (i) By (1), the function $s \mapsto \|f(s,0)\|$ is continuous on $\mathbb{R}$, and it equals 0 when $s = 0$. If $f(t,0) \neq 0$ for some $t > 0$, then $0 < \|f(s,0)\| < \|f(t,0)\|$ for some $s \in (0,t)$, which contradicts (3) applied to $p = f(s,0)$ and $t - s$. Therefore $f(t,0) = 0$ for all $t \geq 0$. By (2), for $t \geq 0$ we have $0 = f(0,0) = f(-t, f(t,0)) = f(-t,0)$, and so $f(-t,0) = 0$ as well.

(ii) This follows from (3) and the fact that f induces a group action of $\mathbb{R}$ on $\mathbb{R}^N$, once we know that $f(t,p)$ is never 0. But if $f(t,p) = 0$ then $f(-t,0) = p$, which contradicts part (i).

(iii) Let $r_1(p)$ and $r_2(p)$ denote the respective limits. By part (ii), both limits exist, where $r_1(p) \in [0,\infty)$ and $r_2(p) \in (0,\infty]$. For any $r \in (0,\infty)$, consider the compact set $K_r := \bigcap_{s\geq 0} f(s, B^N_r)$. By (2), we have $K_r \subset f(t,K_r) = \bigcap_{s\geq t} f(s,B^N_r)$ for any $t \geq 0$. On the other hand, if $q \in K_r$ is a point with maximum norm, and $q \neq 0$, then (3) implies that $q \notin f(t,K_r)$ for any $t > 0$. Thus $K_r = \{0\}$. Taking $r = \|p\|$ implies that $r_1(p) = 0$. Suppose now that $r_2(p) \neq \infty$. Then for any $t \geq 0$, we have $f(-t,p) \in B^N_{r_2(p)}$, i.e. $p \in f(t, B^N_{r_2(p)})$. Thus $p \in K_{r_2(p)} = \{0\}$, a contradiction. $\square$

Lemma 2.1.3. Let $Q \subset \mathbb{R}^N$ be a smooth embedded submanifold of dimension $d \leq N$, and $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ a contractive flow. Suppose that Q is bounded and satisfies the condition

$$f(t,\overline{Q}) \subset Q \quad \text{for } t > 0. \tag{2.1.1}$$

Then the closure $\overline{Q}$ is homeomorphic to a closed ball of dimension d, and $\overline{Q} \setminus Q$ is homeomorphic to a sphere of dimension $d-1$.

Proof. Since Q is bounded, its closure $\overline{Q}$ is compact. By Lemma 2.1.2(iii) and (2.1.1) we have $0 \in \overline{Q}$, and therefore $0 \in Q$. Because Q is smoothly embedded, we can take $r > 0$ sufficiently small so that $B := B_r^N \cap Q$ is homeomorphic to a closed ball of dimension d. We let ∂B denote $(\partial B_r^N) \cap Q$, which is a $(d-1)$-dimensional sphere.

For any $p \in \mathbb{R}^N \setminus \{0\}$, consider the curve $t \mapsto f(t,p)$ starting at p and defined for all $t \in \mathbb{R}$. By Lemma 2.1.2(ii), this curve intersects the sphere ∂B_r^N for a unique $t \in \mathbb{R}$, which we denote by $t_r(p)$. Also, for $p \in \overline{Q} \setminus \{0\}$, define $t_\partial(p) \in (-\infty, 0]$ as follows. Let $T(p) := \{t \in \mathbb{R} : f(t,p) \in \overline{Q}\}$. We have $0 \in T(p)$, and $T(p)$ is bounded from below by Lemma 2.1.2(iii) because $\overline{Q}$ is bounded. By (2.1.1), if $t \in T(p)$ then $[t, \infty) \subset T(p)$. Also, $T(p)$ is closed since it is the preimage of $\overline{Q}$ under the continuous map $t \mapsto f(t,p)$. It follows that $T(p) = [t_\partial(p), \infty)$ for some $t_\partial(p) \in (-\infty, 0]$.

Claim. The functions t_r and t_∂ are continuous on $\overline{Q} \setminus \{0\}$.

Proof. First we prove that t_r is continuous on $\mathbb{R}^N \setminus \{0\}$. It suffices to show that the preimage of any open interval $I \subset \mathbb{R}$ is open. To this end, let $q \in t_r^{-1}(I)$. Take $t_1, t_2 \in I$ with $t_1 < t_r(q) < t_2$. By Lemma 2.1.2(ii), we have $\|f(t_1,q)\| > r > \|f(t_2,q)\|$. Note that the map $\gamma_1 : \mathbb{R}^N \to \mathbb{R}^N, p \mapsto f(t_1,p)$ is continuous and $\mathbb{R}^N \setminus B_r^N$ is open, so $\gamma_1^{-1}(\mathbb{R}^N \setminus B_r^N)$ is an open neighborhood of q. Similarly, defining $\gamma_2 : \mathbb{R}^N \to \mathbb{R}^N, p \mapsto f(t_2,p)$, we have that $\gamma_2^{-1}(\mathrm{int}(B_r^N))$ is an open neighborhood of q. Therefore $\gamma_1^{-1}(\mathbb{R}^N \setminus B_r^N) \cap \gamma_2^{-1}(\mathrm{int}(B_r^N))$ is an open neighborhood of q, whose image under t_r is contained in $(t_1,t_2) \subset I$. This shows that t_r is continuous on $\mathbb{R}^N \setminus \{0\}$.

Next, let us define

$$R := \{f(t,p) : (t,p) \in \mathbb{R} \times \overline{Q}\}. \tag{2.1.2}$$

The map $b : \mathbb{R} \times \partial B \to R \setminus \{0\}$ defined by $(t,p) \mapsto f(t,p)$ is a continuous bijection. (Recall that $\partial B = (\partial B_r^N) \cap Q$.) Its inverse $p \mapsto (-t_r(p), f(t_r(p),p))$ is continuous as well. Therefore b is a homeomorphism. We claim that Q is relatively open in R. Indeed, since $Q \setminus \{0\}$ is a submanifold of $R \setminus \{0\}$ of the same dimension d, we deduce that $Q \setminus \{0\}$ is an open subset of $R \setminus \{0\}$. Also, Q contains the neighborhood $\operatorname{int}(B_r^N) \cap R$ of 0 in R. Thus Q is an open subset of R.

We now prove that the map $t_\partial : \overline{Q} \setminus \{0\} \to \mathbb{R}$ is continuous, by a very similar argument. Let $I \subset \mathbb{R}$ be an open interval and consider a point $q \in t_\partial^{-1}(I)$. Take $t_1, t_2 \in I$ with $t_1 < t_\partial(q) < t_2$. By the definition of t_∂, we have $f(t_1, q) \in R \setminus \overline{Q}$. By (2.1.1), we have $f(t_2, q) \in Q$. Note that the map $\gamma_1 : R \to R, p \mapsto f(t_1, p)$ is continuous and $R \setminus \overline{Q}$ is open in R, so $\gamma_1^{-1}(R \setminus \overline{Q})$ is an open neighborhood of q in R. Similarly, defining $\gamma_2 : R \to R, p \mapsto f(t_2, p)$, we have that $\gamma_2^{-1}(Q)$ is an open neighborhood of q in R. Therefore $\gamma_1^{-1}(R \setminus \overline{Q}) \cap \gamma_2^{-1}(Q) \cap \overline{Q}$ is an open neighborhood of q in $\overline{Q}$, whose image under t_∂ is contained inside $(t_1, t_2) \subset I$. This finishes the proof of the claim. □

Define the maps $\alpha : \overline{Q} \to B$ and $\beta : B \to \overline{Q}$ by

$$\alpha(p) := f(t_r(p) - t_\partial(p), p), \quad \beta(p) := f(t_\partial(p) - t_r(p), p)$$

for $p \neq 0$, and $\alpha(0) := 0$, $\beta(0) := 0$. Let us verify that α sends $\overline{Q}$ inside B and β sends B inside $\overline{Q}$. If $p \in \overline{Q} \setminus \{0\}$, then $f(t_r(p), p) \in B$ and $t_\partial(p) \leq 0$, whence the contractive property (3) implies $\alpha(p) = f(-t_\partial(p), f(t_r(p), p)) \in B$. Similarly, if $p \in B \setminus \{0\}$, then $f(t_\partial(p), p) \in \overline{Q}$ and $t_r(p) \leq 0$, whence (2.1.1) implies $\beta(p) = f(-t_r(p), f(t_\partial(p), p)) \in \overline{Q}$.

Now we check that α and β are inverse maps. For any $p \in \overline{Q}$ and $\Delta t \in \mathbb{R}$ such that $f(\Delta t, p) \in \overline{Q}$, we have

$$t_r(f(\Delta t, p)) = t_r(p) - \Delta t, \quad t_\partial(f(\Delta t, p)) = t_\partial(p) - \Delta t.$$

Taking $\Delta t := t_\partial(p) - t_r(p)$, we find

$$\alpha(\beta(p)) = \alpha(f(\Delta t, p)) = f(t_r(f(\Delta t, p)) - t_\partial(f(\Delta t, p)), f(\Delta t, p)) = f(-\Delta t, f(\Delta t, p)) = p.$$

We can similarly verify that $\beta(\alpha(p)) = p$, by instead taking $\Delta t := t_r(p) - t_\partial(p)$.

By the claim, t_r and t_∂ are continuous on $\overline{Q} \setminus \{0\}$, so α is continuous everywhere except possibly at 0. Also, $t_r(p) > t_\partial(p)$ for all $p \in \overline{Q} \setminus \{0\}$, so α is continuous at 0 by Lemma 2.1.2(ii).

Thus α is a continuous bijection from a compact space to a Hausdorff space, so it is a homeomorphism. This shows that $\overline{Q}$ is homeomorphic to a closed d-dimensional ball.

It remains to prove that $\overline{Q} \setminus Q$ is homeomorphic to a $(d-1)$-dimensional sphere. We claim that α restricts to a homeomorphism from $\overline{Q} \setminus Q$ to ∂B. We need to check that α sends $\overline{Q} \setminus Q$ inside ∂B, and β sends ∂B inside $\overline{Q} \setminus Q$. To this end, let $p \in \overline{Q} \setminus Q$. By condition (2), we have $p = f(-t_\partial(p), f(t_\partial(p), p))$. Hence if $t_\partial(p) < 0$, then (2.1.1) implies $p \in Q$, a contradiction. Therefore $t_\partial(p) = 0$, and $\alpha(p) = f(t_r(p), p) \in \partial B$. Now let $q \in \partial B$. We have $t_r(q) = 0$, so $\beta(q) = f(t_\partial(q), q)$. If $\beta(q) \in Q$, then $f(t_\partial(q) - t, q) \in Q$ for $t > 0$ sufficiently small (as Q is open in R from (2.1.2)), contradicting the definition of $t_\partial(q)$. Thus $\beta(q) \in \overline{Q} \setminus Q$. $\square$

2.2 The totally nonnegative Grassmannian

Let $\mathrm{Gr}(k,n)$ denote the *real Grassmannian*, the space of all k-dimensional subspaces of $\mathbb{R}^n$. We set $[n] := \{1, \dots, n\}$, and let $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$. For $X \in \mathrm{Gr}(k,n)$, we denote by $(\Delta_I(X))_{I \in \binom{[n]}{k}} \in \mathbb{P}^{\binom{n}{k}-1}$ the *Plücker coordinates* of X: $\Delta_I(X)$ is the $k \times k$ minor of X (viewed as a $k \times n$ matrix modulo row operations) with column set I.

Recall that $\mathrm{Gr}_{\geq 0}(k,n)$ is the subset of $\mathrm{Gr}(k,n)$ where all Plücker coordinates are nonnegative (up to a common scalar). We also define the *totally positive Grassmannian* $\mathrm{Gr}_{>0}(k,n)$ as the subset of $\mathrm{Gr}_{\geq 0}(k,n)$ where all Plücker coordinates are positive.

2.2.1 Global coordinates for $\mathrm{Gr}_{\geq 0}(k,n)$

For each k and n, we introduce several distinguished linear operators on $\mathbb{R}^n$. Define the *left cyclic shift* $S \in \mathfrak{gl}_n(\mathbb{R}) = \mathrm{End}(\mathbb{R}^n)$ by $S(v_1, \dots, v_n) := (v_2, \dots, v_n, (-1)^{k-1}v_1)$. The sign $(-1)^{k-1}$ can be explained as follows: if we pretend that S is an element of $\mathrm{GL}_n(\mathbb{R})$, then the action of S on $\mathrm{Gr}(k,n)$ preserves $\mathrm{Gr}_{\geq 0}(k,n)$ (it acts on Plücker coordinates by rotating the index set $[n]$).

Note that the transpose S^T of S is the *right cyclic shift* given by $S^T(v_1, \dots, v_n) = ((-1)^{k-1}v_n, v_1, \dots, v_{n-1})$. Let $\tau := S + S^T \in \mathrm{End}(\mathbb{R}^n)$. We endow $\mathbb{R}^n$ with the standard inner product, so that τ (being symmetric) has an orthogonal basis of eigenvectors

$u_1, \ldots, u_n \in \mathbb{R}^n$ corresponding to real eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$. Let $X_0 \in \mathrm{Gr}(k,n)$ be the linear span of $u_1, \ldots, u_k$. The following lemma implies that X_0 is totally positive and does not depend on the choice of eigenvectors $u_1, \ldots, u_n$.

Lemma 2.2.1.

(i) The eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ are given as follows, depending on the parity of k:

- if k is even, $\lambda_1 = \lambda_2 = 2\cos(\frac{\pi}{n})$, $\lambda_3 = \lambda_4 = 2\cos(\frac{3\pi}{n})$, $\lambda_5 = \lambda_6 = 2\cos(\frac{5\pi}{n})$, $\ldots$;
- if k is odd, $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = 2\cos(\frac{2\pi}{n})$, $\lambda_4 = \lambda_5 = 2\cos(\frac{4\pi}{n})$, $\ldots$.

In either case, we have

$$\lambda_k = 2\cos\left(\tfrac{k-1}{n}\pi\right) > 2\cos\left(\tfrac{k+1}{n}\pi\right) = \lambda_{k+1}.$$

(ii) [Sco79] The Plücker coordinates of X_0 are given by

$$\Delta_I(X_0) = \prod_{i,j \in I,\ i<j} \sin\left(\tfrac{j-i}{n}\pi\right) > 0 \quad \text{for all } I \in \tbinom{[n]}{k}. \tag{2.2.1}$$

For an example in the case of $\mathrm{Gr}(2,4)$, see Section 2.2.5. (We remark that in the example, the Plücker coordinates of X_0 are scaled by a factor of 2 compared to the formula above.)

Proof. In this proof, we work over $\mathbb{C}$. Let $\zeta \in \mathbb{C}$ be an nth root of $(-1)^{k-1}$. There are n such values of ζ, each of the form $\zeta = e^{i\pi m/n}$ for some integer m congruent to $k-1$ modulo 2. Let $z_m := (1, \zeta, \zeta^2, \ldots, \zeta^{n-1}) \in \mathbb{C}^n$. We have $S(z_m) = \zeta z_m$ and $S^T(z_m) = \zeta^{-1} z_m$, so

$$\tau(z_m) = (\zeta + \zeta^{-1}) z_m = 2\cos(\tfrac{\pi m}{n}) z_m. \tag{2.2.2}$$

The n distinct z_m's are linearly independent (they form an $n \times n$ Vandermonde matrix with nonzero determinant), so they give a basis of $\mathbb{C}^n$ of eigenvectors of τ.

We deduce part (i) from (2.2.2). For part (ii), we apply Vandermonde's determinantal identity, following an argument outlined by Scott [Sco79]. That is, by (2.2.2), the $\mathbb{C}$-linear span of $u_1, \ldots, u_k$ is the same as the span of $z_{-k+1}, z_{-k+3}, z_{-k+5}, \ldots, z_{k-1}$. Let M be the matrix whose rows are $z_{-k+1}, z_{-k+3}, z_{-k+5}, \ldots, z_{k-1}$, i.e.

$$M_{r,j} = e^{i\pi(-k-1+2r)(j-1)/n} \qquad \text{for } 1 \le r \le k \text{ and } 1 \le j \le n.$$

Then the Plücker coordinates of X_0 are the $k \times k$ minors of M (up to a common nonzero complex scalar), which can be computed explicitly by Vandermonde's identity after appropriately rescaling the columns. We refer the reader to [Kar18] for details. □

Denote by $\mathrm{Mat}(k, n-k)$ the vector space of real $k \times (n-k)$ matrices. Define a map $\phi : \mathrm{Mat}(k, n-k) \to \mathrm{Gr}(k,n)$ by

$$\phi(A) := \mathrm{span}(u_i + \textstyle\sum_{j=1}^{n-k} A_{i,j} u_{k+j} : 1 \le i \le k). \tag{2.2.3}$$

In other words, the entries of A are the usual coordinates on the big Schubert cell of $\mathrm{Gr}(k,n)$ with respect to the basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$, this Schubert cell being

$$\phi(\mathrm{Mat}(k, n-k)) = \{X \in \mathrm{Gr}(k,n) : X \cap \mathrm{span}(u_{k+1}, \ldots, u_n) = 0\}.$$

In particular, ϕ is a smooth embedding, and it sends the zero matrix to X_0. For an example in the case of $\mathrm{Gr}(2,4)$, see Section 2.2.5.

Proposition 2.2.2. The image $\phi(\mathrm{Mat}(k, n-k))$ contains $\mathrm{Gr}_{\ge 0}(k,n)$.

Proof. Let $X \in \mathrm{Gr}_{\ge 0}(k,n)$ be a totally nonnegative subspace. We need to show that $X \cap \mathrm{span}(u_{k+1}, \ldots, u_n) = 0$. Suppose otherwise that there exists a nonzero vector v in this intersection. Extend v to a basis of X, and write this basis as the rows of a $k \times n$ matrix M. Because X is totally nonnegative, the nonzero $k \times k$ minors of M all have the same sign (and at least one minor is nonzero, since M has rank k). Also let M_0 be the $k \times n$ matrix with rows $u_1, \ldots, u_k$. By Lemma 2.2.1(ii), all $k \times k$ minors of M_0 are nonzero and have the same sign. The vectors $u_1, \ldots, u_n$ are orthogonal, so v is orthogonal to the rows of M_0. Hence the first column of $M_0 M^T$ is zero, and we obtain $\det(M_0 M^T) = 0$. On the other hand, the

Cauchy–Binet identity implies

$$\det(M_0 M^T) = \sum_{I \in \binom{[n]}{k}} \det((M_0)_I)\det(M_I),$$

where A_I denotes the matrix A restricted to the columns I. Each summand has the same sign and at least one summand is nonzero, contradicting $\det(M_0 M^T) = 0$. $\square$

We have shown that the restriction of ϕ^{-1} to $\mathrm{Gr}_{\geq 0}(k,n)$ yields an embedding

$$\mathrm{Gr}_{\geq 0}(k,n) \hookrightarrow \mathrm{Mat}(k, n-k) \simeq \mathbb{R}^{k(n-k)}$$

whose restriction to $\mathrm{Gr}_{>0}(k,n)$ is smooth.

2.2.2 Flows on $\mathrm{Gr}(k,n)$

For $g \in \mathrm{GL}_n(\mathbb{R})$, we let g act on $\mathrm{Gr}(k,n)$ by taking the subspace X to $g \cdot X := \{g(v) : v \in X\}$. We let $1 \in \mathrm{GL}_n(\mathbb{R})$ denote the identity matrix, and for $x \in \mathfrak{gl}_n(\mathbb{R})$ we let $\exp(x) := \sum_{j=0}^{\infty} \frac{x^j}{j!} \in \mathrm{GL}_n(\mathbb{R})$ denote the matrix exponential of x.

We examine the action of $\exp(tS)$ and $\exp(t\tau)$ on $\mathrm{Gr}(k,n)$.

Lemma 2.2.3. For $X \in \mathrm{Gr}_{\geq 0}(k,n)$ and $t > 0$, we have $\exp(tS) \cdot X \in \mathrm{Gr}_{>0}(k,n)$.

Proof. We claim that it suffices to prove the following two facts:

(i) for $X \in \mathrm{Gr}_{\geq 0}(k,n)$ and $t \geq 0$, we have $\exp(tS) \cdot X \in \mathrm{Gr}_{\geq 0}(k,n)$; and

(ii) for $X \in \mathrm{Gr}_{\geq 0}(k,n) \setminus \mathrm{Gr}_{>0}(k,n)$, we have $\exp(tS) \cdot X \notin \mathrm{Gr}_{\geq 0}(k,n)$ for all $t < 0$ sufficiently close to zero.

To see why this is sufficient, let $X \in \mathrm{Gr}_{\geq 0}(k,n)$ and $t > 0$. By part (i), we have $\exp(tS) \cdot X \in \mathrm{Gr}_{\geq 0}(k,n)$, so we just need to show that $\exp(tS) \cdot X \in \mathrm{Gr}_{>0}(k,n)$. Suppose otherwise that $\exp(tS) \cdot X \notin \mathrm{Gr}_{>0}(k,n)$. Then applying part (ii) to $\exp(tS) \cdot X$, we get that $\exp((t + t')S) \cdot X \notin \mathrm{Gr}_{\geq 0}(k,n)$ for $t' < 0$ sufficiently close to zero. But by part (i), we know that $\exp((t+t')S) \cdot X \in \mathrm{Gr}_{\geq 0}(k,n)$ for all t' in the interval $[-t, 0]$. This is a contradiction.

Now we prove parts (i) and (ii). We will make use of the operator $1 + tS$, which belongs to $\mathrm{GL}_n(\mathbb{R})$ for $|t| < 1$. Note that if $[M_1 \mid \cdots \mid M_n]$ is a $k \times n$ matrix representing X, then a $k \times n$ matrix representing $(1 + tS) \cdot X$ is

$$M' = [M_1 + tM_2 \mid M_2 + tM_3 \mid \cdots \mid M_{n-1} + tM_n \mid M_n + (-1)^{k-1}tM_1].$$

We can evaluate the $k \times k$ minors of M' using multilinearity of the determinant. We obtain

$$\Delta_I((1 + tS) \cdot X) = \sum_{\epsilon \in \{0,1\}^k} t^{\epsilon_1+\cdots+\epsilon_k} \Delta_{\{i_1+\epsilon_1,\ldots,i_k+\epsilon_k\}}(X) \quad \text{for } I = \{i_1, \ldots, i_k\} \subset [n], \quad (2.2.4)$$

where $i_1 + \epsilon_1, \ldots, i_k + \epsilon_k$ are taken modulo n. Therefore $(1 + tS) \cdot X \in \mathrm{Gr}_{\geq 0}(k, n)$ for $X \in \mathrm{Gr}_{\geq 0}(k, n)$ and $t \in [0, 1)$. Since $\exp(tS) = \lim_{j\to\infty} \left(1 + \frac{tS}{j}\right)^j$ and $\mathrm{Gr}_{\geq 0}(k, n)$ is closed, we obtain $\exp(tS) \cdot X \in \mathrm{Gr}_{\geq 0}(k, n)$ for $t \geq 0$. This proves part (i).

To prove part (ii), first note that $\exp(tS) = 1 + tS + O(t^2)$. By (2.2.4), we have

$$\Delta_I(\exp(tS) \cdot X) = \Delta_I(X) + t \sum_{I'} \Delta_{I'}(X) + O(t^2) \quad \text{for } I \in \tbinom{[n]}{k}, \quad (2.2.5)$$

where the sum is over all $I' \in \binom{[n]}{k}$ obtained from I by increasing exactly one element by 1 modulo n. If we can find such I and I' with $\Delta_I(X) = 0$ and $\Delta_{I'}(X) > 0$, then $\Delta_I(\exp(tS) \cdot X) < 0$ for all $t < 0$ sufficiently close to zero, thereby proving part (ii). In order to do this, we introduce the directed graph D with vertex set $\binom{[n]}{k}$, where $J \to J'$ is an edge of D if and only if we can obtain J' from J by increasing exactly one element by 1 modulo n. Note that for any two vertices K and K' of D, there exists a directed path from K to K':

- we can get from $[k]$ to any $\{i_1 < \cdots < i_k\}$ by shifting k to i_k, $k - 1$ to i_{k-1}, etc.;
- similarly, we can get from any $\{i_1 < \cdots < i_k\}$ to $\{n - k + 1, n - k + 2, \ldots, n\}$;
- we can get from $\{n - k + 1, \ldots, n\}$ to $[k]$ by shifting n to k, $n - 1$ to $k - 1$, etc.

Now take $K, K' \in \binom{[n]}{k}$ with $\Delta_K(X) = 0$ and $\Delta_{K'}(X) > 0$, and consider a directed path from K to K'. It goes through an edge $I \to I'$ with $\Delta_I(X) = 0$ and $\Delta_{I'}(X) > 0$, as desired. $\square$

Now we consider $\exp(t\tau) = \exp(t(S + S^T))$. Recall that S and S^T are the left and right cyclic shift maps, so by symmetry Lemma 2.2.3 holds with S replaced by S^T. Also, S and S^T commute, so $\exp(t\tau) = \exp(tS)\exp(tS^T)$. We obtain the following.

Corollary 2.2.4. For $X \in \mathrm{Gr}_{\geq 0}(k, n)$ and $t > 0$, we have $\exp(t\tau) \cdot X \in \mathrm{Gr}_{>0}(k, n)$.

Let us see how $\exp(t\tau)$ acts on matrices $A \in \mathrm{Mat}(k, n-k)$. Note that $\tau(u_i) = \lambda_i u_i$ for $1 \leq i \leq n$, so $\exp(t\tau)(u_i) = e^{t\lambda_i} u_i$. Therefore $\exp(t\tau)$ acts on the basis of $\phi(A)$ in (2.2.3) by

$$\exp(t\tau)(u_i + \textstyle\sum_{j=1}^{n-k} A_{i,j} u_{k+j}) = e^{t\lambda_i}(u_i + \textstyle\sum_{j=1}^{n-k} e^{t(\lambda_{k+j} - \lambda_i)} A_{i,j} u_{k+j})$$

for all $1 \leq i \leq k$. Thus $\exp(t\tau) \cdot \phi(A) = \phi(f(t, A))$, where by definition $f(t, A) \in \mathrm{Mat}(k, n-k)$ is the matrix with entries

$$(f(t, A))_{i,j} := e^{t(\lambda_{k+j} - \lambda_i)} A_{i,j} \quad \text{for } 1 \leq i \leq k \text{ and } 1 \leq j \leq n-k. \tag{2.2.6}$$

2.2.3 Proof of Theorem 1.1.1

Consider the map $f : \mathbb{R} \times \mathrm{Mat}(k, n-k) \to \mathrm{Mat}(k, n-k)$ defined by (2.2.6). We claim that f is a contractive flow on $\mathrm{Mat}(k, n-k)$ equipped with the Euclidean norm

$$\|A\|^2 = \sum_{i=1}^{k} \sum_{j=1}^{n-k} A_{i,j}^2.$$

Indeed, parts (1) and (2) of Definition 2.1.1 hold for f. To see that part (3) holds, note that for any $1 \leq i \leq k$ and $1 \leq j \leq n-k$ with $A_{i,j} \neq 0$, we have

$$|(f(t, A))_{i,j}| = |e^{t(\lambda_{k+j} - \lambda_i)} A_{i,j}| = e^{t(\lambda_{k+j} - \lambda_i)} |A_{i,j}| < |A_{i,j}| \quad \text{for } t > 0,$$

using the fact that $\lambda_i \geq \lambda_k > \lambda_{k+1} \geq \lambda_{k+j}$ from Lemma 2.2.1(i). Therefore $\|f(t, A)\| < \|A\|$ if $A \neq 0$, verifying part (3).

Let us now apply Lemma 2.1.3 with $\mathbb{R}^N = \mathrm{Mat}(k, n-k)$ and $Q = \phi^{-1}(\mathrm{Gr}_{>0}(k, n))$. We need to know that $\mathrm{Gr}_{\geq 0}(k, n)$ is the closure of $\mathrm{Gr}_{>0}(k, n)$. This was proved by Postnikov [Pos07, Section 17]; it also follows directly from Corollary 2.2.4, since we can express any $X \in \mathrm{Gr}_{\geq 0}(k,n)$ as a limit of totally positive subspaces:

$$X = \lim_{t \to 0+} \exp(t\tau) \cdot X.$$

Therefore $\overline{Q} = \phi^{-1}(\mathrm{Gr}_{\geq 0}(k,n))$. Moreover, $\mathrm{Gr}_{\geq 0}(k,n)$ is closed inside the compact space $\mathbb{P}^{\binom{n}{k}-1}$, and is therefore also compact. So, $\overline{Q}$ is compact (and hence bounded). Finally, the property (2.1.1) in this case is precisely Corollary 2.2.4. We have verified all the hypotheses of Lemma 2.1.3, and conclude that $\overline{Q}$ (and also $\mathrm{Gr}_{\geq 0}(k,n)$) is homeomorphic to a $k(n-k)$-dimensional closed ball. $\square$

2.2.4 Related work

Lusztig [Lus98b, Section 4] used a flow similar to $\exp(t\tau)$ to show that $(G/P)_{\geq 0}$ is contractible. In fact, we used his flow to show that $(G/P)_{\geq 0}$ is a ball in [GKL18]. Our flow can be thought of as an affine (or loop group) analogue of his flow, and is closely related to the whirl matrices of [LP12]. We also remark that Ayala, Kliemann, and San Martin [AKSM04] used the language of control theory to give an alternative development in type A of Lusztig's theory of total positivity. In that context, $\exp(t\tau)$ $(t > 0)$ lies in the interior of the *compression semigroup* of $\mathrm{Gr}_{\geq 0}(k,n)$, and X_0 is its *attractor*.

Marsh and Rietsch defined and studied a *superpotential* on the Grassmannian in the context of mirror symmetry [MR15, Section 6]. It follows from results of Rietsch [Rie08] (as explained in [Kar18]) that X_0 is, rather surprisingly, also the unique totally nonnegative critical point of the $q = 1$ specialization of the superpotential. However, the superpotential is not defined on the boundary of $\mathrm{Gr}_{\geq 0}(k,n)$. The precise relationship between τ and the gradient flow of the superpotential remains mysterious.

2.2.5 Example: the case $\mathrm{Gr}(2,4)$

The matrix $\tau = S + S^T \in \mathfrak{gl}_4(\mathbb{R})$ and an orthogonal basis of real eigenvectors u_1, u_2, u_3, u_4 are

$$\tau = \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{bmatrix}, \qquad \begin{aligned} u_1 &= (0,1,\sqrt{2},1), & \lambda_1 &= \sqrt{2}, \\ u_2 &= (-\sqrt{2},-1,0,1), & \lambda_2 &= \sqrt{2}, \\ u_3 &= (\sqrt{2},-1,0,1), & \lambda_3 &= -\sqrt{2}, \\ u_4 &= (0,1,-\sqrt{2},1), & \lambda_4 &= -\sqrt{2}. \end{aligned}$$

The embedding $\phi : \mathrm{Mat}(2,2) \hookrightarrow \mathrm{Gr}(2,4)$ sends the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to

$$\phi(A) = X = \begin{bmatrix} u_1 + au_3 + bu_4 \\ u_2 + cu_3 + du_4 \end{bmatrix} = \begin{bmatrix} \sqrt{2}a & 1-a+b & \sqrt{2}-\sqrt{2}b & 1+a+b \\ -\sqrt{2}+\sqrt{2}c & -1-c+d & -\sqrt{2}d & 1+c+d \end{bmatrix}.$$

Above we are identifying $X \in \mathrm{Gr}(2,4)$ with a 2×4 matrix whose rows form a basis of X. In terms of Plücker coordinates $\Delta_{ij} = \Delta_{\{i,j\}}(X)$, the map ϕ is given by

$$\begin{aligned} &\Delta_{12} = \sqrt{2}(1-2a+b-c+ad-bc), \\ &\Delta_{23} = \sqrt{2}(1-2d-b+c+ad-bc), \qquad \Delta_{13} = 2(1-b-c-ad+bc), \\ &\Delta_{34} = \sqrt{2}(1+2d-b+c+ad-bc), \qquad \Delta_{24} = 2(1+b+c-ad+bc), \\ &\Delta_{14} = \sqrt{2}(1+2a+b-c+ad-bc), \end{aligned} \tag{2.2.7}$$

and its inverse is given by

$$\begin{aligned} a &= (2\Delta_{14} - 2\Delta_{12})/\delta, \qquad b = (\Delta_{12} - \Delta_{23} - \Delta_{34} + \Delta_{14} - \sqrt{2}\Delta_{13} + \sqrt{2}\Delta_{24})/\delta, \\ d &= (2\Delta_{34} - 2\Delta_{23})/\delta, \qquad c = (-\Delta_{12} + \Delta_{23} + \Delta_{34} - \Delta_{14} - \sqrt{2}\Delta_{13} + \sqrt{2}\Delta_{24})/\delta, \end{aligned}$$
$$\text{where } \delta = \Delta_{12} + \Delta_{23} + \Delta_{34} + \Delta_{14} + \sqrt{2}\Delta_{13} + \sqrt{2}\Delta_{24}.$$

The point $X_0 = \phi(0) = \mathrm{span}(u_1, u_2) \in \mathrm{Gr}_{>0}(2,4)$ has Plücker coordinates

$$\Delta_{12} = \Delta_{23} = \Delta_{34} = \Delta_{14} = \sqrt{2}, \quad \Delta_{13} = \Delta_{24} = 2,$$

which agrees with Lemma 2.2.1(ii). The image of ϕ is the subset of $\mathrm{Gr}(2,4)$ where $\delta \neq 0$, which we see includes $\mathrm{Gr}_{\geq 0}(2,4)$, verifying Proposition 2.2.2 in this case. Restricting ϕ^{-1} to $\mathrm{Gr}_{\geq 0}(2,4)$ gives a homeomorphism onto the subset of $\mathbb{R}^4$ of points (a,b,c,d) where the 6 polynomials Δ_{ij} in (2.2.7) are nonnegative. By Theorem 1.1.1, these spaces are both homeomorphic to 4-dimensional closed balls. The closures of cells in the cell decomposition of $\mathrm{Gr}_{\geq 0}(2,4)$ are obtained in $\mathbb{R}^4$ by taking an intersection with the zero locus of some subset of the 6 polynomials. The 0-dimensional cells (corresponding to points of $\mathrm{Gr}_{\geq 0}(2,4)$ with only one nonzero Plücker coordinate) are

$$(a,b,c,d) = (-2,1,-1,0),\ (0,-1,1,-2),\ (0,-1,1,2),\ (2,1,-1,0),\ (0,-1,-1,0),\ (0,1,1,0).$$

In general, using the embedding ϕ we can describe $\mathrm{Gr}_{\geq 0}(k,n)$ as the subset of $\mathbb{R}^{k(n-k)}$ where some $\binom{n}{k}$ polynomials of degree at most k are nonnegative.

2.3 The cyclically symmetric amplituhedron

Let k, m, n be nonnegative integers with $k + m \leq n$ and m even, and let $S, \tau \in \mathfrak{gl}_n(\mathbb{R})$ be the operators from Section 2.2.1. Let $\lambda_1 \geq \cdots \geq \lambda_n \in \mathbb{R}$ be the eigenvalues of τ corresponding to orthogonal eigenvectors $u_1, \ldots, u_n$. In this section, we assume that these eigenvectors have norm 1. Recall from Lemma 2.2.1(i) that $\lambda_k > \lambda_{k+1}$. Since m is even, we have $(-1)^{k+m-1} = (-1)^{k-1}$ and $\lambda_{k+m} > \lambda_{k+m+1}$.

Let Z_0 denote the $(k+m)\times n$ matrix whose rows are $u_1, \ldots, u_{k+m}$. By Lemma 2.2.1(ii), the $(k+m)\times(k+m)$ minors of Z_0 are all positive (perhaps after replacing u_1 with $-u_1$). We may also think of Z_0 as a linear map $\mathbb{R}^n \to \mathbb{R}^{k+m}$. Since the vectors $u_1, \ldots, u_n$ are orthonormal, this map takes u_i to the ith unit vector $e_i \in \mathbb{R}^{k+m}$ if $i \leq k+m$, and to 0 if $i > k+m$. Recall from Section 1.4 that Z_0 induces a map $(Z_0)_{\mathrm{Gr}} : \mathrm{Gr}_{\geq 0}(k,n) \to \mathrm{Gr}(k, k+m)$, whose image is the *cyclically symmetric amplituhedron* $\mathcal{A}_{n,k,m}(Z_0)$. We remark that if $g \in \mathrm{GL}_{k+m}(\mathbb{R})$, then $\mathcal{A}_{n,k,m}(gZ_0)$ and $\mathcal{A}_{n,k,m}(Z_0)$ are related by the automorphism g of $\mathrm{Gr}(k,k+m)$, so the topology of $\mathcal{A}_{n,k,m}(Z_0)$ depends only on the row span of Z_0 in $\mathrm{Gr}(k+m,n)$.

Proof of Theorem 1.4.1. We consider the map $\phi : \mathrm{Mat}(k,n-k) \to \mathrm{Gr}(k,n)$ defined in (2.2.3). We write each $k\times(n-k)$ matrix $A \in \mathrm{Mat}(k,n-k)$ as $[A' \mid A'']$, where A' and A'' are the $k\times m$ and $k\times(n-k-m)$ submatrices of A with column sets $\{1,\dots,m\}$ and $\{m+1,\dots,n-k\}$, respectively. We introduce a projection map

$$\pi : \mathrm{Mat}(k,n-k) \to \mathrm{Mat}(k,m), \quad A = [A' \mid A''] \mapsto A'.$$

We claim that there exists an embedding $\gamma : \mathcal{A}_{n,k,m}(Z_0) \hookrightarrow \mathrm{Mat}(k,m)$ making the following diagram commute:

$$\begin{tikzcd} \mathrm{Mat}(k,n-k) \arrow[r, "\pi"] & \mathrm{Mat}(k,m) \\ \mathrm{Gr}_{\geq 0}(k,n) \arrow[u, hook, "\phi^{-1}"] \arrow[r, "(Z_0)_{\mathrm{Gr}}"] & \mathcal{A}_{n,k,m}(Z_0) \arrow[u, hook, dashed, "\gamma"'] \end{tikzcd} . \tag{2.3.1}$$

Let $A = [A' \mid A''] \in \mathrm{Mat}(k,n-k)$ be a matrix such that $\phi(A) \in \mathrm{Gr}_{\geq 0}(k,n)$. Then the element $(Z_0)_{\mathrm{Gr}}(\phi(A))$ of $\mathrm{Gr}(k,k+m)$ is the row span of the $k\times(k+m)$ matrix $[\mathrm{Id}_k \mid A']$, where Id_k denotes the $k\times k$ identity matrix. Thus $\mathcal{A}_{n,k,m}(Z_0) = (Z_0)_{\mathrm{Gr}}(\mathrm{Gr}_{\geq 0}(k,n))$ lies inside the Schubert cell

$$\{Y \in \mathrm{Gr}(k,k+m) : \Delta_{[k]}(Y) \neq 0\}.$$

Every element Y of this Schubert cell is the row span of $[\mathrm{Id}_k \mid A']$ for a unique A', and we define $\gamma(Y) := A'$. Thus γ embeds $\mathcal{A}_{n,k,m}(Z_0)$ inside $\mathrm{Mat}(k,m)$, and (2.3.1) commutes.

Now we define

$$Q_0 := \pi(\phi^{-1}(\mathrm{Gr}_{>0}(k,n))) \subset \mathrm{Mat}(k,m).$$

We know from Section 2.2.3 that $\phi^{-1}(\mathrm{Gr}_{>0}(k,n))$ is an open subset of $\mathrm{Mat}(k,n)$ whose closure $\phi^{-1}(\mathrm{Gr}_{\geq 0}(k,n))$ is compact. Note that π is an open map (since it is essentially a projection $\mathbb{R}^{k(n-k)} \to \mathbb{R}^{km}$), so Q_0 is an open subset of $\mathrm{Mat}(k,m)$. Note that any open subset of $\mathbb{R}^N$ is a smooth embedded submanifold of dimension N. The closure $\overline{Q_0} = \pi(\phi^{-1}(\mathrm{Gr}_{\geq 0}(k,n)))$ of Q_0 is compact. By (2.3.1), $\overline{Q_0}$ is homeomorphic to $\mathcal{A}_{n,k,m}(Z_0)$.

Let $f : \mathbb{R}\times\mathrm{Mat}(k,n-k) \to \mathrm{Mat}(k,n-k)$ be the map defined by (2.2.6), and define a similar map $f_0 : \mathbb{R} \times \mathrm{Mat}(k,m) \to \mathrm{Mat}(k,m)$ by

$$f_0(t, A')_{i,j} := e^{t(\lambda_{k+j}-\lambda_i)} A'_{i,j} \quad \text{for } 1 \le i \le k \text{ and } 1 \le j \le m.$$

That is, $f_0(t, \pi(A)) = \pi(f(t,A))$ for all $t \in \mathbb{R}$ and $A \in \mathrm{Mat}(k, n-k)$. We showed in Section 2.2.3 that f is a contractive flow, so f_0 is also a contractive flow. We also showed that

$$f(t, \phi^{-1}(\mathrm{Gr}_{\ge 0}(k,n))) \subset \phi^{-1}(\mathrm{Gr}_{>0}(k,n)) \quad \text{for } t > 0,$$

and applying π to both sides shows that

$$f_0(t, \overline{Q_0}) \subset Q_0 \quad \text{for } t > 0.$$

Thus Lemma 2.1.3 applies to Q_0 and f_0, showing that $\overline{Q_0}$ (and hence $\mathcal{A}_{n,k,m}(Z_0)$) is homeomorphic to a km-dimensional closed ball. □

Example 2.3.1. Let $k = 1$, $n = 4$, $m = 2$. We have

$$\tau = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \qquad \begin{aligned} u_1 &= \left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\right), & \lambda_1 &= 2, \\ u_2 &= \left(\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}, 0\right), & \lambda_2 &= 0, \\ u_3 &= \left(0, \tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\right), & \lambda_3 &= 0, \\ u_4 &= \left(\tfrac{1}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, -\tfrac{1}{2}\right), & \lambda_4 &= -2, \end{aligned} \qquad Z_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Note that this τ differs in the top-right and bottom-left entries from the one in Section 2.2.5, because k is odd rather than even. Also, here the eigenvectors are required to have norm 1. The embedding $\phi : \mathrm{Mat}(1,3) \hookrightarrow \mathrm{Gr}(1,4)$ sends a matrix $A := \begin{bmatrix} a & b & c \end{bmatrix}$ to the line $\phi(A)$ in $\mathrm{Gr}(1,4)$ spanned by the vector

$$v = u_1 + au_2 + bu_3 + cu_4 = \frac{1}{2}\left(1 + \sqrt{2}a + c, 1 + \sqrt{2}b - c, 1 - \sqrt{2}a + c, 1 - \sqrt{2}b - c\right).$$

This line gets sent by $(Z_0)_{\mathrm{Gr}}$ to the row span of the matrix $v \cdot Z_0^T = \begin{bmatrix} 1 & a & b \end{bmatrix}$. Finally, γ sends this element of $\mathrm{Gr}(1,3)$ to the matrix $\begin{bmatrix} a & b \end{bmatrix}$, so (2.3.1) indeed commutes.

In order for $\phi(A)$ to land in $\mathrm{Gr}_{\geq 0}(1,4)$, the coordinates of v must all have the same sign, and since their sum is 2, they must all be nonnegative:

$$1+\sqrt{2}a+c \geq 0, \quad 1+\sqrt{2}b-c \geq 0, \quad 1-\sqrt{2}a+c \geq 0, \quad 1-\sqrt{2}b-c \geq 0.$$

These linear inequalities define a tetrahedron in $\mathbb{R}^3 \simeq \mathrm{Mat}(1,3)$ with the four vertices $\left(0, \pm\sqrt{2}, -1\right), \left(\pm\sqrt{2}, 0, 1\right)$. The projection $\pi = \gamma \circ (Z_0)_{\mathrm{Gr}} \circ \phi$ sends this tetrahedron to a square in $\mathbb{R}^2 \simeq \mathrm{Mat}(1,2)$ with vertices $\left(0, \pm\sqrt{2}\right), \left(\pm\sqrt{2}, 0\right)$. This square is a km-dimensional ball, as implied by Theorem 1.4.1. We note that when $k=1$, the amplituhedron $\mathcal{A}_{n,k,m}(Z)$ (for any $(k+m)\times n$ matrix Z with positive maximal minors) is a cyclic polytope in the projective space $\mathrm{Gr}(1, m+1) = \mathbb{P}^m$ [Stu88], and is therefore homeomorphic to a km-dimensional closed ball. The case of $k \geq 2$ and $Z \neq Z_0$ remains open.

2.4 The space of planar electrical networks

2.4.1 A slice of the totally nonnegative Grassmannian

We recall some background on electrical networks, and refer the reader to [Lam18] and Example 2.4.3 for details. Let $\mathbb{R}^{\binom{2n}{n-1}}$ have basis vectors e_I for $I \in \binom{[2n]}{n-1}$, and let $\mathbb{P}^{\binom{2n}{n-1}-1}$ denote the corresponding projective space. We define

$$[2n]_{\mathrm{odd}} := \{2i-1 : i \in [n]\}, \quad [2n]_{\mathrm{even}} := \{2i : i \in [n]\}.$$

Let $\mathcal{NC}_n$ denote the set of non-crossing partitions of $[2n]_{\mathrm{odd}}$. Each $\sigma \in \mathcal{NC}_n$ comes with a *dual non-crossing partition* (or *Kreweras complement*) $\tilde{\sigma}$ of $[2n]_{\mathrm{even}}$. We call a subset $I \in \binom{[2n]}{n-1}$ *concordant* with σ if every part of σ and every part of $\tilde{\sigma}$ contains exactly one element not in I. Let $A_\sigma \in \mathbb{R}^{\binom{2n}{n-1}}$ be the sum of e_I over all I concordant with σ, and let $\mathcal{H}$ be the linear subspace of $\mathbb{P}^{\binom{2n}{n-1}-1}$ spanned by the images of A_σ for $\sigma \in \mathcal{NC}_n$.

Identifying $\mathrm{Gr}(n-1, 2n)$ with its image under the Plücker embedding, we consider the subvariety $\mathcal{X}_n := \mathrm{Gr}(n-1,2n) \cap \mathcal{H}$. In [Lam18, Theorem 5.8], an embedding

$$\iota : E_n \simeq \mathcal{X}_n \cap \mathrm{Gr}_{\geq 0}(n-1,2n) \hookrightarrow \mathrm{Gr}_{\geq 0}(n-1,2n) \tag{2.4.1}$$

was constructed, identifying the *compactification of the space of planar electrical networks with n boundary vertices* E_n with the compact space $\mathcal{X}_n \cap \mathrm{Gr}_{\geq 0}(n-1,2n)$. We will need the following property of $(E_n)_{>0} := \mathcal{X}_n \cap \mathrm{Gr}_{>0}(n-1,2n)$.

Proposition 2.4.1. The space $(E_n)_{>0}$ is diffeomorphic to $\mathbb{R}_{>0}^{\binom{n}{2}}$, and the inclusion $(E_n)_{>0} \hookrightarrow \mathrm{Gr}_{>0}(n-1,2n)$ is a smooth embedding.

Here $\mathrm{Gr}_{>0}(n-1,2n) \subset \mathrm{Gr}(n-1,2n)$ is an open submanifold diffeomorphic to $\mathbb{R}_{>0}^{(n-1)(n+1)}$.

Proof. We recall from [CIM98, Theorem 4] that each point in $(E_n)_{>0} = \Omega_n^+$ is uniquely represented by assigning a positive real number (the *conductance*) to each edge of a well-connected electrical network Γ with $\binom{n}{2}$ edges. This gives a parametrization $(E_n)_{>0} \simeq \mathbb{R}_{>0}^{\binom{n}{2}}$. The construction $\Gamma \mapsto N(\Gamma)$ of [Lam18, Section 5] sends Γ to a weighted bipartite graph $N(\Gamma)$ embedded into a disk compatibly with the inclusion (2.4.1). The edge weights of $N(\Gamma)$ are monomials in the edge weights of Γ. Furthermore, the underlying bipartite graph G of $N(\Gamma)$ parametrizes $\mathrm{Gr}_{>0}(n-1,2n)$. That is, we can choose a set of $(n-1)(n+1)$ edges of G, so that assigning arbitrary positive edge weights to these edges and weight 1 to the remaining edges induces a parametrization $\mathrm{Gr}_{>0}(n-1,2n) \simeq \mathbb{R}_{>0}^{(n-1)(n+1)}$ (see [Pos07] or [Tal11]). It follows that the inclusion $\mathbb{R}_{>0}^{\binom{n}{2}} \simeq (E_n)_{>0} \hookrightarrow \mathrm{Gr}_{>0}(n-1,2n) \simeq \mathbb{R}_{>0}^{(n-1)(n+1)}$ is a monomial map, and in particular a homomorphism of Lie groups. The result follows. $\square$

2.4.2 Operators acting on non-crossing partitions

For each $i \in [2n]$, we define u_i and d_i in $\mathfrak{gl}_{\binom{2n}{n-1}}(\mathbb{R})$ by

$$u_i(e_I) := \begin{cases} e_{I\cup\{i+1\}\setminus\{i\}}, & \text{if } i \in I,\ i+1 \notin I, \\ 0, & \text{otherwise,} \end{cases} \qquad d_i(e_I) := \begin{cases} e_{I\cup\{i-1\}\setminus\{i\}}, & \text{if } i \in I,\ i-1 \notin I, \\ 0, & \text{otherwise.} \end{cases}$$

Here the indices are taken modulo $2n$.

For $i \in [2n]_{\mathrm{odd}}$, we let $\kappa(i) \in \mathcal{NC}_n$ be the non-crossing partition which has two parts, namely $\{i\}$ and $[2n]_{\mathrm{odd}} \setminus \{i\}$. For $i \in [2n]_{\mathrm{even}}$, we let $\mu(i) \in \mathcal{NC}_n$ be the non-crossing partition with $n-1$ parts, one of which is $\{i-1, i+1\}$ and the rest being singletons. Given $\sigma \in \mathcal{NC}_n$ and $i \in [2n]$, we define the noncrossing partition $\sigma'(i) \in \mathcal{NC}_n$ as the common refinement of σ and $\kappa(i)$ if i is odd, and the common coarsening of σ and $\mu(i)$ if i is even. The following combinatorial lemma is essentially [Lam18, Proposition 5.15], and can be verified directly.

Lemma 2.4.2. For all $i \in [2n]$, we have

$$(u_i + d_i)(A_\sigma) = \begin{cases} 0, & \text{if } \sigma = \sigma'(i), \\ A_{\sigma'(i)}, & \text{otherwise.} \end{cases}$$

Example 2.4.3. Let $n := 3$ and $\sigma := \{\{1,3\},\{5\}\} \in \mathcal{NC}_n$, so that $\tilde{\sigma} = \{\{2\},\{4,6\}\}$, $\sigma'(1) = \sigma'(3) = \{\{1\},\{3\},\{5\}\}$, $\sigma'(2) = \sigma'(5) = \sigma$, and $\sigma'(4) = \sigma'(6) = \{\{1,3,5\}\}$. Abbreviating $e_{\{a,b\}}$ by e_{ab}, we have

$$A_\sigma = e_{14} + e_{16} + e_{34} + e_{36}.$$

Note that $\sigma \neq \sigma'(1)$ and

$$(u_1 + d_1)(A_\sigma) = (e_{24} + e_{26}) + (e_{46}) = A_{\sigma'(1)},$$

in agreement with Lemma 2.4.2 (since the dual of $\sigma'(1)$ is $\{\{2,4,6\}\}$). Similarly, we have $\sigma = \sigma'(2)$ and

$$(u_2 + d_2)(A_\sigma) = 0 + 0 = 0.$$

We define the operator $\Phi := \sum_{i=1}^{2n} u_i + d_i \in \mathfrak{gl}_{\binom{2n}{n-1}}(\mathbb{R})$.

Lemma 2.4.4. Let $X \in E_n$. We have $\exp(t\tau) \cdot X \in (E_n)_{>0}$ for all $t > 0$.

Proof. This follows from Corollary 2.2.4, once we show that $\exp(t\tau) \cdot X \in \mathcal{H}$ for $X \in \mathcal{X}_n$ and $t \in \mathbb{R}$. To do this, we identify $\mathrm{Gr}(n-1, 2n)$ with its image in $\mathbb{P}^{\binom{2n}{n-1}-1}$ under the Plücker embedding sending $X \in \mathrm{Gr}(n-1, 2n)$ to $\sum_{I \in \binom{[2n]}{n-1}} \Delta_I(X) e_I \in \mathbb{P}^{\binom{2n}{n-1}-1}$. Then for any $X \in \mathcal{X}_n$, we have a smooth curve $t \mapsto \exp(t\tau) \cdot X$ in $\mathbb{P}^{\binom{2n}{n-1}-1}$. As in (2.2.5), we find that

$$\exp(t\tau) \cdot X = X + t\Phi(X) + O(t^2) \quad \text{in } \mathbb{P}^{\binom{2n}{n-1}-1}$$

as $t \to 0$. Therefore $\exp(t\tau) \cdot X$ is an integral curve for the smooth vector field on $\mathbb{P}^{\binom{2n}{n-1}-1}$ defined by the infinitesimal action of Φ. By Lemma 2.4.2, this vector field is tangent to $\mathcal{H}$, so $\exp(t\tau) \cdot X \in \mathcal{H}$ for all $t \in \mathbb{R}$. $\square$

Proof of Theorem 1.5.1. We are identifying $(E_n)_{>0}$ as a subset $\mathrm{Gr}_{>0}(n-1, 2n)$ via the smooth embedding of Proposition 2.4.1. In turn, $\mathrm{Gr}_{>0}(n-1, 2n)$ is smoothly embedded inside $\mathrm{Mat}(n-1, n+1)$ by the map ϕ^{-1} defined in (2.2.3). Thus $Q := \phi^{-1}((E_n)_{>0}) \subset \mathrm{Mat}(n-1, n+1)$ is a smoothly embedded submanifold of $\mathrm{Mat}(n-1, n+1)$ of dimension $\binom{n}{2}$. The map ϕ^{-1} sends the compact set E_n homeomorphically onto its image $\phi^{-1}(E_n)$. Since $(E_n)_{>0}$ is dense in E_n, we have that $\phi^{-1}(E_n)$ equals the closure $\overline{Q}$ of Q. Let $f : \mathbb{R} \times \mathrm{Mat}(n-1, n+1) \to \mathrm{Mat}(n-1, n+1)$ be the map defined by (2.2.6). We showed in Section 2.2.3 that f is a contractive flow, and Lemma 2.4.4 implies that (2.1.1) holds for our choice of Q and f. Thus Lemma 2.1.3 applies, completing the proof. $\square$

3

Regularity theorem for totally nonnegative flag varieties

The goal of this chapter is to prove Theorem 1.2.1 and its special case Theorem 1.1.2. We start by giving a quick explanation of the proof idea in Section 3.1, and then give an outline of the rest of this chapter. A more detailed description of the proof can be found in Section 3.2.

3.1 Stars, links, and the Fomin–Shapiro atlas

Rietsch [Rie06] defined a certain poset $(Q_J, \preceq)$ and proved the decomposition $(G/P)_{\geq 0} = \bigsqcup_{g \in Q_J} \Pi_g^{>0}$. She showed that for $h \in Q_J$, the closure $\Pi_h^{\geq 0}$ of $\Pi_h^{>0}$ is given by $\Pi_h^{\geq 0} = \bigsqcup_{g \preceq h} \Pi_h^{>0}$. When $(G/P)_{\geq 0}$ is the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$, this gives the positroid cell decomposition of [Pos07].

Given $g \in Q_J$, define the *star of g in $(G/P)_{\geq 0}$* by

$$\mathrm{Star}_g^{\geq 0} := \bigsqcup_{h \succeq g} \Pi_h^{>0}. \tag{3.1.1}$$

In Section 3.3.1, we define another space $\mathrm{Lk}_g^{\geq 0}$ stratified as $\mathrm{Lk}_g^{\geq 0} = \bigsqcup_{h \succ g} \mathrm{Lk}_{g,h}^{>0}$. We later show in Theorem 3.3.11 that $\mathrm{Lk}_g^{\geq 0}$ is a regular CW complex homeomorphic to a closed ball.

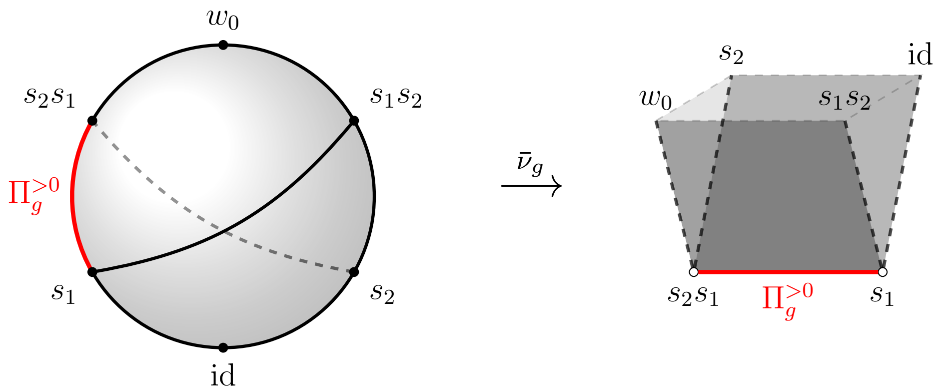


Figure 3-1: The map $\bar{\nu}_g$ for the case $G = \mathrm{SL}_3$ and $P = B$ from Example 3.1.1.

For each $g \in Q_J$, we construct a (stratification-preserving) homeomorphism

$$\bar{\nu}_g : \mathrm{Star}_g^{\geq 0} \xrightarrow{\sim} \Pi_g^{>0} \times \mathrm{Cone}(\mathrm{Lk}_g^{\geq 0}), \tag{3.1.2}$$

Here for a topological space A, we denote by $\mathrm{Cone}(A) := (A \times \mathbb{R}_{\geq 0})/(A \times \{0\})$ the *open cone* over A.

Example 3.1.1. When $G = \mathrm{SL}_n$ and $P = B$ is the standard Borel subgroup, G/B is the *complete flag variety* consisting of flags in $\mathbb{C}^n$, and the Weyl group W is the group S_n of permutations of n elements. The face poset Q_J of $(G/B)_{\geq 0}$ is the set of Bruhat intervals $\{(v, w) \in S_n \times S_n \mid v \leq w\}$ in S_n, and the cell $\Pi_{(v,w)}^{>0} \subset (G/B)_{\geq 0}$ indexed by (v, w) has dimension $\ell(w) - \ell(v)$. For example, when $n = 3$, this gives a cell decomposition of a 3-dimensional ball, see Figure 3-1 (left). For $g := (s_1, s_2s_1)$, $\Pi_g^{>0}$ is an open line segment, and $\mathrm{Star}_g^{\geq 0}$ consists of 4 cells: a line segment $\Pi_g^{>0} = \Pi_{(s_1,s_2s_1)}^{>0}$, two open square faces $\Pi_{(s_1,w_0)}^{>0}$ and $\Pi_{(\mathrm{id},s_2s_1)}^{>0}$, and an open 3-dimensional ball $\Pi_{(\mathrm{id},w_0)}^{>0}$. This union is indeed homeomorphic to $\Pi_g^{>0} \times \mathrm{Cone}(\mathrm{Lk}_g^{\geq 0})$ shown in Figure 3-1 (right). Here $\mathrm{Lk}_g^{\geq 0}$ is a closed line segment whose endpoints are $\mathrm{Lk}_{g,(s_1,w_0)}^{>0}$ and $\mathrm{Lk}_{g,(\mathrm{id},s_2s_1)}^{>0}$, and whose interior is $\mathrm{Lk}_{g,(\mathrm{id},w_0)}^{>0}$.

At the core of our construction lies the notion of a *Fomin–Shapiro atlas*, which is essentially the collection of homeomorphisms $\bar{\nu}_g$ for all $g \in Q_J$. For technical reasons, we require $\bar{\nu}_g$ to be a smooth map defined on an open neighborhood $\mathcal{O}_g$ of $\mathrm{Star}_g^{\geq 0}$, see Section 3.2.1. Our maps are analogous to those introduced in [FS00] for the unipotent radical $U_{\geq 0}$.

In Section 3.3, we prove that constructing a Fomin–Shapiro atlas is sufficient to deduce that $(G/P)_{\geq 0}$ is a regular CW complex. Our topological argument proceeds by induction on the dimension of a cell. It relies on the generalized Poincaré conjecture [Sma61, Fre82, Per02, Per03a, Per03b] combined with the result of Williams [Wil07] that the face poset Q_J of $(G/P)_{\geq 0}$ is shellable.

In order to construct the Fomin–Shapiro atlas, for each $g \in Q_J$ we give an isomorphism $\bar{\varphi}_u$ between an open dense subset $\mathcal{O}_g \subset G/P$ and a certain subset of the *affine flag variety* $\mathcal{G}/\mathcal{B}$ of the loop group $\mathcal{G}$ associated with G. The map $\bar{\varphi}_u$ sends the projected Richardson stratification [KLS14] of G/P to the affine Richardson stratification of its image inside $\mathcal{G}/\mathcal{B}$. The hardest part of the proof consists of showing that the subset $\mathcal{O}_g \subset G/P$ contains $\mathrm{Star}_g^{\geq 0}$. See Section 3.2.2 for a more in-depth overview of our proof.

Remark 3.1.2. The map $\bar{\varphi}_u$ generalizes (up to changing signs of some entries) the map of Snider [Sni10] from $\mathrm{Gr}(k,n)$ to all G/P, see Remark 3.9.9. A different approach to give such a generalization is due to He–Knutson–Lu [HKL19], which led them to the notion of a *Bruhat atlas*. See [Ele16] for the definition. We call our map $\bar{\varphi}_u$ an *affine Bruhat atlas* since its target space is always an affine flag variety, while Bruhat atlases of [HKL19] necessarily involve more general *Kac–Moody flag varieties*.

Outline

In Section 3.2, we introduce (abstract) *totally nonnegative spaces* and define Fomin–Shapiro atlases. We state in Theorem 3.2.4 that every totally nonnegative space that admits a Fomin–Shapiro atlas is a regular CW complex, and prove it in Section 3.3. We give background on G/P in Section 3.4, and study subtraction-free Marsh–Rietsch parametrizations in Section 3.5. We then apply our results on such parametrizations to prove Theorem 3.6.4 that will later imply that the above open subset $\mathcal{O}_g$ contains $\mathrm{Star}_g^{\geq 0}$. We introduce affine Bruhat atlases in Section 3.7 and use them to construct a Fomin–Shapiro atlas for G/P in Section 3.8. Theorem 3.2.5 (which implies our main result Theorem 1.2.1) is proved in Section 3.8.3. Section 3.9 is devoted to specializing our construction to type A (when $G = \mathrm{SL}_n$), with a special focus on the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$. We illustrate many of our constructions by examples in Section 3.9, and we encourage the reader to consult this section while studying other parts of the paper. We discuss some conjectures and further directions in Section 3.10. Finally, we give additional background on Kac–Moody flag varieties in Section 3.A.

3.2 Overview of the proof

We formulate our results in the abstract language of *totally nonnegative spaces*, since we expect that they can be applied in other contexts, see Section 3.10.

3.2.1 Totally nonnegative spaces

We refer the reader to Section 3.3.2 for background on posets and regular CW complexes. For a finite poset $(Q, \preceq)$, we denote by $\widehat{Q} := Q \sqcup \{\hat{0}\}$ the poset obtained from Q by adjoining a minimum $\hat{0}$. Björner showed [Bjö84, Prop. 4.5(a)] that if $\widehat{Q}$ is *graded*, *thin*, and *shellable*, then Q is isomorphic to the face poset of some regular CW complex. If $\widehat{Q}$ is a graded poset, we let $\dim : Q \to \mathbb{Z}_{\geq 0}$ denote the rank function of Q.

Definition 3.2.1. We say that a triple $(\mathcal{Y}, \mathcal{Y}^{\geq 0}, Q)$ is a *totally nonnegative space* (or *TNN space* for short) if the following conditions are satisfied.

(TNN1) $(\widehat{Q}, \preceq)$ is graded, thin, shellable, and contains a unique maximal element $\hat{1} \in Q$.

(TNN2) $\mathcal{Y}$ is a smooth manifold, stratified into embedded submanifolds $\mathcal{Y} = \bigsqcup_{g \in Q} \mathring{\mathcal{Y}}_g$, and for each $h \in Q$, $\mathring{\mathcal{Y}}_h$ has dimension $\dim(h)$ and closure $\mathcal{Y}_h := \bigsqcup_{g \preceq h} \mathring{\mathcal{Y}}_g$.

(TNN3) $\mathcal{Y}^{\geq 0}$ is a compact subset of $\mathcal{Y}$.

(TNN4) For $g \in Q$, $\mathcal{Y}_g^{>0} := \mathring{\mathcal{Y}}_g \cap \mathcal{Y}^{\geq 0}$ is a connected component of $\mathring{\mathcal{Y}}_g$ diffeomorphic to $\mathbb{R}_{>0}^{\dim(g)}$.

(TNN5) The closure of $\mathcal{Y}_h^{>0}$ inside $\mathcal{Y}$ equals $\mathcal{Y}_h^{\geq 0} := \bigsqcup_{g \preceq h} \mathcal{Y}_g^{>0}$.

For the case $\mathcal{Y} = G/P$, the smooth submanifolds $\mathring{\mathcal{Y}}_g$ are the *open projected Richardson varieties* of [KLS14].

Definition 3.2.2. Let $N \geq 0$, and denote by $\|\cdot\|$ the Euclidean norm on $\mathbb{R}^N$. We say that a pair (Z, ϑ) is a *smooth cone* if $Z \subset \mathbb{R}^N$ is a closed embedded submanifold and $\vartheta : \mathbb{R}_{>0} \times Z \to Z$ a smooth map such that

(SC1) ϑ gives an $(\mathbb{R}_{>0}, \cdot)$-action on $\mathbb{R}^N$ that restricts to an $(\mathbb{R}_{>0}, \cdot)$-action on Z.

(SC2) $\frac{\partial}{\partial t}\|\vartheta(t, x)\| > 0$ for all $t \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}^N \setminus \{0\}$.

The map ϑ is a smooth analog of a *contractive flow* from Definition 2.1.1.

For $g \in Q$, define $\mathrm{Star}_g := \bigsqcup_{h \succeq g} \mathring{\mathcal{Y}}_h$ and $\mathrm{Star}_g^{\geq 0} := \bigsqcup_{h \succeq g} \mathcal{Y}_h^{>0}$, cf. (3.1.1).

Definition 3.2.3. We say that a TNN space $(\mathcal{Y}, \mathcal{Y}^{\geq 0}, Q)$ admits a *Fomin–Shapiro atlas* if for each $g \in Q$, there exists an open subset $\mathcal{O}_g \subset \mathrm{Star}_g$, a smooth cone (Z_g, ϑ_g), and a diffeomorphism

$$\bar{\nu}_g : \mathcal{O}_g \xrightarrow{\sim} (\mathring{\mathcal{Y}}_g \cap \mathcal{O}_g) \times Z_g \tag{3.2.1}$$

satisfying the following conditions.

(FS1) For each $h \succeq g$ we are given $\mathring{Z}_{g,h} \subset Z_g$ such that $Z_g = \bigsqcup_{h \succeq g} \mathring{Z}_{g,h}$ and $\mathring{Z}_{g,g} = \{0\}$.

(FS2) For all $h \succeq g$ and $t \in \mathbb{R}_{>0}$, we have $\vartheta_g(t, \mathring{Z}_{g,h}) = \mathring{Z}_{g,h}$.

(FS3) For all $h \succeq g$, we have $\bar{\nu}_g(\mathring{\mathcal{Y}}_h \cap \mathcal{O}_g) = (\mathring{\mathcal{Y}}_g \cap \mathcal{O}_g) \times \mathring{Z}_{g,h}$.

(FS4) For all $y \in \mathring{\mathcal{Y}}_g \cap \mathcal{O}_g$, we have $\bar{\nu}_g(y) = (y, 0)$.

(FS5) $\mathrm{Star}_g^{\geq 0} \subset \mathcal{O}_g$.

Theorem 3.2.4. *Suppose that a TNN space $(\mathcal{Y}, \mathcal{Y}^{\geq 0}, Q)$ admits a Fomin–Shapiro atlas. Then $\mathcal{Y}^{\geq 0} = \bigsqcup_{h \in Q} \mathcal{Y}_h^{>0}$ is a regular CW complex. In particular, for each $h \in Q$, $\mathcal{Y}_h^{\geq 0}$ is homeomorphic to a closed ball of dimension* $\dim(h)$.

Thus Theorem 1.2.1 follows as a corollary of the following result.

Theorem 3.2.5. *$(G/P, (G/P)_{\geq 0}, Q_J)$ is a TNN space that admits a Fomin–Shapiro atlas.*

3.2.2 Plan of the proof

We give a brief outline of the proof of Theorem 3.2.5. See Section 3.4 for background on G/P and see Sections 3.A and 3.7 for background on $\mathcal{G}/\mathcal{B}$. We deduce that $(G/P, (G/P)_{\geq 0}, Q_J)$ is a TNN space from known results in Corollary 3.4.20. In order to construct a Fomin–Shapiro atlas, we consider the (infinite-dimensional) *affine flag variety* $\mathcal{G}/\mathcal{B}$ associated to G. It is stratified into (finite-dimensional) affine Richardson varieties $\mathcal{G}/\mathcal{B} = \bigsqcup_{\tilde h \leq \tilde f \in \tilde W} \mathring{\mathcal{R}}_{\tilde h}^{\tilde f}$, where $\tilde W$ is the affine Weyl group and $\leq$ denotes its Bruhat order. There exists an order-reversing injective map $\psi : Q_J \to \tilde W$, defined by [HL15], see (3.7.7). The set of minimal elements of Q_J equals $\{(u, u) \mid u \in W^J\}$, where W^J is the set of minimal length parabolic coset representatives of the Weyl group, see Section 3.4.6. For each minimal element $f := (u, u) \in Q_J$, ψ identifies the interval $[f, \hat 1]$ of Q_J with (the dual of) a certain interval $[\tau_\lambda^J, \tau_{u\lambda}] \subset \tilde W$. For the case $G/P = \mathrm{Gr}(k, n)$, elements of Q_J are in bijection with *Le diagrams* of [Pos07], and ψ sends a Le diagram indexing a positroid cell to the corresponding *bounded affine permutation* of [KLS14], see Example 3.9.6.

In Section 3.7.3, we lift ψ to the geometric level: given a minimal element $f := (u, u) \in Q_J$, we introduce a map $\bar\varphi_u : C_u^{(J)} \to \mathcal{G}/\mathcal{B}$ defined on an open dense subset $C_u^{(J)} \subset G/P$. We show in Theorem 3.7.2 that for $g \in Q_J$ such that $g \succeq f$, $\bar\varphi_u$ sends $C_u^{(J)} \cap \mathring{\Pi}_g$ isomorphically to the affine Richardson cell $\mathring{\mathcal{R}}_{\psi(g)}^{\psi(f)}$.

For every $\tilde g \in \tilde W$, we consider an open dense subset $\mathcal{C}_{\tilde g} \subset \mathcal{G}/\mathcal{B}$ defined by $\mathcal{C}_{\tilde g} := \tilde g \cdot \mathcal{B}_- \cdot \mathcal{B}/\mathcal{B}$, as well as affine Schubert and opposite Schubert cells $\mathring{\mathcal{X}}^{\tilde g} = \bigsqcup_{\tilde h \leq \tilde g} \mathring{\mathcal{R}}_{\tilde h}^{\tilde g}$, $\mathring{\mathcal{X}}_{\tilde g} = \bigsqcup_{\tilde g \leq \tilde f} \mathring{\mathcal{R}}_{\tilde g}^{\tilde f}$. In Proposition 3.8.2, we give a natural isomorphism

$$\mathcal{C}_{\tilde g} \xrightarrow{\sim} \mathring{\mathcal{X}}_{\tilde g} \times \mathring{\mathcal{X}}^{\tilde g}, \quad \text{which restricts to} \quad (\mathcal{C}_{\tilde g} \cap \mathring{\mathcal{R}}_{\tilde h}^{\tilde f}) \xrightarrow{\sim} \mathring{\mathcal{R}}_{\tilde g}^{\tilde f} \times \mathring{\mathcal{R}}_{\tilde h}^{\tilde g} \quad \text{for all } \tilde h \leq \tilde g \leq \tilde f. \tag{3.2.2}$$

A finite-dimensional analog of this map is due to [KWY13], and similar maps have been considered by [KL79, FS00]. The action of ϑ on $\mathring{\mathcal{X}}^{\tilde g}$ essentially amounts to multiplying by an element of the affine torus, and thus preserves $\mathring{\mathcal{R}}_{\tilde h}^{\tilde g}$ for all $\tilde h \leq \tilde g$.

Let us now fix $g \in Q_J$, and choose some minimal element $f := (u, u) \in Q_J$ such that $f \preceq g$. Then the map $\bar\varphi_u$ is defined on an open dense subset $C_u^{(J)} \subset G/P$, and let us denote by $\mathcal{O}_g \subset C_u^{(J)}$ the preimage of $\mathcal{C}_{\psi(g)}$ under $\bar\varphi_u$. The diffeomorphism (3.2.1) is obtained by conjugating the isomorphism (3.2.2) by the map $\bar{\varphi}_u$. The smooth cone (Z_g, ϑ_g) is extracted from the corresponding structure on $\mathring{\mathcal{X}}^{\psi(g)}$. As we have already mentioned, the hardest step in the proof consists of showing (FS5). To achieve this, we study subtraction-free parametrizations of partial flag varieties in Section 3.5, and then use them to show that some generalized minors of a particular group element $\zeta_{u,v}^{(J)}(x)$ from Section 3.6 do not vanish for all $x \in \mathrm{Star}_g^{\geq 0}$. The definition of $\zeta_{u,v}^{(J)}(x)$ is quite technical, but we conjecture in Section 3.9 that in the Grassmannian case, these generalized minors specialize to simple functions on $\mathrm{Gr}(k,n)$ that we call *u-truncated minors*. We complete the proof of Theorem 3.2.5 in Section 3.8.3.

3.3 Topological results

Throughout this section, we assume that $(\mathcal{Y}, \mathcal{Y}^{\geq 0}, Q)$ is a TNN space that admits a Fomin–Shapiro atlas. Thus for each $g \in Q$, we have the objects $\mathcal{O}_g$, Z_g, ϑ_g, and $\bar{\nu}_g$ from Definition 3.2.3. Additionally, we assume some familiarity with basic theory of smooth manifolds, see e.g. [Lee13].

3.3.1 Links

Throughout, we denote the two components of the map $\bar{\nu}_g$ from (3.2.1) by $\bar{\nu}_g = (\bar{\nu}_{g,1}, \bar{\nu}_{g,2})$, where $\bar{\nu}_{g,1} : \mathcal{O}_g \to \mathring{\mathcal{Y}}_g \cap \mathcal{O}_g$ and $\bar{\nu}_{g,2} : \mathcal{O}_g \to Z_g$. We set $\mathrm{Star}_{g,h}^{\geq 0} := \mathcal{Y}_h^{\geq 0} \cap \mathrm{Star}_g^{\geq 0} = \bigsqcup_{g \preceq g' \preceq h} \mathcal{Y}_{g'}^{>0}$.

Definition 3.3.1. Let $g \preceq h$ and assume that $Z_g \subset \mathbb{R}^{N_g}$. Denote

$$Z_g^{\geq 0} := \bar{\nu}_{g,2}\left(\mathrm{Star}_g^{\geq 0}\right), \qquad Z_{g,h}^{\geq 0} := \bar{\nu}_{g,2}\left(\mathrm{Star}_{g,h}^{\geq 0}\right), \qquad Z_{g,h}^{>0} := Z_g^{\geq 0} \cap \mathring{Z}_{g,h},$$

$$S_g := \{x \in \mathbb{R}^{N_g} : \|x\| = 1\}, \qquad \mathrm{Lk}_{g,h}^{\geq 0} := Z_{g,h}^{\geq 0} \cap S_g, \qquad \mathrm{Lk}_{g,h}^{>0} := Z_{g,h}^{>0} \cap S_g.$$

Notice that $\mathrm{Lk}_{g,g}^{>0} \subset S_g \cap \mathring{Z}_{g,g}$, but $\mathring{Z}_{g,g} = \{0\}$ by (FS1), thus we have a stratification

$$\mathrm{Lk}_{g,h}^{\geq 0} = \bigsqcup_{g \prec g' \preceq h} \mathrm{Lk}_{g,g'}^{>0}. \tag{3.3.1}$$

Recall that $\mathrm{Cone}(A) := (A \times \mathbb{R}_{\geq 0})/(A \times \{0\})$ is the open cone over A. We denote by $c := (*, 0) \in \mathrm{Cone}(A)$ its *cone point*.

Lemma 3.3.2. *Let $g \prec h \in Q$.*

(i) *For all $x \in \mathcal{O}_g$, we have $x \in \mathcal{Y}_h^{>0}$ if and only if $\bar{\nu}_g(x) \in \mathcal{Y}_g^{>0} \times Z_{g,h}^{>0}$.*

(ii) *$Z_{g,h}^{>0}$ is an embedded submanifold of Z_g that intersects S_g transversely.*

(iii) *$\mathrm{Lk}_{g,h}^{>0}$ is a contractible smooth manifold of dimension $\dim(h) - \dim(g) - 1$.*

(iv) *$\mathrm{Lk}_{g,h}^{\geq 0}$ is a compact subset of Z_g.*

Before we prove these properties, let us state some simple preliminary results on smooth manifolds. Given smooth manifolds A, B and a smooth map $f : A \to B$, a point $a \in A$ is called a *regular point* of f if the differential of f at a is surjective. Similarly, $b \in B$ is called a *regular value* of f if $f^{-1}(b)$ consists of regular points. In this case $f^{-1}(b)$ is a closed embedded submanifold of A of dimension $\dim(A) - \dim(B)$.

Lemma 3.3.3. *Suppose that A, B are smooth manifolds and $B' \subset B$ is such that $A \times B'$ is an embedded submanifold of $A \times B$. Then B' is an embedded submanifold of B.*

Proof. Choose $a \in A$. Clearly a is a regular value of the projection map $A \times B' \to A$, therefore $\{a\} \times B'$ is an embedded submanifold of $\{a\} \times B$. The projection map $\{a\} \times B \to B$ is a diffeomorphism. □

Proof of Lemma 3.3.2. (i): We prove this more generally for $g \preceq h$. The set $\mathrm{Star}_g^{\geq 0}$ is connected since it contains a connected dense subset $\mathcal{Y}_{\hat{1}}^{>0}$. Therefore $\bar{\nu}_{g,1}(\mathrm{Star}_g^{\geq 0})$ is a connected subset of $\mathring{\mathcal{Y}}_g \cap \mathcal{O}_g$. By (FS4), it contains $\mathcal{Y}_g^{>0}$, therefore $\bar{\nu}_{g,1}(\mathrm{Star}_g^{\geq 0}) = \mathcal{Y}_g^{>0}$ by (TNN4). By definition, $\bar{\nu}_{g,2}(\mathrm{Star}_{g,h}^{\geq 0}) = Z_{g,h}^{\geq 0}$, thus $\bar{\nu}_g(\mathrm{Star}_{g,h}^{\geq 0}) \subset \mathcal{Y}_g^{>0} \times Z_{g,h}^{\geq 0}$. By (FS3), we get $\bar{\nu}_g(\mathcal{Y}_h^{>0}) \subset \mathcal{Y}_g^{>0} \times Z_{g,h}^{>0}$. In particular, $Z_{g,h}^{>0} = \bar{\nu}_{g,2}(\mathcal{Y}_h^{>0})$ is a connected subset of $\mathring{Z}_{g,h}$. Let C be the connected component of $\mathring{Z}_{g,h}$ containing $Z_{g,h}^{>0}$. By (FS3), $\bar{\nu}_g^{-1}(\mathcal{Y}_g^{>0} \times C)$ is a connected subset of $\mathring{\mathcal{Y}}_h \cap \mathcal{O}_g$, which contains $\mathcal{Y}_h^{>0}$ as we have just shown. Therefore we must have $\bar{\nu}_g^{-1}(\mathcal{Y}_g^{>0} \times C) = \mathcal{Y}_h^{>0}$ by (TNN4), which shows that $Z_{g,h}^{>0} = C$ is a connected component of $\mathring{Z}_{g,h}$. Thus indeed $\bar{\nu}_g(\mathcal{Y}_h^{>0}) = \mathcal{Y}_g^{>0} \times Z_{g,h}^{>0}$.

(ii): Applying Lemma 3.3.3 to $A := \mathcal{Y}_g^{>0}$ and $B := Z_g$, and $B' := Z_{g,h}^{>0}$, we get that $Z_{g,h}^{>0}$ is an embedded submanifold of Z_g by (i). Moreover, it follows from (FS2) that $\vartheta_g(t, Z_{g,h}^{>0}) = Z_{g,h}^{>0}$ for all $t \in \mathbb{R}_{>0}$ since $Z_{g,h}^{>0}$ is a connected component of $\mathring{Z}_{g,h}$. Thus 1 is a regular value of the restriction $\| \cdot \| : Z_{g,h}^{>0} \to \mathbb{R}_{>0}$, so the manifolds S_g and $Z_{g,h}^{>0}$ intersect transversely inside $\mathbb{R}^{N_g}$.

(iii): By (ii), $\mathrm{Lk}^{>0}_{g,h} = Z^{>0}_{g,h} \cap S_g$ is an embedded submanifold of Z_g of dimension $\dim(h) - \dim(g) - 1$. To show that it is contractible, note that $\mathcal{Y}^{>0}_h$ is contractible, therefore so is $\bar{\nu}_g(\mathcal{Y}^{>0}_h) = \mathcal{Y}^{>0}_g \times Z^{>0}_{g,h}$. We see that $Z^{>0}_{g,h}$ is contractible since $\{x\} \times Z^{>0}_{g,h}$ is a retract of $\mathcal{Y}^{>0}_g \times Z^{>0}_{g,h}$ for any $x \in \mathcal{Y}^{>0}_g$, cf. [Hat02, Ex. 0.9].

For each $x \in \mathbb{R}^{N_g} \setminus \{0\}$, the function $t \mapsto \|\vartheta_g(t,x)\|$ is a diffeomorphism $\mathbb{R}_{>0} \xrightarrow{\sim} \mathbb{R}_{>0}$ by Lemma 2.1.2, so there exists a unique $t_x \in \mathbb{R}_{>0}$ such that $\|\vartheta_g(t_x,x)\| = 1$, and we get a continuous map $\mathbb{R}^{N_g} \setminus \{0\} \to S_g$ sending $x \mapsto \vartheta_g(t_x, x)$. It gives a retraction $Z^{>0}_{g,h} \to \mathrm{Lk}^{>0}_{g,h}$, finishing the proof of (iii).

(iv): Clearly $\mathrm{Star}^{\geq 0}_{g,h} = \mathcal{Y}^{\geq 0}_h \cap \mathrm{Star}^{\geq 0}_g = \mathcal{Y}^{\geq 0}_h \cap \mathcal{O}_g$ is a closed subset of $\mathcal{O}_g$. Thus $\bar{\nu}_g(\mathrm{Star}^{\geq 0}_{g,h})$ is a closed subset of $\mathcal{Y}^{>0}_g \times Z_g$. Since $\bar{\nu}_g(\mathrm{Star}^{\geq 0}_{g,h}) = \mathcal{Y}^{>0}_g \times Z^{\geq 0}_{g,h}$, we get that $Z^{\geq 0}_{g,h}$ is a closed subset of Z_g. It follows that $\mathrm{Lk}^{\geq 0}_{g,h} = Z^{\geq 0}_{g,h} \cap S_g$ is a closed bounded subset of Z_g, which is closed in $\mathbb{R}^{N_g}$ by Definition 3.2.2. □

Proposition 3.3.4. *Let $g \prec h \in Q$.*

(i) *There exists a homeomorphism $Z^{\geq 0}_{g,h} \xrightarrow{\sim} \mathrm{Cone}(\mathrm{Lk}^{\geq 0}_{g,h})$ sending 0 to the cone point c. For $g \prec g' \preceq h$, it sends $Z^{>0}_{g,g'}$ to $\mathrm{Lk}^{>0}_{g,g'} \times \mathbb{R}_{>0}$.*

(ii) *We have a homeomorphism*

$$\mathrm{Star}^{\geq 0}_{g,h} \xrightarrow{\sim} \mathcal{Y}^{>0}_g \times \mathrm{Cone}(\mathrm{Lk}^{\geq 0}_{g,h}). \tag{3.3.2}$$

Proof. (i): Consider the map $\xi : Z^{\geq 0}_{g,h} \to \mathrm{Cone}(\mathrm{Lk}^{\geq 0}_{g,h})$ sending $0 \mapsto c$ and $x \mapsto \left(\vartheta_g(t_x,x), \frac{1}{t_x}\right)$ for $x \in Z^{\geq 0}_{g,h} \setminus \{0\}$, where t_x was defined in the proof Lemma 3.3.2(iii). We claim that ξ is a homeomorphism. We have already shown that $\vartheta_g(t, Z^{>0}_{g,g'}) = Z^{>0}_{g,g'}$ for all $g \preceq g'$. In particular, for $x \in Z^{\geq 0}_{g,h}$, we find that $\xi(t_x, x) \in \mathrm{Cone}(\mathrm{Lk}^{\geq 0}_{g,h})$. It is easy to see that ξ is a bijection whose inverse sends $c \mapsto 0$ and $(y,t) \mapsto \vartheta_g(t,y)$ for $(y,t) \in \mathrm{Cone}(\mathrm{Lk}^{\geq 0}_{g,h}) \setminus \{c\} = \mathrm{Lk}^{\geq 0}_{g,h} \times \mathbb{R}_{>0}$. Clearly ξ is continuous on $Z^{\geq 0}_{g,h} \setminus \{0\}$ while its inverse is continuous on $\mathrm{Lk}^{\geq 0}_{g,h} \times \mathbb{R}_{>0}$.

Suppose that $(x_n)_{n\geq 0}$ is a sequence of elements of $Z^{\geq 0}_{g,h} \setminus \{0\}$ converging to 0. We claim that $t_{x_n} \to \infty$ as $n \to \infty$. Otherwise, after passing to a subsequence, we may assume that $t_{x_n} \leq R$ for some $R \in \mathbb{R}_{>0}$ and all $n \geq 0$. The limit of $\vartheta_g(R, x_n)$ is $\vartheta_g(R, 0) = 0$, which gives a contradiction since $\|\vartheta_g(R,x_n)\| \geq \|\vartheta_g(t_{x_n}, x_n)\| = 1$ for all $n \geq 0$. This shows that ξ is continuous at 0.

Suppose now that $((y_n, t_n))_{n\geq 0}$ is a sequence of elements of $\mathrm{Lk}_{g,h}^{\geq 0} \times \mathbb{R}_{>0}$ that converges to c, which is equivalent to $t_n \to 0$. Let $x_n := \vartheta_g(t_n, y_n) = \xi^{-1}(y_n, t_n)$. Suppose that $\|x_n\|$ does not converge to 0. The function $D(t) := \max_{x\in S_g} \|\vartheta_g(t, x)\|$ is strictly increasing in t by compactness of S_g and (SC2). Since for each $x \in S_g$, $\lim_{t\to 0+} \|\vartheta_g(t, x)\| = 0$ by Lemma 2.1.2, we get $\lim_{t\to 0+} D(t) = 0$ by compactness of S_g (more precisely, by *Dini's theorem*). This gives a contradiction, showing that ξ^{-1} is continuous at c.

(ii): By Lemma 3.3.2(i), $\bar{\nu}_g$ restricts to a homeomorphism $\mathrm{Star}_{g,h}^{\geq 0} \xrightarrow{\sim} \mathcal{Y}_g^{>0} \times Z_{g,h}^{\geq 0}$. By (i), $Z_{g,h}^{\geq 0}$ is homeomorphic to $\mathrm{Cone}(\mathrm{Lk}_{g,h}^{\geq 0})$. $\square$

Our next aim is to analyze the local structure of the space $\mathrm{Lk}_{g,h}^{\geq 0}$. For two topological spaces A and B and $a \in A$, $b \in B$, we say that there is a *local homeomorphism between* (A, a) *and* (B, b) if there exist open neighborhoods $a \in U \subset A$, $b \in V \subset B$, and a homeomorphism $U \xrightarrow{\sim} V$ sending a to b.

Lemma 3.3.5. *Let $g \prec p \preceq h \in Q$, $x_p \in \mathrm{Lk}_{g,p}^{>0}$, and $d := \dim(p) - \dim(g) - 1$. Then there exists a local homeomorphism between $\left(\mathrm{Lk}_{g,h}^{\geq 0}, x_p\right)$ and $\left(Z_{p,h}^{\geq 0} \times \mathbb{R}^d, (0, 0)\right)$.*

Proof. Choose some $x_g \in \mathcal{Y}_g^{>0}$ and consider the open subset $H_p \subset Z_g$ defined by $H_p := \{x \in Z_g \mid \bar{\nu}_g^{-1}(x_g, x) \in \mathcal{O}_p\}$. Introduce a map

$$\theta_{g,p} : H_p \cap S_g \to Z_p, \quad x \mapsto \bar{\nu}_{p,2}(\bar{\nu}_g^{-1}(x_g, x)). \tag{3.3.3}$$

Since $x_p \in \mathrm{Lk}_{g,p}^{>0} \subset Z_{g,p}^{>0}$ and $x_g \in \mathcal{Y}_g^{>0}$, we get $\bar{x}_p := \bar{\nu}_g^{-1}(x_g, x_p) \in \mathcal{Y}_p^{>0}$ by Lemma 3.3.2(i). By (FS5), $\mathcal{Y}_p^{>0} \subset \mathrm{Star}_p^{\geq 0} \subset \mathcal{O}_p$, thus $x_p \in H_p$. Since H_p is open in Z_g, $H_p \cap S_g$ is an open subset of $Z_g \cap S_g$, which is nonempty since it contains x_p. Moreover, since $\bar{\nu}_{p,2}(\mathcal{Y}_p^{>0}) = Z_{p,p}^{>0} = \{0\}$ by Lemma 3.3.2(i), we find that $\theta_{g,p}(x_p) = 0$.

We claim that x_p is a regular point of $\theta_{g,p}$. Clearly the differential of $\bar{\nu}_{p,2} : \mathcal{O}_p \to Z_p$ is surjective at $\bar{x}_p$, and its kernel is the tangent space of $\mathring{\mathcal{Y}}_p$ at $\bar{x}_p$. Recall from (TNN4) and (FS5) that $\mathcal{Y}_p^{>0}$ is a connected component of $\mathring{\mathcal{Y}}_p \cap \mathcal{O}_p$, and it contains $\bar{x}_p = \bar{\nu}_g^{-1}(x_g, x_p)$ as we have shown above. Therefore x_p is a regular point of $\theta_{g,p}$ if and only if the manifolds $\mathcal{Y}_p^{>0}$ and $F := \bar{\nu}_g^{-1}\left(\{x_g\} \times (H_p \cap S_g)\right)$ intersect transversely at $\bar{x}_p$. By Lemma 3.3.2(i), we have $\bar{\nu}_g(\mathcal{Y}_p^{>0}) = \mathcal{Y}_g^{>0} \times Z_{g,p}^{>0}$, and clearly $\bar{\nu}_g(F) = \{x_g\} \times (H_p \cap S_g)$. These two manifolds intersect transversely at (x_g, x_p) by Lemma 3.3.2(ii). We have shown that x_p is a regular point of $\theta_{g,p}$.

By the *Submersion theorem* (see e.g. [Kos93, Cor. A(1.3)]), there exist local coordinates centered at $x_p \in H_p \cap S_g$ and $0 \in Z_p$ in which $\theta_{g,p}$ is just the canonical projection $\mathbb{R}^{\dim(H_p \cap S_g)} \to \mathbb{R}^{\dim(Z_p)}$. Recall that Q contains a unique maximal element $\hat{1}$, and we have $\dim(Z_g) = \operatorname{codim}(g) := \dim(\hat{1}) - \dim(g)$. Thus $\dim(H_p \cap S_g) = \operatorname{codim}(g) - 1$, $\dim(Z_p) = \operatorname{codim}(p)$, and $\dim(H_p \cap S_g) - \dim(Z_p) = d$. We have shown that there exist open neighborhoods $x_p \in U \subset H_p \cap S_g$ and $0 \in V \subset Z_p$ and a diffeomorphism $\beta : U \xrightarrow{\sim} V \times \mathbb{R}^d$ sending x_p to $(0,0)$.

In order to complete the proof, we need to show that the image $\beta(U \cap \mathrm{Lk}_{g,h}^{\geq 0})$ equals $(V \cap Z_{p,h}^{\geq 0}) \times \mathbb{R}^d$. We may assume that U is connected. Suppose we are given $x \in U$ and let $r \in Q$ be such that $x' := \bar{\nu}_g^{-1}(x_g, x) \in \mathring{\mathcal{Y}}_r$. Since $U \subset H_p$, x' belongs to $\mathcal{O}_p \subset \mathrm{Star}_p$ by Definition 3.2.3, and therefore $p \preceq r$. By Lemma 3.3.2(i), we have $x \in U \cap \mathrm{Lk}_{g,r}^{>0}$ if and only if $x' \in \mathcal{Y}_r^{>0}$. On the other hand, $\bar{\nu}_{p,1}(\bar{\nu}_g^{-1}(\{x_g\} \times U))$ is a connected subset of $\mathring{\mathcal{Y}}_p \cap \mathcal{O}_p$ that contains $\bar{\nu}_{p,1}(\bar{x}_p) \in \mathcal{Y}_p^{>0}$. Thus $\bar{\nu}_{p,1}(\bar{\nu}_g^{-1}(x_g, U)) \subset \mathcal{Y}_p^{>0}$ by (TNN4). It follows that $x' \in \mathcal{Y}_r^{>0}$ if and only if $\theta_{g,p}(x) = \bar{\nu}_{p,2}(x')$ belongs to $Z_{p,r}^{>0}$. The result follows by taking the union over all $p \preceq r \preceq h$. $\square$

3.3.2 Topological background

Regular CW complexes

We refer to [Hat02, LW69] for background on CW complexes.

Definition 3.3.6. Let X be a Hausdorff space. We call a finite disjoint union $X = \bigsqcup_{\alpha \in P} X_\alpha$ a *regular CW complex* if it satisfies the following two properties.

(CW1) For each $\alpha \in P$, there exists a homeomorphism from the closure $\overline{X_\alpha}$ to a closed ball $\overline{B}$ which sends X_α to the interior of $\overline{B}$;

(CW2) For each $\beta \in P$, $\overline{X_\beta}$ equals $\bigsqcup_{\alpha \in P'} X_\alpha$ for some $P' \subset P$.

The property (CW2) is often omitted from the definition of a regular CW complex, but is necessary in order to apply the arguments of [Bjö84]. We remark that the cell decomposition of $\mathcal{Y}^{\geq 0}$ satisfies (CW2) by (TNN5).

Posets

We review the notions of *thinness* and *shellability* (see e.g. [Bjö80]), though we will not need them in our arguments. Let $(P, \leq)$ be a finite poset. We say that P is *thin* if for all $x \lessdot y \lessdot z$ in P (where $\lessdot$ denotes a covering relation of P), there exists a unique $y' \neq y$ with $x < y' < z$. We say that a pure d-dimensional simplicial complex Δ is *shellable* if its maximal faces can be ordered as $F_1, \dots, F_n$ so that for $2 \leq k \leq n$, $F_k \cap \left(\bigcup_{1 \leq i < k} F_i\right)$ is a nonempty union of $(d-1)$-dimensional faces of F_k. We say that P is *graded* if all maximal chains in P have the same length l, in which case we denote by $\mathrm{ht}(P) := l$ the *height* of P. Each graded poset P gives rise to a pure $(\mathrm{ht}(P) - 1)$-dimensional simplicial complex called the *order complex* Δ_P *of* P. The vertices of Δ_P are the elements of P and the faces of Δ_P are the chains in P. We say that P is *shellable* if its order complex Δ_P is shellable. For $x \leq z \in P$, we denote by $[x, z] := \{y \in P \mid x \leq y \leq z\}$ the corresponding *interval*.

Proposition 3.3.7 ([Bjö80, Prop. 4.2])**.** *If a poset is shellable then so are each of its intervals.*

See [Bjö84, §2,3] for the proof of the following result.

Theorem 3.3.8 ([LW69, DK74, Bjö84])**.** *Suppose that X is a regular CW complex with face poset P. If $P \sqcup \{\hat{0}, \hat{1}\}$ is graded, thin, and shellable, then X is homeomorphic to a sphere of dimension* $\mathrm{ht}(P) - 1$.

Poincaré conjecture

Recall that an *n-dimensional topological manifold C with boundary* is a Hausdorff space such that every point $x \in C$ has an open neighborhood homeomorphic either to $\mathbb{R}^n$ or to $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$. In the latter case, we say that x belongs to the *boundary* ∂C of C.

The following is a well known consequence of the *(generalized) Poincaré conjecture* due to Smale [Sma61], Freedman [Fre82], and Perelman [Per02, Per03a, Per03b]. We refer to [Dav08, Thm. 10.3.3(ii)] for this formulation.

Theorem 3.3.9 ([Sma61, Fre82, Per02, Per03a, Per03b])**.** *Let C be a compact contractible n-dimensional topological manifold with boundary, such that its boundary ∂C is homeomorphic to an $(n-1)$-dimensional sphere. Then C is homeomorphic to a closed n-dimensional ball.*

For $n \geq 6$, Theorem 3.3.9 can be proved using the *topological h-cobordism theorem* [Mil65, KS77]. We sketch another standard argument for deducing Theorem 3.3.9 from the Poincaré conjecture when n is arbitrary. The boundary of C is *collared* by [Bro62, Thm. 2]. Thus we can attach the (collared) boundary of a closed n-dimensional ball to the (collared) boundary of C, obtaining a topological manifold C'. By van Kampen's theorem, C' is simply connected. It is easy to see from the Mayer–Vietoris sequence that C' is a homology sphere. Thus C' must be homeomorphic to a sphere by the Poincaré conjecture. Therefore C is homeomorphic to a closed ball by Brown's Schoenflies theorem [Bro60].

Proposition 3.3.10. *Suppose that C is a topological manifold with boundary ∂C. Then C is homotopy equivalent to its interior* $\operatorname{int}(C) := C \setminus \partial C$.

Proof. By [Bro62, Thm. 2], there exists an open subset $\partial C \subset U \subset C$ that is homeomorphic to $\partial C \times [0, 1)$, which shows the result. □

3.3.3 Link induction

Theorem 3.3.11. *Let $(\mathcal{Y}, \mathcal{Y}^{\geq 0}, Q)$ be a TNN space that admits a Fomin–Shapiro atlas. Suppose that $g \prec h \in Q$. Then $\mathrm{Lk}^{\geq 0}_{g,h}$ is homeomorphic to a closed ball of dimension $d := \dim(h) - \dim(g) - 1$.*

Proof. We proceed by induction on d. For the base case $d = 0$, we see by (3.3.1) that $\mathrm{Lk}^{\geq 0}_{g,h} = \mathrm{Lk}^{>0}_{g,h}$, which is a 0-dimensional contractible manifold by Lemma 3.3.2(iii). Thus $\mathrm{Lk}^{\geq 0}_{g,h}$ is a point and we are done with the base case.

Assume now that $d > 0$. We claim that $\mathrm{Lk}^{\geq 0}_{g,h}$ is a topological manifold with boundary

$$\partial \,\mathrm{Lk}^{\geq 0}_{g,h} = \bigsqcup_{g \prec g' \prec h} \mathrm{Lk}^{>0}_{g,g'} . \tag{3.3.4}$$

Let $x \in \mathrm{Lk}^{\geq 0}_{g,h}$. By (3.3.1), we have $x \in \mathrm{Lk}^{>0}_{g,g'}$ for a unique $g \prec g' \preceq h$. If $g' = h$, then x has an open neighborhood in $\mathrm{Lk}^{\geq 0}_{g,h}$ homeomorphic to $\mathbb{R}^d$ by Lemma 3.3.2(iii). If $g' \prec h$, then by Lemma 3.3.5 we have a local homeomorphism $(\mathrm{Lk}^{\geq 0}_{g,h}, x) \xrightarrow{\sim} (Z^{\geq 0}_{g',h} \times \mathbb{R}^{d'}, (0, 0))$, where $d' := \dim(g') - \dim(g) - 1$. By Proposition 3.3.4(i), we have a homeomorphism $Z^{\geq 0}_{g',h} \xrightarrow{\sim} \mathrm{Cone}(\mathrm{Lk}^{\geq 0}_{g',h})$ which sends 0 to the cone point c. By the induction hypothesis, $\mathrm{Lk}^{\geq 0}_{g',h}$

is homeomorphic to a $(d-d'-1)$-dimensional closed ball, and so we have a homeomorphism $\mathrm{Cone}(\mathrm{Lk}^{\geq 0}_{g',h}) \xrightarrow{\sim} \mathbb{R}_{\geq 0} \times \mathbb{R}^{d-d'-1}$ which sends c to $(0,0)$. Composing gives a local homeomorphism $(\mathrm{Lk}^{\geq 0}_{g,h}, x) \xrightarrow{\sim} (\mathbb{R}_{\geq 0} \times \mathbb{R}^{d-d'-1} \times \mathbb{R}^{d'}, (0,0,0))$. Thus indeed $\mathrm{Lk}^{\geq 0}_{g,h}$ is a topological manifold with boundary given by (3.3.4).

By Lemma 3.3.2(iv), $\mathrm{Lk}^{\geq 0}_{g,h}$ is compact. By Lemma 3.3.2(iii) and Proposition 3.3.10, $\mathrm{Lk}^{\geq 0}_{g,h}$ is contractible. Thus $\mathrm{Lk}^{\geq 0}_{g,h}$ is a compact contractible topological manifold with boundary. In addition, the boundary $\partial\,\mathrm{Lk}^{\geq 0}_{g,h}$ is a regular CW complex by the induction hypothesis. Its face poset is the interval $(g,h) := [g,h] \setminus \{g,h\}$ in Q. The interval $[g,h]$ is graded, thin, and shellable by (TNN1) and Proposition 3.3.7, thus $\partial\,\mathrm{Lk}^{\geq 0}_{g,h}$ is homeomorphic to a $(d-1)$-dimensional sphere by Theorem 3.3.8. We are done by Theorem 3.3.9. $\square$

Proof of Theorem 3.2.4. The proof follows the same structure as the proof of Theorem 3.3.11. We proceed by induction on $\dim(h)$. If $\dim(h) = 0$ then $\mathcal{Y}^{\geq 0}_h = \mathcal{Y}^{>0}_h$ is a point by (TNN4), which finishes the base case.

Let $\dim(h) > 0$. We want to show that $\mathcal{Y}^{\geq 0}_h$ is a topological manifold with boundary

$$\partial \mathcal{Y}^{\geq 0}_h = \bigsqcup_{g \prec h} \mathcal{Y}^{>0}_g. \tag{3.3.5}$$

Let $x \in \mathcal{Y}^{\geq 0}_h$. By (TNN5), we have $x \in \mathcal{Y}^{>0}_g$ for a unique $g \preceq h$. If $g = h$ then x has an open neighborhood in $\mathcal{Y}^{\geq 0}_h$ homeomorphic to $\mathbb{R}^{\dim(h)}$ by (TNN4).

If $g \prec h$ then $\mathrm{Star}^{\geq 0}_g$ is an open subset of $\mathcal{Y}^{\geq 0}$, since its complement is $\bigcup_{g' \not\succeq g} \mathcal{Y}^{\geq 0}_{g'}$, which is closed by (TNN5). Thus $\mathrm{Star}^{\geq 0}_{g,h}$ is an open neighborhood of x in $\mathcal{Y}^{\geq 0}_h$. By Proposition 3.3.4(ii), (TNN4), and Theorem 3.3.11, $\mathrm{Star}^{\geq 0}_{g,h}$ is homeomorphic to $\mathbb{R}_{\geq 0} \times \mathbb{R}^{\dim(h)-1}$. This shows that $\mathcal{Y}^{\geq 0}_h$ is a topological manifold with boundary given by (3.3.5).

By (TNN3) and (TNN5), $\mathcal{Y}^{\geq 0}_h$ is compact. By (TNN4) and Proposition 3.3.10, $\mathcal{Y}^{\geq 0}_h$ is contractible. Thus $\mathcal{Y}^{\geq 0}_h$ is a compact contractible topological manifold with boundary. In addition, the boundary $\partial \mathcal{Y}^{\geq 0}_h$ is a regular CW complex by the induction hypothesis. Its face poset is the interval $(\hat{0}, h)$ in $\widehat{Q}$. The interval $[\hat{0}, h]$ is graded, thin, and shellable by (TNN1) and Proposition 3.3.7, thus $\partial \mathcal{Y}^{\geq 0}_h$ is a $(d-1)$-dimensional sphere by Theorem 3.3.8. We are done by Theorem 3.3.9. $\square$

Remark 3.3.12. We note that Theorems 3.2.5 and 3.3.11 imply the result of Hersh [Her14] (see Corollary 1.3.1) that the link of the identity in the Bruhat decomposition of $U_{\geq 0}$ is a regular CW complex. (Recall that U is the unipotent radical of the standard Borel subgroup $B \subset G$.) Indeed, let $B_- \subset G$ denote the opposite Borel subgroup. Then the natural inclusion $U \hookrightarrow G/B_-$ sends U to the opposite Schubert cell $\mathrm{Star}_{(\mathrm{id},\mathrm{id})}$ indexed by $\mathrm{id} \in W$, and the composition of this map with $\bar{\nu}_{(\mathrm{id},\mathrm{id})}$ sends the link of the identity in $\overline{U^w_{>0}}$ homeomorphically to $\mathrm{Lk}^{\geq 0}_{(\mathrm{id},\mathrm{id}),(\mathrm{id},w)}$ for all $w \in W$.

3.4 G/P: preliminaries

We give some background on partial flag varieties. Throughout, $\mathbb{K}$ denotes an algebraically closed field of characteristic 0, and $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$ denotes its multiplicative group. We work with a simple algebraic group G; the case of a general semisimple group reduces to the simple case by taking products.

3.4.1 Pinnings

We recall some standard notions that can be found in e.g. [Lus94, §1]. Suppose that G is a simple and simply connected algebraic group over $\mathbb{K}$, with $T \subset G$ a maximal torus. Let B, B_- be opposite Borel subgroups satisfying $T = B \cap B_-$. We identify G with its set of $\mathbb{K}$-valued points. When $\mathbb{K} = \mathbb{C}$, we assume that G and T are split over $\mathbb{R}$, and denote by $G(\mathbb{R}) \subset G$ and $T(\mathbb{R}) \subset T$ the sets of their $\mathbb{R}$-valued points. (Thus what was denoted by G in Section 1.2 is from now on denoted by $G(\mathbb{R})$.)

Let $X(T) := \mathrm{Hom}(T, \mathbb{K}^*)$ be *weight lattice*, and for a weight $\gamma \in X(T)$ and $a \in T$, we denote the value of γ on a by a^γ. Let $\Phi \subset X(T)$ be the set of *roots*. We have a decomposition $\Phi = \Phi^+ \sqcup \Phi^-$ of Φ as a union of positive and negative roots corresponding to the choice of B, see [Hum75, §27.3]. For $\alpha \in \Phi$, we write $\alpha > 0$ if $\alpha \in \Phi^+$ and $\alpha < 0$ if $\alpha \in \Phi^-$. Let $\{\alpha_i\}_{i \in I}$ be the simple roots corresponding to the choice of Φ^+. For every $i \in I$, we have a homomorphism $\phi_i : \mathrm{SL}_2 \to G$, and denote

$$x_i(t) := \phi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad y_i(t) := \phi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \dot{s}_i := \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = y_i(1)x_i(-1)y_i(1). \tag{3.4.1}$$

The data $(T, B, B_-, x_i, y_i; i \in I)$ is called a *pinning* for G. Let $W := N_G(T)/T$ be the Weyl group, and for $i \in I$, let $s_i \in W$ be represented by $\dot{s}_i$ above. Then W is generated by $\{s_i\}_{i \in I}$, and $(W, \{s_i\}_{i\in I})$ is a finite Coxeter group. For $w \in W$, we denote by $\ell(w)$ the minimal n such that $w = s_{i_1} \cdots s_{i_n}$ for some $i_1, \dots, i_n \in I$. When $n = \ell(w)$, $\mathbf{i} := (i_1, \dots, i_n)$ is called a *reduced word* for w. The representatives $\{\dot{s}_i\}_{i\in I}$ satisfy the braid relations, so we set $\dot{w} := \dot{s}_{i_1} \cdots \dot{s}_{i_n} \in G$, and this representative does not depend on the choice of $\mathbf{i}$.

Let $Y(T) := \mathrm{Hom}(\mathbb{K}^*, T)$ be the *coweight lattice*. For $i \in I$, we have a simple coroot $\alpha_i^\vee(t) := \phi_i \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in Y(T)$. Denote by $\langle \cdot, \cdot \rangle : X(T) \times Y(T) \to \mathbb{Z}$ the natural pairing so that for $\gamma \in X(T)$, $\beta \in Y(T)$, and $t \in \mathbb{K}^*$, we have $(\beta(t))^\gamma = t^{\langle \gamma, \beta \rangle}$. Let $\{\omega_i\}_{i\in I} \subset X(T)$ be the *fundamental weights*. They form a dual basis to $\{\alpha_i^\vee\}_{i\in I}$: $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $i, j \in I$.

The Weyl group W acts on T by conjugation, which induces an action on $Y(T)$, $X(T)$, and Φ. For $\gamma \in X(T)$, $t \in \mathbb{K}^*$, $a \in T$, and $w \in W$, we have [FZ99, Eqns. (1.2) and (2.5)]

$$(\dot{w}^{-1} a \dot{w})^\gamma = a^{w\gamma}, \quad a x_i(t) a^{-1} = x_i(a^{\alpha_i} t), \quad a y_i(t) a^{-1} = y_i(a^{-\alpha_i} t).b \tag{3.4.2}$$

Following [BZ97, Eqns. (1.6) and (1.7)] (see also [FZ99, Eqns. (2.1) and (2.2)]), we define two involutive anti-automorphisms $x \mapsto x^T$ and $x \mapsto x^\iota$ of G by

$$(x_i(t))^T = y_i(t), \qquad (y_i(t))^T = x_i(t), \qquad \dot{w}^T = \dot{w}^{-1}, \qquad a^T = a \tag{3.4.3}$$

$$(x_i(t))^\iota = x_i(t), \qquad (y_i(t))^\iota = y_i(t), \qquad \dot{w}^\iota = \dot{z}, \qquad a^\iota = a^{-1}, \tag{3.4.4}$$

for all $i \in I$, $t \in \mathbb{K}^*$, $a \in T$, and $w \in W$, where $z := w^{-1}$. We note that when $z = w^{-1} \in W$ and $\mathbf{i} = (i_1, \dots, i_n)$ is a reduced word for w then $\dot{w}^{-1} = \dot{s}_{i_n}^{-1} \cdots \dot{s}_{i_1}^{-1}$ while $\dot{z} = \dot{s}_{i_n} \cdots \dot{s}_{i_1}$.

3.4.2 Subgroups of U

We say that a subset $\Theta \subset \Phi$ is *bracket closed* if whenever $\alpha, \beta \in \Theta$ are such that $\alpha + \beta \in \Phi$, we have $\alpha+\beta \in \Theta$. For a bracket closed subset $\Theta \subset \Phi^+$, define $U(\Theta) \subset U$ to be the subgroup generated by $\{U_\alpha \mid \alpha \in \Theta\}$, where U_α is a *root subgroup* of G, see [Hum75, Theorem 26.3]. For a bracket closed subset $\Theta \subset \Phi^-$, let $U_-(\Theta) := U(-\Theta)^T \subset U_-$.

Given closed subgroups $H_1, \dots, H_n$ of an algebraic group H, we say that $H_1, \cdots, H_n$ *directly span* H if the multiplication map $H_1 \times \cdots \times H_n \to H$ is a biregular isomorphism.

Lemma 3.4.1 ([Hum75, Prop. 28.1])**.** *Let $\Theta \subset \Phi^+$ be a bracket closed subset.*

(i) *If $\Theta = \bigsqcup_{i=1}^n \Theta_i$ and $\Theta, \Theta_1, \dots, \Theta_n \subset \Phi^+$ are bracket closed then $U(\Theta)$ is directly spanned by $U(\Theta_1), \dots, U(\Theta_n)$.*

(ii) *In particular, $U(\Theta)$ is directly spanned by $\{U_\alpha \mid \alpha \in \Theta\}$ in any order, and therefore $U(\Theta) \cong \mathbb{K}^{|\Theta|}$.*

For $\alpha \in \Phi$ and $w \in W$, we have $\dot{w} U_\alpha \dot{w}^{-1} = U_{w\alpha}$. For $w \in W$, define $\mathrm{Inv}(w) := (w^{-1}\Phi^-) \cap \Phi^+$. The subsets $\mathrm{Inv}(w)$ and $\Phi^+ \setminus \mathrm{Inv}(w)$ are bracket closed [Hum75, §28.1], and

$$U(\mathrm{Inv}(w)) = \dot{w}^{-1} U_- \dot{w} \cap U. \tag{3.4.5}$$

3.4.3 Bruhat projections

Let $\Theta \subset \Phi^+$ be bracket closed, and let $w \in W$. Define $\Theta_1 := \Theta \cap \mathrm{Inv}(w)$ and $\Theta_2 := \Theta \setminus \mathrm{Inv}(w)$. Thus both sets are bracket closed and

$$\dot{w} U(\Theta) \dot{w}^{-1} \cap U_- = U_-(w\Theta_1), \quad \dot{w} U(\Theta) \dot{w}^{-1} \cap U = U(w\Theta_2).$$

Denote $U_1 := U_-(w\Theta_1)$ and $U_2 := U(w\Theta_2)$. Then by Lemma 3.4.1(i), the multiplication map gives isomorphisms $\mu_{12} : U_1 \times U_2 \to \dot{w} U(\Theta) \dot{w}^{-1}$ and $\mu_{21} : U_2 \times U_1 \to \dot{w} U(\Theta) \dot{w}^{-1}$. Denote by $\nu_1 : \dot{w} U(\Theta) \dot{w}^{-1} \to U_1$ and $\nu_2 : \dot{w} U(\Theta) \dot{w}^{-1} \to U_2$ the second component of μ_{21}^{-1} and μ_{12}^{-1}, respectively. In other words, given $g \in \dot{w} U(\Theta) \dot{w}^{-1}$, $\nu_1(g)$ is the unique element in $U_1 \cap U_2 g$ and $\nu_2(g)$ is the unique element in $U_2 \cap U_1 g$.

Lemma 3.4.2 ([KWY13, Lemma 2.2]). *The map $(\nu_1, \nu_2) : \dot{w}U(\Theta)\dot{w}^{-1} \to U_1 \times U_2$ is a biregular isomorphism.*

3.4.4 Commutation relations

Let $a, b \in W$ be such that $\ell(ab) = \ell(a) + \ell(b)$. Then

$$\mathrm{Inv}(b) \subset \mathrm{Inv}(ab), \quad b^{-1}\mathrm{Inv}(a) \subset \Phi^+, \quad \text{and} \quad \mathrm{Inv}(ab) = \big(b^{-1}\mathrm{Inv}(a)\big) \sqcup \mathrm{Inv}(b). \tag{3.4.6}$$

Thus by Lemma 3.4.1(i), the multiplication map gives an isomorphism

$$\dot{b}^{-1}U(\mathrm{Inv}(a))\dot{b} \times U(\mathrm{Inv}(b)) \xrightarrow{\sim} U(\mathrm{Inv}(ab)). \tag{3.4.7}$$

We will later need the following trivial consequences of (3.4.7): if $\ell(ab) = \ell(a) + \ell(b)$ then

$$\dot{b}^{-1} \cdot (U_- \cap \dot{a}^{-1}U\dot{a}) \subset (U_- \cap (\dot{a}\dot{b})^{-1}U\dot{a}\dot{b}) \cdot \dot{b}^{-1}, \tag{3.4.8}$$

$$(U \cap \dot{a}^{-1}U_-\dot{a}) \cdot \dot{b} \subset \dot{b} \cdot (U \cap (\dot{a}\dot{b})^{-1}U_-\dot{a}\dot{b}). \tag{3.4.9}$$

Multiplying both sides of (3.4.9) by $\dot{b}^{-1}$ on the left, we get $\dot{b}^{-1}U(\mathrm{Inv}(a))\dot{b} \subset U(\mathrm{Inv}(ab))$, which follows from (3.4.6). Eq. (3.4.8) follows from (3.4.9) by applying the map $x \mapsto x^T$, see (3.4.3).

Lemma 3.4.3. *Let $\alpha \in \Phi^+$ and $i \in I$ be such that $\alpha \neq \alpha_i$. Let $\Psi \subset \Phi$ denote the set of all roots of the form $m\alpha - r\alpha_i$ for integers $m > 0$, $r \geq 0$. Then Ψ is a bracket closed subset of Φ^+, and for all $t \in \mathbb{K}$ we have $y_i(t)U_\alpha y_i(-t) \subset U(\Psi)$.*

Proof. Let $x \in U_\alpha$ and $x' := \dot{s}_i^{-1}x\dot{s}_i \in U_{s_i\alpha}$. Recall from [BB05, Lemma 4.4.3] that s_i permutes $\Phi^+ \setminus \{\alpha_i\}$ (in particular, $s_i\alpha > 0$). Write

$$y_i(t) \cdot x \cdot y_i(-t) = \dot{s}_i x_i(-t)\dot{s}_i^{-1} \cdot x \cdot \dot{s}_i x_i(t)\dot{s}_i^{-1} = \dot{s}_i x_i(-t) \cdot x' \cdot x_i(t)\dot{s}_i^{-1}.$$

Denote by $\Psi' \subset \Phi$ the set of all roots of the form $ms_i\alpha + r\alpha_i$ for integers $m, r > 0$. It is clear that $\Psi' \subset \Phi^+ \setminus \{\alpha_i, s_i\alpha\}$ is a bracket closed subset. By [Hum75, Lemma 32.5], we have

$x_i(-t)x'x_i(t)x'^{-1} \in U(\Psi')$, so $x_i(-t)x'x_i(t) \in U(\Psi')x'$. Thus $\Psi'' := s_i\Psi'$ is also a bracket closed subset of $\Phi^+ \setminus \{\alpha_i, \alpha\}$, and we have $\dot{s}_i U(\Psi')x'\dot{s}_i^{-1} = U(\Psi'')x$. Clearly, $\Psi = \Psi'' \sqcup \{\alpha\}$. We thus have $y_i(t)U_\alpha y_i(-t) \subset U(\Psi'')U_\alpha = U(\Psi)$. □

3.4.5 Flag variety and Bruhat decomposition

Let G/B be the *flag variety* of G (over $\mathbb{K}$). We recall some standard properties of the Bruhat decomposition that can be found in e.g. [Hum75, §28]. Define open Schubert, opposite Schubert, and Richardson varieties:

$$\mathcal{X}^w = B\dot{w}B/B, \quad \mathcal{X}_v = B_-\dot{v}B/B, \quad \mathring{R}_{v,w} := \mathcal{X}_v \cap \mathcal{X}^w \quad (\text{for } v \leq w \in W). \tag{3.4.10}$$

Recall the Bruhat and Birkhoff decompositions:

$$G = \bigsqcup_{w \in W} B\dot{w}B = \bigsqcup_{v \in W} B_-\dot{v}B, \quad \text{where} \tag{3.4.11}$$

$$B_-\dot{v}B \cap B\dot{w}B = \emptyset \quad \text{and} \quad \mathcal{X}_v \cap \mathcal{X}^w = \emptyset \quad \text{for } v \not\leq w \in W. \tag{3.4.12}$$

Let X_v denote the (Zariski) closure of $\mathcal{X}_v$. Similarly, let X^w denote the closure of $\mathcal{X}^w$, and then $R_{v,w} = X_v \cap X^w$ is the closure of $\mathring{R}_{v,w}$ in G/B. We have

$$X_v = \bigsqcup_{v \leq v'} \mathcal{X}_{v'}, \qquad X^w = \bigsqcup_{w' \leq w} \mathcal{X}^{w'}, \tag{3.4.13}$$

$$G/B = \bigsqcup_{v \leq w} \mathring{R}_{v,w}, \qquad R_{v,w} = \bigsqcup_{v \leq v' \leq w' \leq w} \mathring{R}_{v',w'}. \tag{3.4.14}$$

For any $w \in W$, $i \in I$, and $t \in \mathbb{K}^*$, we have

$$x_i(t) \in B_- \dot{s}_i B_-, \quad y_i(t) \in B \dot{s}_i B, \tag{3.4.15}$$

$$B\dot{s}_i B \cdot B\dot{w}B \subset \begin{cases} B\dot{s}_i\dot{w}B, & \text{if } s_i w > w, \\ B\dot{s}_i\dot{w}B \sqcup B\dot{w}B, & \text{if } s_i w < w, \end{cases} \tag{3.4.16}$$

$$B_-\dot{s}_i B_- \cdot B_-\dot{w}B \subset \begin{cases} B_-\dot{s}_i\dot{w}B, & \text{if } s_i w < w, \\ B_-\dot{s}_i\dot{w}B \sqcup B_-\dot{w}B, & \text{if } s_i w > w, \end{cases} \tag{3.4.17}$$

$$B\dot{v}B \cdot B\dot{w}B \subset B\dot{v}\dot{w}B \quad \text{for } v \in W \text{ such that } \ell(vw) = \ell(v) + \ell(w). \tag{3.4.18}$$

For $\mathbf{t} = (t_1, \ldots, t_n) \in (\mathbb{K}^*)^n$ and a reduced word $\mathbf{i} = (i_1, \ldots, i_n)$ for $w \in W$, define

$$\mathbf{x_i}(\mathbf{t}) := x_{i_1}(t_1) \cdots x_{i_n}(t_n), \quad \text{and} \quad \mathbf{y_i}(\mathbf{t}) := y_{i_1}(t_1) \cdots y_{i_n}(t_n). \tag{3.4.19}$$

It follows from (3.4.15), (3.4.16), and (3.4.3) that

$$\mathbf{x_i}(\mathbf{t}) \in B_-\dot{w}B_-, \quad \mathbf{y_i}(\mathbf{t}) \in B\dot{w}B. \tag{3.4.20}$$

3.4.6 Parabolic subgroup W_J of W

Let $J \subset I$, and denote by $W_J \subset W$ the subgroup generated by $\{s_i\}_{i \in J}$. Let w_J be the longest element of W_J, and $w^J := w_0 w_J$ be the maximal element of W^J. Let $\Phi_J \subset \Phi$ consist of roots that are linear combinations of $\{\alpha_i\}_{i \in J}$. Define

$$\Phi_J^+ := \Phi_J \cap \Phi^+, \quad \Phi_J^- := \Phi_J \cap \Phi^-, \quad \Phi_+^{(J)} := \Phi^+ \setminus \Phi_J^+, \quad \Phi_-^{(J)} := \Phi^- \setminus \Phi_J^-.$$

The sets Φ_J^+, $\Phi_+^{(J)}$, Φ_J^-, $\Phi_-^{(J)}$ are clearly bracket closed, so consider subgroups

$$U_J = U(\Phi_J^+), \quad U_J^- = U_-(\Phi_J^-), \quad U^{(J)} = U(\Phi_+^{(J)}), \quad U_-^{(J)} = U_-(\Phi_-^{(J)}).$$

In fact, we have

$$\Phi_J^+ = \mathrm{Inv}(w_J), \quad \Phi_+^{(J)} = \mathrm{Inv}(w^J), \quad \dot{w}_J U_J^- \dot{w}_J^{-1} = U_J. \tag{3.4.21}$$

Let $W^J_{\max} := \{ww_J \mid w \in W^J\}$. By [BB05, Prop. 2.4.4], every $w \in W$ admits a unique *parabolic factorization* $w = w_1 w_2$ for $w_1 \in W^J$ and $w_2 \in W_J$, and this factorization is length-additive. We state some standard facts on parabolic factorizations for later use.

Lemma 3.4.4.

(i) *If $u \in W^J$ and $s_i u < u$ then $s_i u \in W^J$.*

(ii) *Given $u \in W^J$ and $r, r' \in W_J$, we have $ur \le ur'$ if and only if $r \le r'$.*

Proof. Part (i) follows since if $s_i u \notin W^J$ then $s_i u s_j < s_i u$ for some $j \in J$, therefore $s_i u s_j < s_i u < u < u s_j$, which contradicts $\ell(u s_j) = \ell(s_i u s_j) \pm 1$. For (ii), see [BB05, Exercise 2.21]. □

Lemma 3.4.5. *For any $w \in W^J$, we have $\mathrm{Inv}(w) \subset \Phi_+^{(J)}$. In particular, $w\Phi_J^+ \subset \Phi^+$, $\dot{w} U_J \dot{w}^{-1} \subset U$, and $\dot{w} U_J^- \dot{w}^{-1} \subset U^-$.*

Proof. Let $\alpha \in \Phi^+$ be a positive root. Then it can be written as $\alpha = \sum_{i \in I} c_i \alpha_i$ for $c_i \in \mathbb{Z}_{\ge 0}$. Since $w \in W^J$, we have $w\alpha_i > 0$ for all $i \in J$. Thus if $w\alpha < 0$, we must have $c_i \neq 0$ for some $i \notin J$, so $\alpha \in \Phi_+^{(J)}$. □

Lemma 3.4.6 ([He09])**.** *Let $x, y \in W$.*

(i) *The set $\{uv \mid u \le x,\ v \le y\}$ contains a unique maximal element, denoted $x * y$. The set $\{xv \mid v \le y\}$ contains a unique minimal element, denoted $x \triangleleft y$.*

(ii) *There exist elements $u' \le x$ and $v' \le y$ such that $x * y = xv' = u'y$, and these factorizations are both length-additive.*

(iii) *If $x' \le x$, then $x' * y \le x * y$ and $x' \triangleleft y \le x \triangleleft y$.*

(iv) *If xy is length-additive, then $x * y = xy$ and $(xy) \triangleleft y^{-1} = x$.*

The operation $*$ is known as the *Demazure product*.

Proof. The first three parts were shown in [He09, §1.3], with the caveat that our $\triangleleft$ is the 'mirror image' of his $\triangleright$. Part (iv) follows from the definitions of $*$ and $\triangleleft$. □

Definition 3.4.7. Let $Q_J = \{(v, w) \in W \times W^J \mid v \le w\}$. We define an order relation $\preceq$ on Q_J as follows: for $(v, w), (v', w') \in Q_J$, we write $(v, w) \preceq (v', w')$ if and only if there exists $r \in W_J$ such that vr is length-additive and $v' \le vr \le wr \le w'$. For $(v, w) \in Q_J$, denote

$$Q_J^{\succeq (v,w)} := \{(v', w') \in Q_J \mid (v, w) \preceq (v', w')\}, \quad Q_J^{\preceq (v,w)} := \{(v', w') \in Q_J \mid (v', w') \preceq (v, w)\}.$$

Lemma 3.4.8.

(i) *Suppose that* $(v, w), (v', w') \in Q_J$, $r \in W_J$, *and* $v' \le vr \le wr \le w'$. *Then* $(v, w) \preceq (v', w')$.

(ii) *Let* $(u, u), (v, w), (v', w') \in Q_J$. *Then* $(u, u) \preceq (v, w) \preceq (v', w')$ *if and only if*

$$v' \le vr' \le ur \le wr' \le w' \quad \textit{for some } r, r' \in W_J \textit{ such that } vr' \textit{ is length-additive.} \tag{3.4.22}$$

Proof. (i): By Lemma 3.4.6, there exists $r' \le r$ such that $v * r = vr' \ge vr$, and vr' is length-additive. We have $vr' \le wr'$ by Lemma 3.4.6(iii), and $wr' \le wr$ by Lemma 3.4.4(ii). Therefore $v' \le vr \le vr' \le wr' \le wr \le w'$, so $(v, w) \preceq (v', w')$.

(ii) ($\Rightarrow$): Suppose that $(u, u) \preceq (v, w) \preceq (v', w')$. Then by Definition 3.4.7, there exist $r', r'' \in W_J$ such that vr' is length-additive, $v' \le vr' \le wr' \le w'$, and $v \le ur'' \le w$. Define $r \in W_J$ by the equality $(ur'') * r' = ur$. Then applying $*r'$ on the right to $v \le ur'' \le w$, by Lemma 3.4.6(iii)–(iv), we obtain $vr' \le ur \le wr'$. Therefore (3.4.22) holds.

(ii) ($\Leftarrow$): Suppose that (3.4.22) holds. Then $(v, w) \preceq (v', w')$. Define $r'' \in W_J$ by the equality $(ur) \triangleleft r'^{-1} = ur''$. Then applying $\triangleleft (r')^{-1}$ on the right to $vr' \le ur \le wr'$, by Lemma 3.4.6(iii)–(iv), we obtain $v \le ur'' \le w$. Therefore $(u, u) \preceq (v, w)$. $\square$

Remark 3.4.9. By Lemma 3.4.8(i), Definition 3.4.7 remains unchanged if we omit the condition that vr is length-additive. It follows that Q_J coincides with the poset studied in [HL15, §2.4]. Therefore by [HL15, Appendix], Q_J is also isomorphic to the posets studied in [Rie06, GY09, KLS13].

3.4.7 The partial flag variety G/P

Fix $J \subset I$ as before. Let $P \subset G$ be the subgroup generated by B and $\{y_i(t) \mid t \in \mathbb{K}^*,\ i \in J\}$. We denote by G/P the *partial flag variety* corresponding to J, and let $\pi_J : G/B \to G/P$ be the natural projection map. Let $L_J \subset P$ be the *Levi subgroup* of P. It is generated by T and $\{x_i(t), y_i(t) \mid i \in J,\ t \in \mathbb{K}^*\}$. Let P_- be the parabolic subgroup opposite to P, with $L_J = P \cap P_-$.

For $(v, w) \in Q_J$ we introduce $\mathring{\Pi}_{v,w} := \pi_J(\mathring{R}_{v,w}) \subset G/P$, and define the *projected Richardson variety* $\Pi_{v,w} \subset G/P$ to be the closure of $\mathring{\Pi}_{v,w}$ in the Zariski topology. By [KLS14, Prop. 3.6], we have

$$G/P = \bigsqcup_{(v,w)\in Q_J} \mathring{\Pi}_{v,w}, \quad \text{and} \quad \Pi_{v,w} = \bigsqcup_{(v',w')\in Q_J^{\preceq(v,w)}} \mathring{\Pi}_{v',w'}. \tag{3.4.23}$$

Let now $\mathbb{K} = \mathbb{C}$. The varieties $\mathcal{X}^w$, $\mathcal{X}_v$, X_w, X^w, $\mathring{R}_{v,w}$, and $R_{v,w}$ are defined over $\mathbb{R}$. The map π_J is defined over $\mathbb{R}$ as well, thus so are $\mathring{\Pi}_{v,w}$ and $\Pi_{v,w}$. We let $(G/B)_\mathbb{R} := \{gB \mid g \in G(\mathbb{R})\} \subset G/B$, $\mathring{R}^\mathbb{R}_{v,w} := (G/B)_\mathbb{R} \cap \mathring{R}_{v,w}$, and $R^\mathbb{R}_{v,w} := (G/B)_\mathbb{R} \cap R_{v,w}$. Additionally, set $(G/P)_\mathbb{R} := \{xP \mid x \in G(\mathbb{R})\} \subset G/P$, $\mathring{\Pi}^\mathbb{R}_{v,w} := \mathring{\Pi}_{v,w} \cap (G/P)_\mathbb{R}$, and $\Pi^\mathbb{R}_{v,w} := \Pi_{v,w} \cap (G/P)_\mathbb{R}$. It follows that the decomposition (3.4.23) is valid over $\mathbb{R}$:

$$(G/P)_\mathbb{R} = \bigsqcup_{(v,w)\in Q_J} \mathring{\Pi}^\mathbb{R}_{v,w}, \qquad \Pi^\mathbb{R}_{v,w} = \bigsqcup_{(v',w')\in Q_J^{\preceq(v,w)}} \mathring{\Pi}^\mathbb{R}_{v',w'}. \tag{3.4.24}$$

3.4.8 Total positivity

Assume $\mathbb{K} = \mathbb{C}$ in this subsection. Recall that for each $i \in I$, we have one-parameter subgroups $x_i(t)$, $y_i(t)$, $\alpha_i^\vee(t)$ (for $t \in \mathbb{C}$), see (3.4.1).

Definition 3.4.10 ([Lus94])**.** Let $G_{\geq 0} \subset G(\mathbb{R})$ be the submonoid generated by $x_i(t), y_i(t), \alpha^\vee(t)$ for $t \in \mathbb{R}_{>0}$. Define $(G/B)_{\geq 0}$ to be the closure of $(G_{\geq 0}/B) \subset (G/B)_\mathbb{R}$ in the analytic topology. For $v \leq w \in W$, let $R^{\geq 0}_{v,w}$ denote the closure of $R^{>0}_{v,w} := \mathring{R}_{v,w} \cap (G/B)_{\geq 0}$ inside $(G/B)_{\geq 0}$.

Definition 3.4.11 ([Lus98a, Rie99])**.** Set $(G/P)_{\geq 0} := \pi_J((G/B)_{\geq 0})$. For $(v, w) \in Q_J$, let $\Pi^{\geq 0}_{v,w}$ denote the closure of $\Pi^{>0}_{v,w} := \pi_J(R^{>0}_{v,w})$ inside $(G/P)_{\geq 0}$.

Thus we denote by $\Pi_{v,w}^{>0}$ what was denoted by $\Pi_{(v,w)}^{>0}$ in Example 3.1.1. We have decompositions

$$(G/P)_{\geq 0} = \bigsqcup_{(v,w)\in Q_J} \Pi_{v,w}^{>0}, \qquad \Pi_{v,w}^{\geq 0} = \bigsqcup_{(v',w')\in Q_J^{\preceq(v,w)}} \Pi_{v',w'}^{>0}. \tag{3.4.25}$$

As a special case of (3.4.25) for $J = \emptyset$, we have

$$(G/B)_{\geq 0} = \bigsqcup_{v\leq w} R_{v,w}^{>0}, \qquad R_{v,w}^{\geq 0} = \bigsqcup_{v\leq v'\leq w'\leq w} R_{v',w'}^{>0}. \tag{3.4.26}$$

Lemma 3.4.12. *(Assume $\mathbb{K} = \mathbb{C}$.) Let $(v,w) \in Q_J$ and $r \in W_J$ be such that vr is length-additive. Then*

$$\mathring{\Pi}_{v,w} = \pi_J(\mathring{R}_{v,w}) = \pi_J(\mathring{R}_{vr,wr}), \qquad \Pi_{v,w}^{>0} = \pi_J(R_{v,w}^{>0}) = \pi_J(R_{vr,wr}^{>0}), \tag{3.4.27}$$

$$\Pi_{v,w} = \pi_J(R_{v,w}) = \pi_J(R_{vr,wr}), \qquad \Pi_{v,w}^{\geq 0} = \pi_J(R_{v,w}^{\geq 0}) = \pi_J(R_{vr,wr}^{\geq 0}). \tag{3.4.28}$$

Proof. By [KLS13, Lemma 3.1], we have $\pi_J(\mathring{R}_{v,w}) = \pi_J(\mathring{R}_{vr,wr}) = \mathring{\Pi}_{v,w}$, and π_J restricts to isomorphisms $\mathring{R}_{v,w} \xrightarrow{\sim} \mathring{\Pi}_{v,w}$, $\mathring{R}_{vr,wr} \xrightarrow{\sim} \mathring{\Pi}_{v,w}$. Thus $\pi_J(R_{v,w}^{>0}) = \pi_J(R_{vr,wr}^{>0}) = \Pi_{v,w}^{>0}$ follows from the equality $\pi_J((G/B)_{\geq 0}) = (G/P)_{\geq 0}$, proving (3.4.27). To show (3.4.28), note that $R_{a,b}$ and $R_{a,b}^{\geq 0}$ are compact for any $a \leq b$, and therefore their images under π_J are closed. $\square$

Recall the definition of $\mathbf{x_i}(\mathbf{t})$ and $\mathbf{y_i}(\mathbf{t})$ from (3.4.19). Choose a reduced word $\mathbf{i} = (i_1, \dots, i_n)$ for $w \in W$ and define

$$U_{>0}(w) := \{\mathbf{x_i}(\mathbf{t}) \mid \mathbf{t} \in \mathbb{R}_{>0}^n\}, \quad U_{>0}^-(w) := \{\mathbf{y_i}(\mathbf{t}) \mid \mathbf{t} \in \mathbb{R}_{>0}^n\}.$$

Let $U_{\geq 0} \subset U(\mathbb{R})$ (resp., $U_{\geq 0}^- \subset U_-(\mathbb{R})$) be the submonoid generated by $x_i(t)$ (resp., by $y_i(t)$) for $t \in \mathbb{R}_{>0}$. Then $U_{\geq 0} = \bigsqcup_{w\in W} U_{>0}(w)$ and $U_{\geq 0}^- = \bigsqcup_{w\in W} U_{>0}^-(w)$. We have $U_{>0}(w) = U_{\geq 0} \cap B_-\dot{w}B_-$ and $U_{>0}^-(w) = U_{\geq 0}^- \cap B\dot{w}B$, and these sets do not depend on the choice of the reduced word $\mathbf{i}$ for w, see [Lus94, Prop. 2.7].

3.4.9 MR-parametrizations

Assume that $\mathbb{K}$ is algebraically closed. Given $w \in W$, an *expression* $\mathbf{w}$ for w is a sequence $\mathbf{w} = (w_{(0)}, \dots, w_{(n)})$ such that $w_{(0)} = \mathrm{id}$, $w_{(n)} = w$, and for $j = 1, \dots, n$, either $w_{(j)} = w_{(j-1)}$ or $w_{(j)} = w_{(j-1)}s_{i_j}$ for some $i_j \in I$. In the latter case we require $w_{(j-1)} < w_{(j)}$, unlike [MR04]. We denote $J^+_{\mathbf{w}} := \{1 \le j \le n \mid w_{(j-1)} < w_{(j)}\}$ and $J^\circ_{\mathbf{w}} := \{1 \le j \le n \mid w_{(j-1)} = w_{(j)}\}$ so that $J^+_{\mathbf{w}} \sqcup J^\circ_{\mathbf{w}} = \{1, 2, \dots, n\}$. Every reduced word $\mathbf{i} = (i_1, \dots, i_n)$ for w gives rise to a *reduced expression* $\mathbf{w} = \mathbf{w}(\mathbf{i}) = (w_{(0)}, \dots, w_{(n)})$ with $w_{(j)} = w_{(j-1)}s_{i_j}$ for $j = 1, \dots, n$.

Lemma 3.4.13 ([MR04, Lemma 3.5])**.** *Let $v \le w \in W$, and consider a reduced expression $\mathbf{w} = (w_{(0)}, \dots, w_{(n)})$ for w corresponding to a reduced word $\mathbf{i} = (i_1, \dots, i_n)$. Then there exists a unique* positive subexpression $\mathbf{v}$ for v inside $\mathbf{w}$, *i.e., an expression $\mathbf{v} = (v_{(0)}, \dots, v_{(n)})$ for v such that for $j = 1, \dots, n$, we have $v_{(j-1)} < v_{(j-1)}s_{i_j}$. This positive subexpression can be constructed inductively by setting $v_{(n)} := v$ and*

$$v_{(j-1)} := \begin{cases} v_{(j)}s_{i_j}, & \textit{if } v_{(j)}s_{i_j} < v_{(j)}, \\ v_{(j)}, & \textit{otherwise,} \end{cases} \qquad \textit{for } j = n, \dots, 1. \tag{3.4.29}$$

Corollary 3.4.14. *In the above setting, if $v_{(1)} = s_i$ for some $i \in I$ then $v \not\le s_i w$.*

Proof. Indeed, if $v \le s_i w < w$ then there exists a positive subexpression $\mathbf{v}' = (v'_{(0)}, \dots, v'_{(n-1)})$ for v inside $\mathbf{w}(\mathbf{i}')$, where $\mathbf{i}' = (i_2, \dots, i_n)$. By (3.4.29), we have $v'_{(j)} = v_{(j+1)}$ for $j = 0, 1, \dots, n-1$, which contradicts the fact that $v'_{(0)} = 1$ while $v_{(1)} = s_i$. $\square$

For $w \in W$, let $\mathrm{Red}(w) := \{\mathbf{w} \mid \mathbf{w} \text{ is a reduced expression for } w\}$. For $v \le w \in W$, let

$$\mathrm{Red}(v, w) := \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{w} \in \mathrm{Red}(w),\ \mathbf{v} \text{ is a positive subexpression for } v \text{ inside } \mathbf{w}\}.$$

Thus for all $v \le w$, the sets $\mathrm{Red}(w)$ and $\mathrm{Red}(v, w)$ have the same cardinality. Let $v \le w \in W$ and $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$. Given a collection $\mathbf{t} = (t_k)_{k \in J^\circ_{\mathbf{v}}} \in (\mathbb{K}^*)^{J^\circ_{\mathbf{v}}}$, define

$$\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) := g_1 \cdots g_n, \quad \text{where} \quad g_k := \begin{cases} y_{i_k}(t_k), & \text{if } k \in J^\circ_{\mathbf{v}}, \\ \dot{s}_{i_k}, & \text{if } k \in J^+_{\mathbf{v}}. \end{cases} \tag{3.4.30}$$

MR-parametrizations of $(G/B)_{\geq 0}$

In this section, we assume $\mathbb{K} = \mathbb{C}$. Let v, w, $\mathbf{v}$, and $\mathbf{w}$ be as above. Define a subset $G^{>0}_{\mathbf{v},\mathbf{w}} \subset G(\mathbb{R})$ by

$$G^{>0}_{\mathbf{v},\mathbf{w}} := \{\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) \mid \mathbf{t} \in \mathbb{R}^{J^\circ_{\mathbf{v}}}_{>0}\}.$$

Theorem 3.4.15 ([MR04, Theorem 11.3])**.** *The map $G(\mathbb{R}) \to (G/B)_{\mathbb{R}}$ sending $g \mapsto gB$ restricts to an isomorphism of real semialgebraic varieties*

$$G^{>0}_{\mathbf{v},\mathbf{w}} \xrightarrow{\sim} R^{>0}_{v,w}.$$

Proposition 3.4.16 ([Lus94, Prop. 8.12])**.** *We have $G_{\geq 0} \cdot (G/B)_{\geq 0} \subset (G/B)_{\geq 0}$.*

Lemma 3.4.17. *Suppose that $g \in G_{\geq 0}$ and $x \in G$ are such that $xB \in R^{>0}_{v,w}$ for some $v \leq w \in W$. Then $gxB \in R^{>0}_{v',w'}$ for some $v' \leq v \leq w \leq w'$.*

Proof. By Proposition 3.4.16, we have $gxB \in (G/B)_{\geq 0}$, so it suffices to show that $gx \in B\dot{w}'B \cap B_-\dot{v}'B$ for some $v' \leq v \leq w \leq w'$. Note that we have $x \in B\dot{w}B \cap B_-\dot{v}B$. By Definition 3.4.10, it is enough to consider the cases $g = x_i(t)$ and $g = y_i(t)$ for $i \in I$ and $t \in \mathbb{R}_{>0}$.

Suppose that $g = y_i(t)$. We clearly have $gx \in B_-\dot{v}B$. If $s_iw > w$ then by (3.4.16) we have $gx \in B\dot{s}_i\dot{w}B$. Thus we may assume that $s_iw < w$. By Theorem 3.4.15, we can also assume $x = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = g_1 \cdots g_n$ for $\mathbf{t} \in \mathbb{R}^{J^\circ_{\mathbf{v}}}_{>0}$ and some choice of $(\mathbf{v},\mathbf{w}) \in \mathrm{Red}(v,w)$ such that $\mathbf{w} = (w_{(0)}, \dots w_{(n)})$ satisfies $w_{(1)} = s_i$. Let $\mathbf{v} = (v_{(0)}, \dots, v_{(n)})$. If $v_{(1)} \neq s_i$ then $g_1 = y_i(t')$ so $gx \in G^{>0}_{\mathbf{v},\mathbf{w}}$ and we are done. If $v_{(1)} = s_i$ then by Corollary 3.4.14 we have $v \not\leq s_iw$. Recall that $gx \in B_-\dot{v}B$ and by (3.4.16), $gx \in B\dot{s}_i\dot{w}B \sqcup B\dot{w}B$. But $B_-\dot{v}B \cap B\dot{s}_i\dot{w}B = \emptyset$ by (3.4.12). Therefore we must have $gx \in B\dot{w}B$, finishing the proof in this case.

The case $g = x_i(t)$ follows analogously using a "dual" Marsh–Rietsch parametrization [Rie06, Section 3.4], where for $(\mathbf{v},\mathbf{w}) \in \mathrm{Red}(v,w)$, every element of $R^{>0}_{ww_0,vw_0}$ is parametrized as

$$g_1 \cdots g_n \dot{w}_0 B, \quad \text{where} \quad g_k := \begin{cases} x_{i_k}(t_k), & \text{if } k \in J^\circ_{\mathbf{v}}, \\ \dot{s}_{i_k}^{-1}, & \text{if } k \in J^+_{\mathbf{v}}. \end{cases} \qquad \square$$

We will use the following consequence of Theorem 3.4.15 in Section 3.9.11

Corollary 3.4.18 (cf. [KLS14, Prop. 3.3]). *Let $u \in W^J$, $r \in W_J$, and $v \in W$ be such that $v \leq ur$. Then*

$$\pi_J(R^{>0}_{v,ur}) = \pi_J(R^{>0}_{v \triangleleft r^{-1},u}) = \Pi^{>0}_{v \triangleleft r^{-1},u}.$$

Proof. Let $\mathbf{i} = (i_1, \dots, i_n)$ be a reduced word for $w := ur$, such that $(i_{\ell(u)+1}, \dots, i_n)$ is a reduced word for r. Let $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$ be such that $\mathbf{w}$ corresponds to $\mathbf{i}$. Then it is clear from Lemma 3.4.13 that after setting $\mathbf{v}' := (v_{(0)}, \dots, v_{(\ell(u))})$ and $\mathbf{u} := (w_{(0)}, \dots, w_{(\ell(u))})$, we get $(\mathbf{v}', \mathbf{u}) \in \mathrm{Red}(v \triangleleft r^{-1}, u)$. Moreover, the indices $i_{\ell(u)+1}, \dots, i_n$ clearly belong to J, so if $g_1 \dots g_n \in G^{>0}_{\mathbf{v},\mathbf{w}}$ then $g_1 \dots g_{\ell(u)} \in G^{>0}_{\mathbf{v}',\mathbf{u}}$ and $\pi_J(g_1 \dots g_n B) = \pi_J(g_1 \dots g_{\ell(u)} B)$. We are done by Theorem 3.4.15. □

3.4.10 G/P is a TNN space

We show that the triple $((G/P)_{\mathbb{R}}, (G/P)_{\geq 0}, Q_J)$ is a TNN space in the sense of Definition 3.2.1. We start by recalling several well known facts.

Proposition 3.4.19.

(i) *The poset $\widehat{Q}_J := Q_J \sqcup \{\hat{0}\}$ is graded, thin, and shellable.*

(ii) *$(G/P)_{\mathbb{R}}$ is a smooth manifold, and each $\mathring{\Pi}^{\mathbb{R}}_{v,w}$ is a smooth embedded submanifold of $(G/P)_{\mathbb{R}}$.*

(iii) *For $(v, w) \in Q_J$, $\Pi^{>0}_{v,w}$ is a connected component of $\mathring{\Pi}^{\mathbb{R}}_{v,w}$.*

Proof. For (i), see [Wil07]. For (ii), $(G/P)_{\mathbb{R}}$ is a smooth manifold because it is a homogeneous space of a real Lie group, and $\mathring{\Pi}^{\mathbb{R}}_{v,w}$ is a smooth embedded manifold because it is the set of real points of a smooth algebraic subvariety $\mathring{\Pi}_{v,w}$ of G/P, see [KLS14, Cor. 3.2] or [Lus98a, Rie06]. Part (iii) is due to [Rie99]. □

Corollary 3.4.20. *$((G/P)_{\mathbb{R}}, (G/P)_{\geq 0}, Q_J)$ is a TNN space.*

Proof. Let us check each part of Definition 3.2.1.

(TNN1): Follows from Proposition 3.4.19(i). The maximal element $\hat{1} \in Q_J$ is given by (id, w^J), see Section 3.4.6.

(TNN2): Follows from Proposition 3.4.19(ii) and (3.4.24).

(TNN3): Clear since $(G/P)_{\mathbb{R}}$ is compact and $\Pi_{v,w}^{\geq 0} \subset G/P$ is closed.

(TNN4): Follows from Proposition 3.4.19(iii) combined with Theorem 3.4.15.

(TNN5): This result is due to [Rie06], see (3.4.25). $\square$

3.4.11 Gaussian decomposition

Assume $\mathbb{K}$ is algebraically closed. Let us define

$$G_0^{\mp} := B_-B, \quad G_0^{\pm} := BB_-.$$

For $i \in I$, let $\Delta_i^{\mp} : G_0^{\mp} \to \mathbb{K}$ and $\Delta_i^{\pm} : G_0^{\pm} \to \mathbb{K}$ be defined as follows. Given $(x_-, x_0, x_+) \in U_- \times T \times U$, we have $x_-x_0x_+ \in G_0^{\mp}$ and $x_+x_0x_- \in G_0^{\pm}$, and we set $\Delta_i^{\mp}(x_-x_0x_+) := x_0^{\omega_i}$, $\Delta_i^{\pm}(x_+x_0x_-) := x_0^{w_0\omega_i}$. For a finite set A, let $\mathbb{P}^{\binom{2n}{n-1}-1^A}$ denote the $(|A|-1)$-dimensional projective space over $\mathbb{K}$, with coordinates indexed by elements of A.

Lemma 3.4.21.

(i) *The multiplication map gives biregular isomorphisms:*

$$U_- \times T \times U \xrightarrow{\sim} G_0^{\mp}, \quad U \times T \times U_- \xrightarrow{\sim} G_0^{\pm}.$$

(ii) *The maps $\Delta_i^{\mp}$ and $\Delta_i^{\pm}$ extend to regular functions $G \to \mathbb{K}$.*

(iii) *We have $G_0^{\mp} = \{x \in G \mid \Delta_i^{\mp}(x) \neq 0 \ \forall i \in I\}$ and $G_0^{\pm} = \{x \in G \mid \Delta_i^{\pm}(x) \neq 0 \ \forall i \in I\}$.*

(iv) *Fix $i \in I$ and let $W\omega_i := \{w\omega_i \mid w \in W\}$ denote the W-orbit of the corresponding fundamental weight. Then there exists a regular map $\Delta_i^{\mathrm{flag}} : G/B \to \mathbb{P}^{\binom{2n}{n-1}-1^{W\omega_i}}$ such that for $w \in W$ and $x \in G$, the $w\omega_i$-th coordinate of $\Delta_i^{\mathrm{flag}}(xB)$ equals $\Delta_i^{\mp}(\dot{w}^{-1}x)$.*

Proof. For (i), see [Hum75, Prop. 28.5]. Parts (ii) and (iii) are well known when $\mathbb{K} = \mathbb{C}$, see [FZ99, Prop. 2.4 and Cor. 2.5]. We give a proof for arbitrary algebraically closed $\mathbb{K}$, using a standard argument that relies on representation theory. We refer to [Hum75, §31] for the necessary notation and background.

We have $G_0^{\pm} = \dot{w}_0^{-1}G_0^{\mp}\dot{w}_0$ and $\Delta_i^{\pm}(\dot{w}_0^{-1}g\dot{w}_0) = \Delta_i^{\mp}(g)$ for all $g \in G_0^{\mp}$. Thus it suffices to give a proof for $\Delta_i^{\mp}$ and $G_0^{\mp}$. For $i \in I$, there exists a regular function $c_{\omega_i} : G \to \mathbb{K}$ that coincides with $\Delta_i^{\mp}$ on $G_0^{\mp}$, see [Hum75, §31.4]. This shows (ii). Explicitly, c_{ω_i} is given as follows: consider the highest weight module V_{ω_i} for G, and let $v_+ \in V_{\omega_i}$ be its highest weight vector. We have a direct sum of vector spaces $V_{\omega_i} = \mathbb{K}v_+ \oplus V'$, where V' is spanned by weight vectors of weight other than ω_i. Let $r^+ : V_{\omega_i} \to \mathbb{K}$ denote the linear function such that $r^+(v_+) = 1$ and $r^+(V') = \{0\}$, then $c_{\omega_i}(g) := r^+(gv_+)$ for all $g \in G$. The decomposition $V_{\omega_i} = \mathbb{K}v_+ \oplus V'$ is such that for $(x_-, x_0, x_+) \in U_- \times T \times U$ and $w \in W$, we have $x_+v_+ = v_+$, $x_0v_+ = Mv_+$ for some $M \in \mathbb{K}^*$, $x_-v_+ \in v_+ + V'$, $x_-V' \subset V'$, and $\dot{w}v_+ \in V'$ if $w\omega_i \neq \omega_i$. Thus if $g \in G_0^{\mp}$ then $c_{\omega_i}(g) \neq 0$ for all $i \in I$. Conversely, if $x \notin G_0^{\pm}$ then by (3.4.11), there exists a unique $w \neq \mathrm{id} \in W$ such that $g \in U_-\dot{w}TU$. For $i \in I$ such that $w\omega_i \neq \omega_i$, we get $c_{\omega_i}(g) = 0$. This proves (iii). For (iv), let $V_{\omega_i} = V_1 \oplus V_2$ where V_1 is spanned by all weight vectors of weight in $W\omega_i$, and V_2 is spanned by the remaining weight vectors. Let $\pi_1 : V_{\omega_i} \to V_1$ denote the projection along V_2. It follows that for all $g \in G$, $\pi_1(gv_+) \neq 0$. Then Δ_i^{flag} is the natural morphism $G/B \to \mathbb{P}^{\binom{2n}{n-1}-1}(V_1)$, sending gB to $[\pi_1(gv_+)]$. $\square$

Lemma 3.4.22. *Define $G_0^{(J)} := P_-P$.*

(i) *We have $G_0^{(J)} = P_-B$ and $P = \bigsqcup_{r \in W_J} B\dot{r}B$.*

(ii) *For $p \in P$, we have $pU^{(J)}p^{-1} = U^{(J)}$. Similarly, for $p \in P_-$, we have $pU_-^{(J)}p^{-1} = U_-^{(J)}$. In particular, for $p \in L_J$, we have $pU^{(J)}p^{-1} = U^{(J)}$ and $pU_-^{(J)}p^{-1} = U_-^{(J)}$.*

(iii) *The multiplication map gives a biregular isomorphism $U_-^{(J)} \times L_J \times U^{(J)} \xrightarrow{\sim} G_0^{(J)}$. In particular, every element $x \in G_0^{(J)}$ can be uniquely factorized as $[x]_-^{(J)} \cdot [x]_J \cdot [x]_+^{(J)} \in U_-^{(J)} \cdot L_J \cdot U^{(J)}$. The map $G_0^{(J)} \to L_J$ sending $x \mapsto [x]_J$ satisfies $[p_-xp_+]_J = [p_-]_J[x]_J[p_+]_J$ for all $x \in G_0^{(J)}$, $p_- \in P_-$, and $p_+ \in P$.*

(iv) *The map $b \to [b]_J$ gives group homomorphisms $U \to U_J$ and $U_- \to U_J^-$, sending*

$$x_i(t) \mapsto [x_i(t)]_J = \begin{cases} x_i(t), & \text{if } i \in J, \\ 1, & \text{otherwise,} \end{cases} \qquad y_i(t) \mapsto [y_i(t)]_J = \begin{cases} y_i(t), & \text{if } i \in J, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. By [Hum75, §30.2], $U^{(J)}$ is the *unipotent radical* (in particular, a normal subgroup) of P and $U_-^{(J)}$ is the unipotent radical of P_-. This shows (ii). It follows that $P = L_JU^{(J)} = L_JB$, therefore $G_0^{(J)} = P_-B$. By [Hum75, §30.1] and (3.4.11), $P = \bigsqcup_{r \in W_J} B\dot{r}B$, which proves (i).

By [Bor91, Prop. 14.21(iii)], the multiplication map gives a biregular isomorphism $U_-^{(J)} \times$

$P \to G_0^{(J)}$. By [Hum75, §30.2], the multiplication map gives a biregular isomorphism $L_J \times U^{(J)} \to P$. Thus we get a biregular isomorphism $U_-^{(J)} \times L_J \times U^{(J)} \xrightarrow{\sim} G_0^{(J)}$. It is clear from the definition that $[p_- x p_+]_J = [p_-]_J [x]_J [p_+]_J$, since we can factorize $p_- = [p_-]_-^{(J)} [p_-]_J$ and $p_+ = [p_+]_J [p_+]_+^{(J)}$. Thus we are done with (iii), and (iv) follows by repeatedly applying (iii). $\square$

3.4.12 Affine charts

For $u \in W^J$, denote $C_u^{(J)} := \dot{u} G_0^{(J)}/P \subset G/P$. The following maps are biregular isomorphisms for $u \in W^J$ and $v, w \in W$, see [Bor91, Prop. 14.21(iii)], [Spr98, Prop. 8.5.1(ii)], and [FH91, Cor. 23.60]:

$$\dot{u} U_-^{(J)} \dot{u}^{-1} \xrightarrow{\sim} C_u^{(J)}, \qquad g^{(J)} \mapsto g^{(J)} \dot{u} P, \tag{3.4.31}$$

$$\dot{v} U_- \dot{v}^{-1} \cap U_- \xrightarrow{\sim} \mathcal{X}_v, \qquad g \mapsto g \dot{v} B, \tag{3.4.32}$$

$$\dot{w} U_- \dot{w}^{-1} \cap U \xrightarrow{\sim} \mathcal{X}^w, \qquad g \mapsto g \dot{w} B. \tag{3.4.33}$$

As a consequence of (3.4.32) and (3.4.33), we get

$$B_- \dot{v} B = (\dot{v} U_- \cap U_- \dot{v}) \cdot B, \quad B \dot{w} B = (\dot{w} U_- \cap U \dot{w}) \cdot B. \tag{3.4.34}$$

The isomorphism in (3.4.31) identifies an open dense subset $C_u^{(J)}$ of G/P with the group $\dot{u} U_-^{(J)} \dot{u}^{-1}$. We now combine this with Lemma 3.4.2.

Definition 3.4.23. Let $U_1^{(J)} := \dot{u} U_-^{(J)} \dot{u}^{-1} \cap U$ and $U_2^{(J)} := \dot{u} U_-^{(J)} \dot{u}^{-1} \cap U_-$. For $x \in \dot{u} G_0^{(J)}$, consider the element $g^{(J)} \in \dot{u} U_-^{(J)} \dot{u}^{-1}$ such that $g^{(J)} \dot{u} \in xP \cap \dot{u} U_-^{(J)}$, unique by (3.4.31). Further, let $h_1^{(J)}, g_1^{(J)} \in U_1^{(J)}$ and $h_2^{(J)}, g_2^{(J)} \in U_2^{(J)}$ be the elements such that $h_2^{(J)} g^{(J)} = g_1^{(J)}$ and $h_1^{(J)} g^{(J)} = g_2^{(J)}$. By (3.4.31), the map $x \mapsto g^{(J)}$ is regular, and the map $g^{(J)} \to (g_1^{(J)}, g_2^{(J)}, h_1^{(J)}, h_2^{(J)})$ is regular by Lemma 3.4.2. Let us denote by $\kappa : \dot{u} G_0^{(J)} \to U_2^{(J)}$ the map $x \mapsto \kappa_x := h_2^{(J)}$. It descends to a regular map $\kappa : C_u^{(J)} \to U_2^{(J)}$ sending $xP \mapsto \kappa_x$.

3.5 Subtraction-free parametrizations

We study subtraction-free analogs of Marsh–Rietsch parametrizations of $(G/B)_{\geq 0}$.

3.5.1 Subtraction-free subsets

Given some fixed collection $\mathbf{t}$ of variables of size $|\mathbf{t}|$, let $\mathbb{R}[\mathbf{t}]$ be the ring of polynomials in $\mathbf{t}$, and $\mathbb{R}_{>0}[\mathbf{t}] \subset \mathbb{R}[\mathbf{t}]$ be the semiring of nonzero polynomials in $\mathbf{t}$ with positive real coefficients. Let $\mathcal{F} := \mathbb{R}(\mathbf{t})$ be the field of rational functions in $\mathbf{t}$. Define

$$\mathcal{F}_{\mathrm{sf}}^* := \{R(\mathbf{t})/Q(\mathbf{t}) \mid R(\mathbf{t}), Q(\mathbf{t}) \in \mathbb{R}_{>0}[\mathbf{t}]\}, \quad \mathcal{F}_{\mathrm{sf}} := \{0\} \sqcup \mathcal{F}_{\mathrm{sf}}^*,$$

$$\mathcal{F}^\diamond := \{R(\mathbf{t})/Q(\mathbf{t}) \mid R(\mathbf{t}) \in \mathbb{R}[\mathbf{t}],\ Q(\mathbf{t}) \in \mathbb{R}_{>0}[\mathbf{t}]\}.$$

We call elements of $\mathcal{F}_{\mathrm{sf}}$ *subtraction-free rational expressions in* $\mathbf{t}$. In this section, we assume that $\mathbb{K} = \overline{\mathcal{F}}$ is the algebraic closure of $\mathcal{F}$.

Definition 3.5.1. Let $T^{\mathrm{sf}} \subset T$ be the submonoid generated by $\alpha_i^\vee(t)$ for $i \in I$ and $t \in \mathcal{F}_{\mathrm{sf}}^*$. Let $G^\diamond \subset G$ be the subgroup generated by

$$\{x_i(t), y_i(t) \mid i \in I,\ t \in \mathcal{F}^\diamond\} \cup \{\dot{w} \mid w \in W\} \cup T^{\mathrm{sf}}.$$

We define subgroups $U^\diamond := U \cap G^\diamond$, $U_-^\diamond := U_- \cap G^\diamond$, $B^{\mathrm{sf}} := T^{\mathrm{sf}} U^\diamond = U^\diamond T^{\mathrm{sf}}$ and $B_-^{\mathrm{sf}} = T^{\mathrm{sf}} U_-^\diamond = U_-^\diamond T^{\mathrm{sf}}$ (cf. Lemma 3.5.2 below). We also put $U^\diamond(\Theta) := U^\diamond \cap U(\Theta)$ (resp., $U_-^\diamond(\Theta) := U_-^\diamond \cap U_-(\Theta)$) for a bracket closed subset Θ of Φ^+ (resp., of Φ^-). Given a reduced word $\mathbf{i}$ for $w \in W$, define

$$U_{\mathrm{sf}}(w) := \{\mathbf{x_i}(\mathbf{t}') \mid \mathbf{t}' \in (\mathcal{F}_{\mathrm{sf}}^*)^n\}, \quad U_{\mathrm{sf}}^-(w) := \{\mathbf{y_i}(\mathbf{t}') \mid \mathbf{t}' \in (\mathcal{F}_{\mathrm{sf}}^*)^n\}. \tag{3.5.1}$$

These subsets do not depend on the choice of $\mathbf{i}$, see [BZ97].

For two subsets H_1, H_2 of G, we say that H_1 *commutes with* H_2 if $H_1 \cdot H_2 = H_2 \cdot H_1$. We say that H_1 *commutes with* $g \in G$ if $H_1 \cdot g = g \cdot H_1$.

Lemma 3.5.2. *T^{sf} commutes with B^{sf}, U, U_-, $U^\diamond(\Theta)$, $U_-^\diamond(\Theta)$, $U_{\mathrm{sf}}(w)$, $U_{\mathrm{sf}}^-(w)$ and $\dot{w}$.*

Proof. It follows from (3.4.2) that T^{sf} commutes with B^{sf}, U, U_-, $U_{\mathrm{sf}}(w)$, $U_{\mathrm{sf}}^-(w)$ and $\dot{w}$. For $U^\diamond(\Theta)$, $U_-^\diamond(\Theta)$, we use a generalization of (3.4.2): for $\alpha \in \Phi^+$, $i \in I$, and $w \in W$ such that $w\alpha_i = \alpha$, write $x_\alpha(t) := \dot{w} x_i(t) \dot{w}^{-1} \in U^\diamond(\{\alpha\})$ and $y_\alpha(t) := \dot{w} y_i(t) \dot{w}^{-1} \in U_-^\diamond(\{-\alpha\})$ for $t \in \mathcal{F}^\diamond$. Then (3.4.2) implies $a x_\alpha(t) a^{-1} = x_\alpha(a^\alpha t)$ and $a y_\alpha(t) a^{-1} = y_\alpha(a^{-\alpha} t)$. $\square$

Let us now introduce subtraction-free analogs of MR parametrizations. Let $v \le w \in W$ and $(\mathbf{v},\mathbf{w}) \in \mathrm{Red}(v,w)$. Recall that for $\mathbf{t}' = (t'_k)_{k\in J^\circ_{\mathbf{v}}} \in (\mathbb{K}^*)^{J^\circ_{\mathbf{v}}}$, $\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}') = g_1\cdots g_n$ is defined in (3.4.30). Define $G^{\mathrm{sf}}_{\mathbf{v},\mathbf{w}} := \{\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}') \mid \mathbf{t}' \in (\mathcal{F}^*_{\mathrm{sf}})^{J^\circ_{\mathbf{v}}}\} \subset G^\diamond$. The following result is closely related to [MR04, Lemma 11.8].

Lemma 3.5.3. *Let $v \le w \in W$ and $(\mathbf{v},\mathbf{w}) \in \mathrm{Red}(v,w)$. Let $\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}')$ be as in (3.4.30) for $\mathbf{t}' \in (\mathcal{F}^*_{\mathrm{sf}})^{J^\circ_{\mathbf{v}}}$. Then for each $k = 0,1,\dots,n$ and for all $x \in U^\diamond \cap \dot{v}_{(k)}^{-1} U_- \dot{v}_{(k)}$, we have*

$$g_1\cdots g_k \cdot x \cdot g_{k+1}\cdots g_n \in g_1\cdots g_n \cdot U^\diamond. \tag{3.5.2}$$

Proof. We prove this by induction on k. For $k = n$, the result is trivial, so suppose that $k < n$. Let $x \in U^\diamond \cap \dot{v}_{(k)}^{-1} U_- \dot{v}_{(k)}$. If $g_{k+1} = \dot{s}_i$ for some $i \in I$ then $\ell(v_{(k+1)}) = \ell(v_{(k)}) + \ell(s_i)$, so we use (3.4.9) to show that $x \cdot g_{k+1} = g_{k+1} \cdot x'$ for some $x' \in U \cap \dot{v}_{(k+1)}^{-1} U_- \dot{v}_{(k+1)}$. Since $x' = \dot{s}_i^{-1} x \dot{s}_i$ and each term belongs to $G^\diamond$, we see that $x' \in U^\diamond \cap \dot{v}_{(k+1)}^{-1} U_- \dot{v}_{(k+1)}$, so we are done by induction.

Suppose now that $g_{k+1} = y_i(t)$ for some $i \in I$ and $t \in \mathcal{F}^*_{\mathrm{sf}}$. Write

$$x \cdot g_{k+1} = g_{k+1} \cdot g_{k+1}^{-1} x g_{k+1} = g_{k+1} \cdot y_i(-t) x y_i(t).$$

By (3.4.5), $U^\diamond \cap \dot{v}_{(k)}^{-1} U_- \dot{v}_{(k)} = U^\diamond(\mathrm{Inv}(v_{(k)}))$. Clearly again $y_i(-t)xy_i(t) \in G^\diamond$, and we claim that $y_i(-t)xy_i(t) \in U(\mathrm{Inv}(v_{(k)}))$ for all $x \in U(\mathrm{Inv}(v_{(k)}))$. First, using Lemma 3.4.1(ii), we can assume that $x \in U_\alpha$ for some $\alpha \in \mathrm{Inv}(v_{(k)})$. Since $v_{(k)}s_i > v_{(k)}$, we have $\alpha_i \notin \mathrm{Inv}(v_{(k)})$, so $\alpha \ne \alpha_i$. Let $\Psi = \{m\alpha - r\alpha_i\} \subset \Phi^+$ be the set of roots as in Lemma 3.4.3. Our goal is to show that $\Psi \subset \mathrm{Inv}(v_{(k)})$. Let $\gamma := m\alpha - r\alpha_i \in \Psi$ for some $m > 0$ and $r \ge 0$. We now show that $\gamma \in \mathrm{Inv}(v_{(k)})$, which is equivalent to saying that $v_{(k)}\gamma < 0$. Indeed, $v_{(k)}\gamma = mv_{(k)}\alpha - rv_{(k)}\alpha_i$. Since $\alpha \in \mathrm{Inv}(v_{(k)})$, $v_{(k)}\alpha < 0$. Since $\alpha_i \notin \mathrm{Inv}(v_{(k)})$, $v_{(k)}\alpha_i > 0$. Thus $v_{(k)}\gamma < 0$, because $-v_{(k)}\gamma$ is a positive linear combination of positive roots. We have shown that $\Psi \subset \mathrm{Inv}(v_{(k)})$, thus by Lemma 3.4.3, we find $y_i(-t)xy_i(t) \in U(\mathrm{Inv}(v_{(k)}))$. Since $v_{(k)} = v_{(k+1)}$, we get

$$y_i(-t)xy_i(t) \in U^\diamond(\mathrm{Inv}(v_{(k)})) = U^\diamond \cap \dot{v}_{(k)}^{-1} U_- \dot{v}_{(k)} = U^\diamond \cap \dot{v}_{(k+1)}^{-1} U_- \dot{v}_{(k+1)},$$

and we are done by induction. $\square$

Proposition 3.5.4. *For $v \le w \in W$, the set $G^{\mathrm{sf}}_{\mathbf{v},\mathbf{w}} \cdot U^\diamond \subset G^\diamond$ does not depend on the choice of $(\mathbf{v},\mathbf{w}) \in \mathrm{Red}(v,w)$. In other words: let $(\mathbf{v}_0,\mathbf{w}_0),(\mathbf{v}_1,\mathbf{w}_1) \in \mathrm{Red}(v,w)$. Then for any $\mathbf{t}_0 \in (\mathcal{F}^*_{\mathrm{sf}})^{J^\circ_{\mathbf{v}_0}}$ there exists $\mathbf{t}_1 \in (\mathcal{F}^*_{\mathrm{sf}})^{J^\circ_{\mathbf{v}_1}}$ and $x \in U^\diamond$ such that $\mathbf{g}_{\mathbf{v}_0,\mathbf{w}_0}(\mathbf{t}_0) = \mathbf{g}_{\mathbf{v}_1,\mathbf{w}_1}(\mathbf{t}_1) \cdot x$.*

Proof. Recall that for each $\mathbf{w}_0 \in \mathrm{Red}(w)$ there exists a unique positive subexpression $\mathbf{v}_0$ for v such that $(\mathbf{v}_0,\mathbf{w}_0) \in \mathrm{Red}(v,w)$. We need to show that choosing a different reduced expression $\mathbf{w}_1$ for w results in a subtraction-free coordinate change $\mathbf{t}_0 \mapsto \mathbf{t}_1$ of the parameters in Theorem 3.4.15. Any two reduced expressions for w are related by a sequence of braid moves, so it suffices to assume that $\mathbf{w}_0$ and $\mathbf{w}_1$ differ in a single braid move.

The explicit formulae for the corresponding coordinate transformations can be found in the proof of [Rie08, Prop. 7.2], however, an extra step is needed to show that those formulae indeed give the correct coordinate transformations. More precisely, suppose that Φ' is a root subsystem of Φ of rank 2 (i.e., Φ' is of type $A_1 \times A_1$, A_2, B_2, or G_2), and let W' be its Weyl group. Then it was checked in the proof of [Rie08, Prop. 7.2] that for any $v' \le w' \in W'$, any $(\mathbf{v}'_0,\mathbf{w}'_0),(\mathbf{v}'_1,\mathbf{w}'_1) \in \mathrm{Red}(v',w')$, and any $\mathbf{t}'_0 \in (\mathcal{F}^*_{\mathrm{sf}})^{J^\circ_{\mathbf{v}'_0}}$, there exists $\mathbf{t}'_1 \in (\mathcal{F}^*_{\mathrm{sf}})^{J^\circ_{\mathbf{v}'_1}}$ and $x \in U$ such that $\mathbf{g}_{\mathbf{v}'_0,\mathbf{w}'_0}(\mathbf{t}'_0) = \mathbf{g}_{\mathbf{v}'_1,\mathbf{w}'_1}(\mathbf{t}'_1) \cdot x$.

Since $\mathbf{g}_{\mathbf{v}'_0,\mathbf{w}'_0}(\mathbf{t}'_0)$ and $\mathbf{g}_{\mathbf{v}'_1,\mathbf{w}'_1}(\mathbf{t}'_1)$ belong to $G^\diamond$, we must have $x \in U^\diamond$. Note that the only non-trivial cases to check are the ones where $w' = (s_i s_j)^{m_{ij}}$ is the longest element of W' and $v' \in W'$ is arbitrary. If Φ' is of type G_2 then we must have $\Phi = \Phi'$ and $w' = w = w_0$, so we are done. Using a computer algebra system [Sag16], we were able to additionally check in each of the remaining cases (i.e. Φ' being of type $A_1 \times A_1$, A_2, or B_2) that $x \in \dot{v}'^{-1} U_- \dot{v}'$.

Let us now complete the proof of Proposition 3.5.4 (as well as of [Rie08, Prop. 7.2]). Suppose that $\mathbf{w}_0$ and $\mathbf{w}_1$ differ in a braid move along a subword $g_{k+1} \cdots g_{k+m}$ of $g_1 \cdots g_n$. Here $g_{k+1} \cdots g_{k+m} = \mathbf{g}_{\mathbf{v}'_0,\mathbf{w}'_0}(\mathbf{t}'_0)$ as above. Applying a move from [Rie08], we transform $g_{k+1} \cdots g_{k+m}$ into $g'_{k+1} \cdots g'_{k+m} x$ for some $x \in U$ and $g'_{k+1} \cdots g'_{k+m} = \mathbf{g}_{\mathbf{v}'_1,\mathbf{w}'_1}(\mathbf{t}'_1)$. Thus

$$g_1 \cdots g_n = g_1 \cdots g_k \cdot g'_{k+1} \cdots g'_{k+m} \cdot x \cdot g_{k+m+1} \cdots g_n.$$

We have checked that $x \in U^\diamond$, and if $k+m < n$ then $x \in U^\diamond \cap v'^{-1} U_- v'$, where $v' \in W$ satisfies $v_{(k+m)} = v_{(k)} \cdot v'$ and $\ell(v_{(k+m)}) = \ell(v_{(k)}) + \ell(v')$. It follows by (3.4.6) that $x \in$

$U^\diamond \cap v_{(k+m)}^{-1} U_- v_{(k+m)}$, so by Lemma 3.5.3, we have

$$g_1 \cdots g_n \in g_1 \cdots g_k \cdot g'_{k+1} \cdots g'_{k+m} \cdot g_{k+m+1} \cdots g_n \cdot U^\diamond. \qquad \square$$

Definition 3.5.5. From now on we denote $R_{v,w}^{\mathrm{sf}} := G_{\mathbf{v},\mathbf{w}}^{\mathrm{sf}} B^{\mathrm{sf}} \subset G^\diamond$. By Proposition 3.5.4, the set $R_{v,w}^{\mathrm{sf}}$ does not depend on the choice of $(\mathbf{v}, \mathbf{w}) \in \mathrm{Red}(v, w)$. As we discuss in Section 3.5.4, $R_{v,w}^{\mathrm{sf}}$ is the "subtraction-free" analog of $R_{v,w}^{>0}$.

3.5.2 Collision moves

Assume $\mathbb{K} = \overline{\mathcal{F}}$. By [FZ99, Eq. (2.13)], for each $t \in \mathcal{F}_{\mathrm{sf}}^*$ there exist $t_+ \in \mathcal{F}_{\mathrm{sf}}^*$, $a_+ \in T^{\mathrm{sf}}$, and $t_- \in \mathcal{F}^\diamond$ satisfying

$$\dot{s}_i x_i(t) = a_+ x_i(t_-) y_i(t_+), \quad x_i(t) \dot{s}_i = y_i(t_+) x_i(t_-) a_+, \tag{3.5.3}$$

$$\dot{s}_i^{-1} y_i(t) = a_+ y_i(t_-) x_i(t_+), \quad y_i(t) \dot{s}_i^{-1} = x_i(t_+) y_i(t_-) a_+. \tag{3.5.4}$$

(Here, each of the four moves yields different t_+, a_+, t_-.) By [FZ99, Eq. (2.11)], for each $t, t' \in \mathcal{F}_{\mathrm{sf}}^*$ there exist $t_+, t'_+ \in \mathcal{F}_{\mathrm{sf}}^*$ and $a_+ \in T^{\mathrm{sf}}$ satisfying

$$x_i(t) y_i(t') = y_i(t'_+) x_i(t_+) a_+, \quad y_i(t') x_i(t) = x_i(t_+) y_i(t'_+) a_+. \tag{3.5.5}$$

We also have [FZ99, Eq. (2.9)]

$$x_i(t) y_j(t') = y_j(t') x_i(t), \quad \text{for } i \neq j. \tag{3.5.6}$$

As a direct consequence of (3.5.5), (3.5.6), and Lemma 3.5.2, for any $v, w \in W$ we get

$$U_{\mathrm{sf}}(v) \cdot U_{\mathrm{sf}}^-(w) \cdot T^{\mathrm{sf}} = U_{\mathrm{sf}}^-(w) \cdot U_{\mathrm{sf}}(v) \cdot T^{\mathrm{sf}}. \tag{3.5.7}$$

Lemma 3.5.6.

(i) *Let $w \in W$. Then*

$$B_-^{\mathrm{sf}} \cdot \dot{w}^{-1} \cdot U_{\mathrm{sf}}^-(w) = B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(w^{-1}) \quad \textit{and} \quad U_{\mathrm{sf}}^-(w) \cdot \dot{w}^{-1} \cdot B_-^{\mathrm{sf}} = U_{\mathrm{sf}}(w^{-1}) \cdot B_-^{\mathrm{sf}}. \tag{3.5.8}$$

(ii) *If $v, w \in W$ are such that $\ell(vw) = \ell(v) + \ell(w)$ then*

$$\dot{w}^{-1}\dot{v}^{-1} \cdot U_{\mathrm{sf}}^-(v) \subset B_-^{\mathrm{sf}} \cdot \dot{w}^{-1} \cdot U_{\mathrm{sf}}(v^{-1}). \tag{3.5.9}$$

(iii) *Let $w_1, \dots, w_k \in W$ be such that $\ell(w_1 \cdots w_k) = \ell(w_1) + \cdots + \ell(w_k)$. Then for any $h \in U_{\mathrm{sf}}^-(w_1 \cdots w_k)$ there exist $b_1 \in U_{\mathrm{sf}}(w_1^{-1}), \dots, b_k \in U_{\mathrm{sf}}(w_k^{-1})$ such that for each $1 \le i \le k$, we have*

$$\dot{w}_i^{-1} \cdots \dot{w}_1^{-1} \cdot h \in B_-^{\mathrm{sf}} \cdot b_i \cdots b_1. \tag{3.5.10}$$

(iv) *Let $v \le w \in W$. Then*

$$\dot{v}^{-1} \cdot U_{\mathrm{sf}}^-(w) \subset B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(v^{-1}). \tag{3.5.11}$$

Proof. Let us prove the following claim: if $vv_1 = w$ and $\ell(w) = \ell(v) + \ell(v_1)$ then

$$\dot{v}^{-1}U_{\mathrm{sf}}^-(w) \subset T^{\mathrm{sf}} \cdot (U_-^{\diamond} \cap \dot{v}^{-1}U\dot{v}) \cdot U_{\mathrm{sf}}^-(v_1) \cdot U_{\mathrm{sf}}(v^{-1}). \tag{3.5.12}$$

We prove this by induction on $\ell(v)$. If $\ell(v) = 0$ then $v = \mathrm{id}$ and (3.5.12) is trivial. Otherwise there exists an $i \in I$ such that $v' := s_i v < v$ and thus $w' := s_i w < w$. Let $\mathbf{y_i}(\mathbf{t}') \in U_{\mathrm{sf}}^-(w)$. Using (3.5.4), we see that for some $t_1 \in \mathcal{F}_{\mathrm{sf}}^*$, $t_+ \in \mathcal{F}_{\mathrm{sf}}^*$ and $t_- \in \mathcal{F}^{\diamond}$,

$$\dot{v}^{-1} \cdot \mathbf{y_i}(\mathbf{t}') \in \dot{v}'^{-1} \cdot \dot{s}_i^{-1} y_i(t_1) \cdot U_{\mathrm{sf}}^-(w') \subset T^{\mathrm{sf}} \dot{v}'^{-1} \cdot y_i(t_-) x_i(t_+) \cdot U_{\mathrm{sf}}^-(w').$$

By (3.5.7), $x_i(t_+) \cdot U_{\mathrm{sf}}^-(w') \subset T^{\mathrm{sf}} \cdot U_{\mathrm{sf}}^-(w') \cdot U_{\mathrm{sf}}(s_i)$. Clearly $s_i v' > v'$, so $y' := \dot{v}'^{-1} y_i(t_-) \dot{v}' \in U_-$. On the other hand, $\dot{v} y' \dot{v}^{-1} = \dot{s}_i^{-1} y_i(t_-) \dot{s}_i = x_i(-t_-) \in U$. Thus $y' \in U_- \cap \dot{v}^{-1}U\dot{v}$, and it is also clear that $y' \in G^{\diamond}$. We have shown that

$$\dot{v}^{-1} \cdot \mathbf{y_i}(\mathbf{t}') \in T^{\mathrm{sf}} \cdot y' \cdot \dot{v}'^{-1} \cdot U_{\mathrm{sf}}^-(w') \cdot U_{\mathrm{sf}}(s_i) \subset T^{\mathrm{sf}} \cdot (U_-^{\diamond} \cap \dot{v}^{-1}U\dot{v}) \cdot \dot{v}'^{-1} \cdot U_{\mathrm{sf}}^-(w') \cdot U_{\mathrm{sf}}(s_i). \tag{3.5.13}$$

We have $v'v_1 = w'$, so by induction,

$$\dot{v}'^{-1} \cdot U_{\rm sf}^{-}(w') \subset T^{\rm sf} \cdot (U_{-}^{\diamond} \cap \dot{v}'^{-1} U \dot{v}') \cdot U_{\rm sf}^{-}(v_1) \cdot U_{\rm sf}(v'^{-1}).$$

Since $U_{\rm sf}(v'^{-1}) \cdot U_{\rm sf}(s_i) = U_{\rm sf}(v^{-1})$, we have shown that

$$\dot{v}^{-1}\mathbf{y_i}(\mathbf{t}') \in T^{\rm sf} \cdot (U_{-}^{\diamond} \cap \dot{v}^{-1} U \dot{v}) \cdot (U_{-}^{\diamond} \cap \dot{v}'^{-1} U \dot{v}') \cdot U_{\rm sf}^{-}(v_1) \cdot U_{\rm sf}(v^{-1}).$$

By (3.4.6) applied to $a = s_i$, $b = v'$, $ab = v$, we get $\mathrm{Inv}(v') \subset \mathrm{Inv}(v)$, so $(U_{-}^{\diamond} \cap \dot{v}'^{-1} U \dot{v}') \subset (U_{-}^{\diamond} \cap \dot{v}^{-1} U \dot{v})$, and we have finished the proof of (3.5.12). Combining (3.5.12) with (3.4.8), we obtain (3.5.9). Next, (3.5.10) can be shown by induction: the case $k = 0$ is trivial. For $k \geq 1$, we can write $h = h_1 \cdots h_k \in U_{\rm sf}^{-}(w_1) \cdots U_{\rm sf}^{-}(w_k)$. By (3.5.9), we have

$$\dot{w}_i^{-1} \cdots \dot{w}_1^{-1} \cdot h_1 \cdots h_k \in B_{-}^{\rm sf} \cdot \dot{w}_i^{-1} \cdots \dot{w}_2^{-1} \cdot b_1' \cdot h_2 \cdots h_k$$

for some $b_1' \in U_{\rm sf}(w_1)$ that does not depend on i. Using (3.5.7), we write $b_1' \cdot h_2 \cdots h_k = h_2' \cdots h_k' \cdot b_1 \in U_{\rm sf}^{-}(w_2) \cdots U_{\rm sf}^{-}(w_k) \cdot U_{\rm sf}(w_1)$, and then proceed by induction.Let us state several further corollaries of (3.5.12):

$$\dot{w}^{-1} \cdot U_{\rm sf}^{-}(w) \subset T^{\rm sf} \cdot (U_{-}^{\diamond} \cap \dot{w}^{-1} U \dot{w}) \cdot U_{\rm sf}(w^{-1}), \tag{3.5.14}$$

$$U_{\rm sf}^{-}(w) \cdot \dot{w}^{-1} \subset U_{\rm sf}(w^{-1}) \cdot (U_{-}^{\diamond} \cap \dot{w} U \dot{w}^{-1}) \cdot T^{\rm sf}, \tag{3.5.15}$$

$$\dot{w} \cdot U_{\rm sf}(w^{-1}) \subset (U^{\diamond} \cap \dot{w} U_{-} \dot{w}^{-1}) \cdot U_{\rm sf}^{-}(w) \cdot T^{\rm sf}. \tag{3.5.16}$$

Indeed, specializing (3.5.12) to $v = w$, we obtain (3.5.14). Eq. (3.5.15) is obtained from (3.5.14) by replacing w with $z := w^{-1}$ and then applying the involution $x \mapsto x^{\iota}$ of (3.4.4), while (3.5.16) is obtained from (3.5.15) by applying the involution $x \mapsto x^{T}$ of (3.4.3).

To show (3.5.8), observe that the inclusion $B_{-}^{\rm sf} \cdot \dot{w}^{-1} \cdot U_{\rm sf}^{-}(w) \subset B_{-}^{\rm sf} \cdot U_{\rm sf}(w^{-1})$ follows from (3.5.14). To show the reverse inclusion, we use (3.5.16) to write

$$B_{-}^{\rm sf} \cdot U_{\rm sf}(w^{-1}) = B_{-}^{\rm sf} \cdot \dot{w}^{-1} \cdot \dot{w} \cdot U_{\rm sf}(w^{-1}) \subset B_{-}^{\rm sf} \cdot \dot{w}^{-1} \cdot (U^{\diamond} \cap \dot{w} U_{-} \dot{w}^{-1}) \cdot U_{\rm sf}^{-}(w).$$

Since $\dot{w}^{-1} \cdot (U^\diamond \cap \dot{w} U_- \dot{w}^{-1}) \subset U_-^\diamond \dot{w}^{-1}$, we obtain $B_-^{\mathrm{sf}} \cdot \dot{w}^{-1} \cdot U_{\mathrm{sf}}^-(w) = B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(w^{-1})$, which is the first part of (3.5.8). The second part follows by applying the involution $x \mapsto x^\iota$ of (3.4.4).

It remains to show (3.5.11). We argue by induction on $\ell(w)$, and the base case $\ell(w) = 0$ is clear. Suppose that $v \le w$, and let $w' := s_i w < w$ for some $i \in I$. If $v' := s_i v < v$ then by the same argument as in the proof of (3.5.13), we get

$$\dot{v}^{-1} \cdot U_{\mathrm{sf}}^-(w) \subset B_-^{\mathrm{sf}} \cdot \dot{v}'^{-1} \cdot U_{\mathrm{sf}}^-(w') \cdot U_{\mathrm{sf}}(s_i).$$

Since $v' \le w'$, we can apply the induction hypothesis to write $\dot{v}'^{-1} \cdot U_{\mathrm{sf}}^-(w') \subset B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(v'^{-1})$. We thus obtain

$$\dot{v}^{-1} \cdot U_{\mathrm{sf}}^-(w) \subset B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(v'^{-1}) \cdot U_{\mathrm{sf}}(s_i) = B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(v^{-1}),$$

finishing the induction step in the case $s_i v < v$. But if $s_i v > v$ then $\dot{v}^{-1} y_i(t_1) \dot{v} \in U_-^\diamond$, so in this case we have $\dot{v}^{-1} U_{\mathrm{sf}}^-(w) \subset U_-^\diamond \cdot \dot{v}^{-1} \cdot U_{\mathrm{sf}}^-(w')$, and the result follows by applying the induction hypothesis to the pair $v \le w'$. $\square$

3.5.3 Alternative parametrizations for the top cell

The following two lemmas are subtraction-free versions of [Rie06, Lemmas 4.2 and 4.3].

Lemma 3.5.7. *Let $v \in W$. Then we have*

$$R_{v,w_0}^{\mathrm{sf}} = U_{\mathrm{sf}}(v w_0) \cdot \dot{w}_0 \cdot B^{\mathrm{sf}}.$$

Proof. Recall from Definition 3.5.5 that $R_{v,w}^{\mathrm{sf}} = G_{\mathbf{v},\mathbf{w}}^{\mathrm{sf}} \cdot B^{\mathrm{sf}}$. We have $w = w_0$, so choose a reduced expression $\mathbf{w}_0$ for w_0 that ends with v. With this choice, $G_{\mathbf{v},\mathbf{w}_0}^{\mathrm{sf}} = U_{\mathrm{sf}}^-(w_0 v^{-1}) \cdot \dot{v}$. Thus we can write

$$R_{v,w_0}^{\mathrm{sf}} = G_{\mathbf{v},\mathbf{w}_0}^{\mathrm{sf}} \cdot B^{\mathrm{sf}} = U_{\mathrm{sf}}^-(w_0 v^{-1}) \cdot \dot{v} \cdot B^{\mathrm{sf}} = U_{\mathrm{sf}}^-(w_0 v^{-1}) \cdot \dot{v} \dot{w}_0^{-1} \cdot \dot{w}_0 \cdot B^{\mathrm{sf}}.$$

Let $z := w_0 v^{-1}$. Using (3.5.8) and $B_-^{\mathrm{sf}} \cdot \dot{w}_0 = \dot{w}_0 \cdot B^{\mathrm{sf}}$, we have

$$U_{\mathrm{sf}}^-(w_0 v^{-1}) \cdot \dot{v}\dot{w}_0^{-1} \cdot \dot{w}_0 \cdot B^{\mathrm{sf}} = U_{\mathrm{sf}}^-(z) \cdot \dot{z}^{-1} \cdot \dot{w}_0 \cdot B^{\mathrm{sf}} = U_{\mathrm{sf}}(z^{-1}) \cdot \dot{w}_0 \cdot B^{\mathrm{sf}}.$$

Combining the above equations, we find $R_{v,w_0}^{\mathrm{sf}} = U_{\mathrm{sf}}(z^{-1}) \cdot \dot{w}_0 \cdot B^{\mathrm{sf}}$, and it remains to note that $z^{-1} = v w_0^{-1} = v w_0$. $\square$

Lemma 3.5.8. *Let $v \leq w \in W$. Then we have*

$$U_{\mathrm{sf}}(v^{-1}) \cdot U_{\mathrm{sf}}^-(w_0 w^{-1}) \cdot R_{v,w}^{\mathrm{sf}} = R_{\mathrm{id},w_0}^{\mathrm{sf}} = U_{\mathrm{sf}}^-(w_0) \cdot B^{\mathrm{sf}}. \tag{3.5.17}$$

Proof. It follows from the definition of $G_{\mathbf{v},\mathbf{w}}^{\mathrm{sf}}$ that if $w'w$ is length-additive then $U_{\mathrm{sf}}^-(w')R_{v,w}^{\mathrm{sf}} = R_{v,w'w}^{\mathrm{sf}}$. Applying this to $w' = w_0 w^{-1}$, we get $U_{\mathrm{sf}}^-(w_0 w^{-1}) \cdot R_{v,w}^{\mathrm{sf}} = R_{v,w_0}^{\mathrm{sf}}$. By Lemma 3.5.7, we have $R_{v,w_0}^{\mathrm{sf}} \cdot B^{\mathrm{sf}} = U_{\mathrm{sf}}(vw_0) \cdot \dot{w}_0 \cdot B^{\mathrm{sf}}$. Thus $U_{\mathrm{sf}}(v^{-1}) \cdot U_{\mathrm{sf}}(vw_0) \cdot \dot{w}_0 \cdot B^{\mathrm{sf}} = U_{\mathrm{sf}}(w_0) \cdot \dot{w}_0 \cdot B^{\mathrm{sf}}$, so applying Lemma 3.5.7 again, we find $U_{\mathrm{sf}}(w_0) \cdot \dot{w}_0 \cdot B^{\mathrm{sf}} = R_{\mathrm{id},w_0}^{\mathrm{sf}} \cdot B^{\mathrm{sf}}$. The result follows since $R_{\mathrm{id},w_0}^{\mathrm{sf}} = U_{\mathrm{sf}}^-(w_0) \cdot B^{\mathrm{sf}}$. $\square$

3.5.4 Evaluation

We explain the relationship between $R_{v,w}^{\mathrm{sf}}$ and $R_{v,w}^{>0}$. Given $\mathbf{t}' \in \mathbb{R}_{>0}^{|\mathbf{t}|}$, we denote by $\mathrm{eval}_{\mathbf{t}'} : \mathcal{F}_{\mathrm{sf}} \to \mathbb{R}_{>0}$ the evaluation homomorphism (of semifields) sending $f(\mathbf{t}) \to f(\mathbf{t}')$. It extends to a well defined group homomorphism $\mathrm{eval}_{\mathbf{t}'} : G^\diamond \to G(\mathbb{R})$, and it follows from Theorem 3.4.15 that $\{\mathrm{eval}_{\mathbf{t}'}(g)B \mid g \in R_{v,w}^{\mathrm{sf}}\} = R_{v,w}^{>0}$ as subsets of $(G/B)_{\mathbb{R}}$. It is clear that the following diagram is commutative.

$$\begin{tikzcd}
\mathcal{F} & G^\diamond \arrow[l, "\Delta_i^\mp"'] \arrow[r, "\Delta_i^\pm"] \arrow[d, "\mathrm{eval}_{\mathbf{t}'}"] & \mathcal{F} \arrow[d, dashed, "\mathrm{eval}_{\mathbf{t}'}"] \\
\mathbb{R} \arrow[from=u, dashed, "\mathrm{eval}_{\mathbf{t}'}"'] & G(\mathbb{R}) \arrow[l, "\Delta_i^\mp"] \arrow[r, "\Delta_i^\pm"'] & \mathbb{R}
\end{tikzcd} \tag{3.5.18}$$

Here solid arrows denote regular maps, and dashed arrows denote maps defined on a subset $\mathcal{F}' \subset \mathcal{F}$ given by $\mathcal{F}' := \{R(\mathbf{t})/Q(\mathbf{t}) \mid R(\mathbf{t}), Q(\mathbf{t}) \in \mathbb{R}[\mathbf{t}],\ Q(\mathbf{t}') \neq 0\}$. Since the diagram (3.5.18) is commutative, it follows that the images $\Delta_i^\mp(G^\diamond)$ and $\Delta_i^\pm(G^\diamond)$ belong to $\mathcal{F}'$.

Let $\mathbf{t} = (\mathbf{t}', \mathbf{t}'')$. Observe that any $f(\mathbf{t}', \mathbf{t}'') \in \mathcal{F}_{\mathrm{sf}}^*$ gives rise to a continuous function

$\mathbb{R}_{>0}^{|\mathbf{t}'|} \times \mathbb{R}_{>0}^{|\mathbf{t}''|} \to \mathbb{R}_{>0}$.

Lemma 3.5.9. *Suppose that $f(\mathbf{t}', \mathbf{t}'') \in \mathcal{F}_{\mathrm{sf}}^*$ is such that the corresponding function $\mathbb{R}_{>0}^{|\mathbf{t}'|} \times \mathbb{R}_{>0}^{|\mathbf{t}''|} \to \mathbb{R}_{>0}$ extends to a continuous function $\mathbb{R}_{>0}^{|\mathbf{t}'|} \times \mathbb{R}_{\geq 0}^{|\mathbf{t}''|} \to \mathbb{R}_{\geq 0}$. Then $\lim_{\mathbf{t}'' \to 0} f(\mathbf{t}', \mathbf{t}'')$ can be represented (as a function $\mathbb{R}_{>0}^{|\mathbf{t}'|} \to \mathbb{R}_{\geq 0}$) by a subtraction-free rational expression in $\mathbf{t}'$.*

Proof. By induction, it is enough to prove this when $|\mathbf{t}''| = 1$, where $\mathbf{t}'' = t''$ is a single variable. In this case, $f(\mathbf{t}', t'') = R(\mathbf{t}', t'')/Q(\mathbf{t}', t'')$ where R and Q have positive coefficients. Let us consider R and Q as polynomials in t'' only. After dividing R and Q by $(t'')^k$ for some k, we may assume that one of them is not divisible by t''. Then Q cannot be divisible by t'', otherwise f would not give rise to a continuous function $\mathbb{R}_{>0}^{|\mathbf{t}'|} \times \mathbb{R}_{\geq 0}^{|\mathbf{t}''|} \to \mathbb{R}_{\geq 0}$. We can write $Q(\mathbf{t}', t'') = Q_1(\mathbf{t}', t'')t'' + Q_2(\mathbf{t}')$ and $R(\mathbf{t}', t'') = R_1(\mathbf{t}', t'')t' + R_2(\mathbf{t}')$, where R_1, R_2, Q_1, Q_2 are polynomials with nonnegative coefficients and $Q_2(\mathbf{t}') \neq 0$. Thus $\lim_{t'' \to 0} f(\mathbf{t}', t'')$ can be represented by $R_2(\mathbf{t}')/Q_2(\mathbf{t}')$, which is a subtraction free rational expression in $\mathbf{t}'$. $\square$

Lemma 3.5.10. *(Assume $\mathbb{K} = \mathbb{C}$.) Suppose that $a \leq b \leq c \in W$. Then we cannot have $\Delta^{\mp}(\dot{b}^{-1}x) = 0$ for all $x \in G(\mathbb{R})$ such that $xB \in R_{a,b}^{>0}$.*

Proof. Suppose that $\Delta^{\mp}(\dot{b}^{-1}x) = 0$ for all $x \in G(\mathbb{R})$ such that $xB \in R_{a,b}^{>0}$. Consider the map $\Delta_i^{\mathrm{flag}} : G/B \to \mathbb{P}^{\binom{2n}{n-1}-1}{}^{W\omega_i}$ from Lemma 3.4.21(iv). We get that the $b\omega_i$-th coordinate of Δ_i^{flag} is identically zero on $R_{a,c}^{>0}$. Therefore it must be zero on the Zariski closure of $R_{a,c}^{>0}$ inside G/B. It is well known that $R_{a,c}^{>0}$ is Zariski dense in $\mathring{R}_{a,c}$, so the closure of $R_{a,c}^{>0}$ is $R_{a,c}$. By (3.4.14), $R_{a,c}$ contains $\dot{b}B = \mathring{R}_{b,b}$, thus $\Delta_i^{\mp}(\dot{b}^{-1}\dot{b})$ must be zero. We get a contradiction since by definition $\Delta_i^{\mp}(\dot{b}^{-1}\dot{b}) = 1$. $\square$

3.5.5 Applications to the flag variety

We use the machinery developed in the previous sections to obtain some natural statements about $(G/B)_{\geq 0}$.

Lemma 3.5.11. *(Assume $\mathbb{K} = \overline{\mathcal{F}}$.) Suppose that $a \leq c \in W$ and $b \in W$. Then for any $x \in R_{a,c}^{\mathrm{sf}}$ and $i \in I$,*

$$\Delta_i^{\mp}(\dot{b}^{-1}x) \in \mathcal{F}_{\mathrm{sf}}. \tag{3.5.19}$$

Moreover, if $a \le b \le c$ then

$$\Delta_i^{\mp}(\dot{b}^{-1}x) \in \mathcal{F}_{\mathrm{sf}}^*, \quad \textit{and} \quad x \in \dot{b}B_-B. \tag{3.5.20}$$

Proof. Let $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ for $|\mathbf{t}_1| = \ell(a)$, $|\mathbf{t}_2| = \ell(w_0) - \ell(c)$, $|\mathbf{t}_3| = \ell(c) - \ell(a)$. Choose reduced words $\mathbf{i}$ for a^{-1} and $\mathbf{j}$ for w_0c^{-1}, and let $(\mathbf{a}, \mathbf{c}) \in \mathrm{Red}(a, c)$. Suppose that $x \in \mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}_3)B^{\mathrm{sf}}$ and let

$$g := \mathbf{x}_{\mathbf{i}}(\mathbf{t}_1) \cdot \mathbf{y}_{\mathbf{j}}(\mathbf{t}_2) \cdot \mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}_3) \in U_{\mathrm{sf}}(a^{-1}) \cdot U_{\mathrm{sf}}^-(w_0c^{-1}) \cdot R_{a,c}^{\mathrm{sf}}.$$

By Lemma 3.5.8, $g \in U_{\mathrm{sf}}^-(w_0) \cdot B^{\mathrm{sf}} = U_{\mathrm{sf}}^-(b) \cdot U_{\mathrm{sf}}^-(b^{-1}w_0) \cdot B^{\mathrm{sf}}$. By (3.5.8), we have $\dot{b}^{-1} \cdot U_{\mathrm{sf}}^-(b) \subset B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(b^{-1})$. Therefore

$$\dot{b}^{-1}g \in B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(b^{-1}) \cdot U_{\mathrm{sf}}^-(b^{-1}w_0) \cdot B^{\mathrm{sf}}.$$

By (3.5.7), we get $\dot{b}^{-1}g \in B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}^-(b^{-1}w_0) \cdot U_{\mathrm{sf}}(b^{-1}) \cdot B^{\mathrm{sf}} = B_-^{\mathrm{sf}} \cdot B^{\mathrm{sf}}$, and by definition, $\Delta_i^{\mp}(y) \in \mathcal{F}_{\mathrm{sf}}^*$ for any $y \in B_-^{\mathrm{sf}} \cdot B^{\mathrm{sf}}$. Since $\Delta_i^{\mp}$ is a regular function on G by Lemma 3.4.21(ii), the function $f(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) := \Delta_i^{\mp}(\dot{b}^{-1}g) \in \mathcal{F}_{\mathrm{sf}}^*$ extends to a continuous function on $\mathbb{R}_{\ge 0}^{|\mathbf{t}_1|} \times \mathbb{R}_{\ge 0}^{|\mathbf{t}_2|} \times \mathbb{R}_{>0}^{|\mathbf{t}_3|}$. Therefore by Lemma 3.5.9, $\lim_{\mathbf{t}_1,\mathbf{t}_2 \to 0} f(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ is a subtraction-free rational expression in $\mathbf{t}_3$. Since $\lim_{\mathbf{t}_1,\mathbf{t}_2 \to 0} g = \mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}_3)$, we get that $\Delta_i^{\mp}(\dot{b}^{-1}\mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}_3)) \in \mathcal{F}_{\mathrm{sf}}$. Since $x \in \mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}_3)B^{\mathrm{sf}}$, (3.5.19) follows.

Suppose now that $a \le b \le c$. We would like to show (3.5.20), thus let us assume that for some $i \in I$ and for $x \in R_{a,c}^{\mathrm{sf}}$, we have $\Delta_i^{\mp}(\dot{b}^{-1}x) = 0$. Let $\mathbf{t}' \in (\mathcal{F}_{\mathrm{sf}}^*)^{|\mathbf{t}|}$ and $(\mathbf{a}, \mathbf{c}) \in \mathrm{Red}(a, c)$ be such that $x \in \mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}')B^{\mathrm{sf}}$, and let $y(\mathbf{t}) := \mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t})$. Then we have $\Delta_i^{\mp}(\dot{b}^{-1}y(\mathbf{t})) \in \mathcal{F}_{\mathrm{sf}}$ by (3.5.19). If $\Delta_i^{\mp}(\dot{b}^{-1}y(\mathbf{t}))$ was a nonzero rational function in $\mathbf{t}$ then clearly substituting $\mathbf{t} \mapsto \mathbf{t}'$ for $\mathbf{t}' \in (\mathcal{F}_{\mathrm{sf}}^*)^{|\mathbf{t}|}$ would also produce a nonzero rational function. Since substituting $\mathbf{t} \mapsto \mathbf{t}'$ yields $\Delta_i^{\mp}(\dot{b}^{-1}x) = 0$, we must have $\Delta_i^{\mp}(\dot{b}^{-1}y(\mathbf{t})) = 0$. Therefore $\Delta_i^{\mp}(\dot{b}^{-1}x') = 0$ for all $x' \in R_{a,c}^{\mathrm{sf}}$.

Let now $\mathbf{t}' \in \mathbb{R}_{>0}^{|\mathbf{t}|}$. Recall from Section 3.5.4 that the image of $R_{a,c}^{\mathrm{sf}}$ in $(G/B)_{\mathbb{R}}$ under the map $\mathrm{eval}_{\mathbf{t}'}$ equals $R_{a,c}^{>0}$, thus by (3.5.18), $\Delta_i^{\mp}(\dot{b}^{-1}x') = 0$ for all $x' \in G(\mathbb{R})$ such that $x'B \in R_{a,c}^{>0}$. This contradicts Lemma 3.5.10, hence $\Delta_i^{\mp}(\dot{b}^{-1}x) \in \mathcal{F}_{\mathrm{sf}}^*$, and therefore $x \in \dot{b}B_-B$ follows from Lemma 3.4.21(iii), finishing the proof of (3.5.20). □

Corollary 3.5.12. *(Assume $\mathbb{K} = \mathbb{C}$.) Suppose that $a \leq c \in W$ and $b \in W$. Then for any $(\mathbf{a}, \mathbf{c}) \in \mathrm{Red}(a, c)$ and $\mathbf{t}' \in \mathbb{R}_{>0}^{J_{\mathbf{a}}^{\circ}}$, we have*

$$\Delta_i^{\mp}(\dot{b}^{-1}\mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}')) \geq 0. \tag{3.5.21}$$

Moreover, if $a \leq b \leq c$ then

$$\Delta_i^{\mp}(\dot{b}^{-1}\mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}')) > 0, \quad \text{and} \quad R_{a,c}^{>0} \subset \dot{b}B_{-}B/B. \tag{3.5.22}$$

Proof. By (3.5.19), we know that $\Delta_i^{\mp}(\dot{b}^{-1}\mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t})) \in \mathcal{F}_{\mathrm{sf}}$ for all $i \in I$. Evaluating at $\mathbf{t} = \mathbf{t}'$ (cf. Section 3.5.4), we find that $\Delta_i^{\mp}(\dot{b}^{-1}\mathbf{g}_{\mathbf{a},\mathbf{c}}(\mathbf{t}')) \geq 0$ for all $i \in I$, showing (3.5.21). Similarly, (3.5.22) follows from (3.5.20). □

Proposition 3.5.13. *(Assume $\mathbb{K} = \overline{\mathcal{F}}$.) For all $v, w, v', w' \in W$ and $x \in U_{\mathrm{sf}}(v') \cdot T^{\mathrm{sf}} \cdot U_{\mathrm{sf}}^{-}(w')$, we have $\Delta_i^{\pm}(\dot{v}x\dot{w}^{-1}) \in \mathcal{F}_{\mathrm{sf}}$.*

Proof. Let $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_1', \mathbf{t}_2')$ with $|\mathbf{t}_1| = \ell(v')$, $|\mathbf{t}_2| = \ell(w')$, $|\mathbf{t}_1'| = \ell(w_0) - \ell(v')$, and $|\mathbf{t}_2'| = \ell(w_0) - \ell(w')$. Let $\mathbf{t}_v := (\mathbf{t}_1', \mathbf{t}_1)$ and $\mathbf{t}_w := (\mathbf{t}_2, \mathbf{t}_2')$. Choose reduced words $\mathbf{i}, \mathbf{j}$ for w_0 such that $\mathbf{i}$ ends with a reduced word for v' and $\mathbf{j}$ starts with a reduced word for w'. Set $g = g(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_v, \mathbf{t}_w) := \mathbf{x}_{\mathbf{i}}(\mathbf{t}_v) \cdot a \cdot \mathbf{y}_{\mathbf{j}}(\mathbf{t}_w)$ for some arbitrary element $a \in T^{\mathrm{sf}}$. We get

$$\dot{v}g\dot{w}^{-1} \in \dot{v} \cdot U_{\mathrm{sf}}(w_0) \cdot T^{\mathrm{sf}} \cdot U_{\mathrm{sf}}^{-}(w_0) \cdot \dot{w}^{-1} \subset \dot{v} \cdot U_{\mathrm{sf}}(v^{-1}) \cdot U_{\mathrm{sf}}(vw_0) \cdot T^{\mathrm{sf}} \cdot U_{\mathrm{sf}}^{-}(w_0w^{-1}) \cdot U_{\mathrm{sf}}^{-}(w) \cdot \dot{w}^{-1}.$$

By (3.5.16), (3.5.7), and (3.5.8), we get $\dot{v}g\dot{w}^{-1} \in B^{\mathrm{sf}} \cdot U_{\mathrm{sf}}^{-}(v) \cdot U_{\mathrm{sf}}(w^{-1}) \cdot B_{-}^{\mathrm{sf}}$. By (3.5.7), we can permute $U_{\mathrm{sf}}^{-}(v)$ and $U_{\mathrm{sf}}(w^{-1})$, showing $\dot{v}g\dot{w}^{-1} \in B^{\mathrm{sf}} \cdot B_{-}^{\mathrm{sf}}$. Thus $\Delta_i^{\pm}(\dot{v}g\dot{w}^{-1}) \in \mathcal{F}_{\mathrm{sf}}^{*}$. It gives rise to a continuous function on $\mathbb{R}_{>0}^{|\mathbf{t}_1|} \times \mathbb{R}_{>0}^{|\mathbf{t}_2|} \times \mathbb{R}_{\geq 0}^{|\mathbf{t}_1'|} \times \mathbb{R}_{\geq 0}^{|\mathbf{t}_2'|}$, so sending $\mathbf{t}_1', \mathbf{t}_2' \to 0$ via Lemma 3.5.9 and varying $\mathbf{t}_1$, $\mathbf{t}_2$, and a, we get $\Delta_i^{\pm}(\dot{v}x\dot{w}^{-1}) \in \mathcal{F}_{\mathrm{sf}}$ for all $x \in U_{\mathrm{sf}}(v') \cdot T^{\mathrm{sf}} \cdot U_{\mathrm{sf}}^{-}(w')$. □

3.6 Bruhat projections and total positivity

In this section, we prove a technical result (Theorem 3.6.4) which will be later used to finish the proof of Theorem 3.2.5. Assume $\mathbb{K}$ is algebraically closed and fix $u \in W^J$.

3.6.1 The map $\zeta_{u,v}^{(J)}$

Retain the notation from Definition 3.4.23. Given $v \in W$ and $u \in W^J$, let us introduce a subset

$$G_{u,v}^{(J)} := \{x \in \dot{u}G_0^{(J)} \mid \kappa_x x \in \dot{v}G_0^{(J)}\} \subset G. \tag{3.6.1}$$

Note that if $x \in G_{u,v}^{(J)}$ then $xP \subset G_{u,v}^{(J)}$, see Lemma 3.6.2(iii) below.

Definition 3.6.1. Define a map $\eta : G_{u,v}^{(J)} \to L_J$ sending $x \in G_{u,v}^{(J)}$ to $\eta(x) := [\dot{v}^{-1}\kappa_x x]_J$. Also define a map $\pi_{\dot{u}P_-} : \dot{u}G_0^{(J)} \to \dot{u}P_-$ sending $x \in \dot{u}G_0^{(J)}$ to the unique element $\pi_{\dot{u}P_-}(x) \in \dot{u}P_- \cap xU^{(J)}$. Explicitly (cf. Lemma 3.4.22(iii)), we put

$$\pi_{\dot{u}P_-}(x) := \dot{u}[\dot{u}^{-1}x]_-^{(J)}[\dot{u}^{-1}x]_J = x \cdot ([\dot{u}^{-1}x]_+^{(J)})^{-1}. \tag{3.6.2}$$

Finally, define $\zeta_{u,v}^{(J)} : G_{u,v}^{(J)} \to G$ by $\zeta_{u,v}^{(J)}(x) := \pi_{\dot{u}P_-}(x) \cdot \eta(x)^{-1}$.

Lemma 3.6.2.

(i) *The maps κ and $\pi_{\dot{u}P_-}$ are regular on $\dot{u}G_0^{(J)}$.*

(ii) *The maps η and $\zeta_{u,v}^{(J)}$ are regular on $G_{u,v}^{(J)} \subset \dot{u}G_0^{(J)}$.*

(iii) *If $x \in \dot{u}G_0^{(J)}$ and $x' \in xP$ then $\kappa_{x'} = \kappa_x$.*

(iv) *If $x \in G_{u,v}^{(J)}$ and $x' \in xP$ then $\zeta_{u,v}^{(J)}(x) = \zeta_{u,v}^{(J)}(x')$.*

Proof. Parts (i) and (ii) are clear since each map is a composition of regular maps. Part (iii) follows from Definition 3.4.23, since by construction the map κ starts by applying the isomorphism in (3.4.31), which gives a regular map $C_u^{(J)} \to \dot{u}U_-^{(J)}\dot{u}^{-1}$. To prove (iv), suppose that $x \in G_{u,v}^{(J)}$ and $x' \in xP$ is given by $x' = xp$ for $p \in P$. Then $\pi_{\dot{u}P_-}(x') = \pi_{\dot{u}P_-}(x)[p]_J$ by Lemma 3.4.22(iii). By (iii), $\kappa_{x'} = \kappa_x$, and

$$\eta(x') = [\dot{v}^{-1}\kappa_{x'}x']_J = [\dot{v}^{-1}\kappa_x x]_J[p]_J = \eta(x)[p]_J, \qquad \text{thus}$$
$$\zeta_{u,v}^{(J)}(x') = \pi_{\dot{u}P_-}(x') \cdot \eta(x')^{-1} = \pi_{\dot{u}P_-}(x)[p]_J \cdot [p]_J^{-1}\eta(x)^{-1} = \zeta_{u,v}^{(J)}(x). \qquad \square$$

Lemma 3.6.3. *Let $x \in \dot{u}P_-$.*

(i) *We have $\pi_{\dot{u}P_-}(x) = x$.*

(ii) *If* $x \in G^{(J)}_{u,v}$ *then* $\zeta^{(J)}_{u,v}(x) = x\eta(x)^{-1}$.

Proof. Both parts are clear from Definition 3.6.1. $\square$

The ultimate goal of this section is to prove the following result.

Theorem 3.6.4. *(Assume $\mathbb{K} = \mathbb{C}$.) Let $(u,u) \preceq (v,w) \preceq (v',w') \in Q_J$ and $x \in G$ be such that $xB \in R^{>0}_{v',w'}$. Then $x \in G^{(J)}_{u,v}$ and $\zeta^{(J)}_{u,v}(x) \in BB_-\dot{w}$.*

3.6.2 Properties of κ

We further investigate the element $\kappa_x x$. Denote $\tilde{u} := uw_J \in W^J_{\max}$.

Lemma 3.6.5. *The groups $U^{(J)}$, $U^{(J)}_1$, $U^{(J)}_2$ from Definition 3.4.23 satisfy*

$$\dot{u}U^{(J)}_-\dot{u}^{-1} = \dot{\tilde{u}}U^{(J)}_-\dot{\tilde{u}}^{-1}, \tag{3.6.3}$$

$$U^{(J)}_1 = \dot{u}U^{(J)}_-\dot{u}^{-1} \cap U = \dot{u}U_-\dot{u}^{-1} \cap U, \tag{3.6.4}$$

$$U^{(J)}_2 = \dot{u}U^{(J)}_-\dot{u}^{-1} \cap U_- = \dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} \cap U_-. \tag{3.6.5}$$

Proof. By Lemma 3.4.22(ii), we see that $\dot{w}_J U^{(J)}_- \dot{w}_J{}^{-1} = U^{(J)}_-$, which shows (3.6.3). For (3.6.4), $U^{(J)}_1 = \dot{u}U^{(J)}_-\dot{u}^{-1} \cap U$ is just the definition. By Lemma 3.4.5, we have $\dot{u}U^-_J\dot{u}^{-1} \subset U_-$, so (3.6.4) follows from (3.4.5).

For (3.6.5), observe that $w_J\Phi^+_J = \Phi^-_J$, so $\tilde{u}\Phi^+_J \subset \Phi^-$ by (3.4.6). We thus have $\dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} = (\dot{\tilde{u}}U^-_J\dot{\tilde{u}}^{-1})\cdot(\dot{\tilde{u}}U^{(J)}_-\dot{\tilde{u}}^{-1})$ where $(\dot{\tilde{u}}U^-_J\dot{\tilde{u}}^{-1}) \subset U$, and hence $\dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} \cap U_- = \dot{\tilde{u}}U^{(J)}_-\dot{\tilde{u}}^{-1} \cap U_- = U^{(J)}_2$ by the definition of $U^{(J)}_2$. $\square$

Lemma 3.6.6. *For $x \in \dot{u}G^{(J)}_0$, there exists a unique element $h \in U^{(J)}_2$ such that $hx \in U^{(J)}_1\dot{u}P$, and we have $h = \kappa_x$.*

Proof. Let $g^{(J)} \in U^{(J)}$ and $p \in P$ be such that $g^{(J)}\dot{u} = xp$. We first show that such an $h \in U^{(J)}_2$ exists. By Definition 3.4.23, κ_x is an element of $U^{(J)}_2$ such that $\kappa_x g^{(J)} \in U^{(J)}_1$. In particular, $\kappa_x x = \kappa_x g^{(J)}\dot{u}p^{-1} \in U^{(J)}_1\dot{u}P$, which shows the existence. To show the uniqueness, observe that the action of $\dot{u}U^{(J)}_-\dot{u}^{-1}$ on $\dot{u}G^{(J)}_0/P \subset G/P$ is free by (3.4.31), and in particular the action of $U^{(J)}_2$ is also free. $\square$

Lemma 3.6.7. *If $x \in \dot{u}G_0^{(J)} \cap B\dot{u}\dot{r}B$ for some $r \in W_J$ then $\kappa_x = 1$.*

Proof. By Lemma 3.6.6, it suffices to show that $B\dot{u}\dot{r}B \subset U_1^{(J)}uP$. Write

$$B\dot{u}\dot{r}B \subset B\dot{u}P \subset (B\dot{u}B) \cdot P.$$

By (3.4.34), $B\dot{u}B \subset (\dot{u}U_- \cap U\dot{u}) \cdot B$, therefore we find

$$B\dot{u}\dot{r}B \subset (\dot{u}U_- \cap U\dot{u}) \cdot P = (\dot{u}U_-\dot{u}^{-1} \cap U)\dot{u}P = U_1^{(J)}\dot{u}P,$$

where the last equality follows from (3.6.4). □

Lemma 3.6.8. *Let $a \in T$.*

(i) *The subgroups $\dot{u}U^{(J)}\dot{u}^{-1}$, $U_1^{(J)}$, and $U_2^{(J)}$ are preserved under conjugation by a.*

(ii) *If $x \in \dot{u}G_0^{(J)}$ then $ax \in \dot{u}G_0^{(J)}$ and $\kappa_{ax}ax = a\kappa_x x$.*

(iii) *(Assume $\mathbb{K} = \mathbb{C}$.) For each $w \in W$, there exists $\rho_w^\vee \in Y(T)$ such that for all $x \in \dot{w}B_-B$, $\lim_{t\to 0} \rho_w^\vee(t) \cdot xB = \dot{w}B$ in G/B. If $w \in W^J$ then for all $x \in \dot{w}G_0^{(J)}$, $\lim_{t\to 0} \rho_w^\vee(t) \cdot xP = \dot{w}P$ in G/P.*

Proof. Since $\dot{u} \in N_G(T)$, there exists $b \in T$ such that $a\dot{u} = \dot{u}b$. Thus $a\dot{u}U^{(J)}\dot{u}^{-1}a^{-1} = \dot{u}bU^{(J)}b^{-1}\dot{u}^{-1} = \dot{u}U^{(J)}\dot{u}^{-1}$, which shows (i), and (ii) is a simple consequence of (i). To show (iii), assume $\mathbb{K} = \mathbb{C}$ and choose $\rho^\vee \in Y(T)$ such that $\langle \alpha_i, \rho^\vee \rangle < 0$ for all $i \in I$. Then $\lim_{t\to 0} \rho^\vee(t)y\rho^\vee(t)^{-1} = 1$ for all $y \in U_-$, in particular, for all $y \in U_-^{(J)}$. Set $\rho_w^\vee := w^{-1}\rho^\vee$, thus for $t \in \mathbb{C}^*$, $\rho_w^\vee(t) = \dot{w}\rho^\vee(t)\dot{w}^{-1}$, see (3.4.2). Every $x \in \dot{w}B_-B$ belongs to $\dot{w}yB$ for some $y \in U_-$, so $\rho_w^\vee(t) \cdot x \cdot B = \dot{w}\rho^\vee(t)y\rho^\vee(t)^{-1} \cdot B \to \dot{w}B$ as $t \to 0$. Similarly, if $w \in W^J$ then every $x \in \dot{w}G_0^{(J)}$ belongs to $\dot{w}yP$ for some $y \in U_-^{(J)}$ by (3.4.31), so $\rho_w^\vee(t) \cdot xP \to \dot{w}P$ as $t \to 0$. □

Lemma 3.6.9. *Suppose that $v'' \le ur \le w''$ for some $v'', w'' \in W$ and $r \in W_J$, and let $x \in G$.*

(i) *(Assume $\mathbb{K} = \overline{\mathcal{F}}$.) If $x \in R_{v'',w''}^{\mathrm{sf}}$ then $x \in \dot{u}G_0^{(J)}$.*

(ii) *(Assume $\mathbb{K} = \mathbb{C}$.) If $xB \in R_{v'',w''}^{>0}$ then $x \in \dot{u}G_0^{(J)}$ and $\kappa_x xB \in R_{v'',ur_w}^{>0}$ for some $r_w \in W_J$ such that $r_w \ge r$.*

Proof. When $\mathbb{K} = \overline{\mathcal{F}}$, (3.5.20) shows that $R^{\mathrm{sf}}_{v'',w''} \subset \dot{u}\dot{r}B_-B \subset \dot{u}P_-B$, and by Lemma 3.4.22(i), $P_-B = G_0^{(J)}$, which shows (i). Similarly (for $\mathbb{K} = \mathbb{C}$), by Corollary 3.5.12, we have $x \in \dot{u}\dot{r}B_-B$ for any $x \in R^{>0}_{v'',w''}$, so $R^{>0}_{v'',w''} \subset \dot{u}G_0^{(J)}$.

Assume now that $\mathbb{K} = \mathbb{C}$ and $xB \in R^{>0}_{v'',w''}$. Let $p \in P$ and $g^{(J)} \in \dot{u}U_-^{(J)}\dot{u}^{-1}$ be such that $xp = g^{(J)}\dot{u}$. Then $\kappa_x xp = g_1^{(J)}\dot{u}$ for $g_1^{(J)} \in U_1^{(J)}$. By (3.6.4), $U_1^{(J)}\dot{u} \subset U\dot{u} \subset B\dot{u}B$. By Lemma 3.4.22(i), we have $p^{-1} \in B\dot{r}_wB$ for some $r_w \in W_J$. We get $\kappa_x x = g_1^{(J)}\dot{u} \cdot p^{-1} \in B\dot{u}B \cdot B\dot{r}_wB \subset B\dot{u}\dot{r}_wB$ by (3.4.18). On the other hand, $\kappa_x \in U_-$ and $x \in B_-v''B$, so $\kappa_x x \in B_-v''B$. Therefore $\kappa_x xB \in \mathring{R}_{v'',ur_w}$.

We now show $r_w \geq r$. By (3.5.22), $x \in \dot{u}\dot{r}B_-B$, so by Lemma 3.6.8(iii), we have $\rho^\vee_{ur}(t) \cdot xB \to \dot{u}\dot{r}B$ as $t \to 0$ in G/B. Since $\dot{u}\dot{r} \in \dot{u}G_0^{(J)}$, κ is regular at $\dot{u}\dot{r}B$, and by Lemma 3.6.7, we have $\kappa_{\dot{u}\dot{r}} = 1$. Thus $\kappa_{\rho^\vee_{ur}(t)x}\rho^\vee_{ur}(t)xB \to \dot{u}\dot{r}B$ as $t \to 0$. By Lemma 3.6.8(ii), $\kappa_{\rho^\vee_{ur}(t)x}\rho^\vee_{ur}(t)xB = \rho^\vee_{ur}(t) \cdot \kappa_x xB$, which belongs to $\mathring{R}_{v'',ur_w}$ for all $t \in \mathbb{C}^*$. We see that the closure of $\mathring{R}_{v'',ur_w}$ contains $\dot{u}\dot{r}B$, thus $v'' \leq ur \leq ur_w$ by (3.4.14), so $r \leq r_w$ by Lemma 3.4.4(ii).

Finally, we show $\kappa_x xB \in (G/B)_{\geq 0}$. First, clearly the map κ is defined over $\mathbb{R}$, thus $\kappa_x xB \in (G/B)_{\mathbb{R}}$. Consider the subset $R^{>0}_{v'',[\tilde{u},w_0]} := \bigsqcup_{w'' \geq \tilde{u}} R^{>0}_{v'',w''} \subset (G/B)_{\geq 0}$. It contains $R^{>0}_{v'',w_0}$ as an open dense subset, and therefore $R^{>0}_{v'',[\tilde{u},w_0]}$ is connected. We have already shown that for any $x' \in R^{>0}_{v'',[\tilde{u},w_0]}$, $\kappa_{x'}x'B \in \mathring{R}^{\mathbb{R}}_{v'',\tilde{u}}$ (because we have $r_w \geq r = w_J$). Thus the image of the set $R^{>0}_{v'',[\tilde{u},w_0]}$ under the map $x' \mapsto \kappa_{x'}x'$ must lie inside a single connected component of $\mathring{R}^{\mathbb{R}}_{v'',\tilde{u}}$. However, if $x' \in R^{>0}_{v'',\tilde{u}} \subset R^{>0}_{v'',[\tilde{u},w_0]}$ then $\kappa_{x'} = 1$ by Lemma 3.6.7, so in this case $\kappa_{x'}x' \in R^{>0}_{v'',\tilde{u}}$. We conclude that the image of $R^{>0}_{v'',[\tilde{u},w_0]}$ is contained inside $R^{>0}_{v'',\tilde{u}} \subset (G/B)_{\geq 0}$. It follows by continuity that for arbitrary $v'' \leq ur \leq w''$ and $x \in R^{>0}_{v'',w''}$, we have $\kappa_x xB \in (G/B)_{\geq 0}$. $\square$

We will use the following consequence of Lemma 3.6.9(ii) in Section 3.9.11.

Corollary 3.6.10. *(Assume $\mathbb{K} = \mathbb{C}$.) In the notation of Lemma* 3.6.9(ii)*, we have $\kappa_x xP \in \Pi^{>0}_{\bar{v}'',u}$ for $\bar{v}'' := v'' \triangleleft r_w^{-1}$.*

Proof. Indeed, Lemma 3.6.9(ii) says that $\kappa_x xB \in R^{>0}_{v'',ur_w}$, so applying Corollary 3.4.18, we find that $\pi_J(\kappa_x xB) = \kappa_x xP \in \Pi^{>0}_{\bar{v}'',u}$. $\square$

3.6.3 Proof via subtraction-free parametrizations

In this section, we fix some set $\mathbf{t}$ of variables and assume $\mathbb{K} = \overline{\mathcal{F}}$. Also fix $u \in W^J$ and recall that $\tilde{u} = uw_J \in W^J_{\max}$.

By Definition 3.4.23, the map κ is defined on $\dot{u}G_0^{(J)}$. By Lemma 3.6.9(i), we have $R^{\mathrm{sf}}_{v'',w''} \subset \dot{u}G_0^{(J)}$ whenever $v'' \leq ur \leq w''$ for some $r \in W_J$. In particular, κ is defined on $U^-_{\mathrm{sf}}(w'') \subset R^{\mathrm{sf}}_{\mathrm{id},w''}$ for all $w'' \geq \tilde{u}$.

Proposition 3.6.11. *Let $q \in W$ be such that $\ell(\tilde{u}q) = \ell(\tilde{u}) + \ell(q)$. Then for $h \in U^-_{\mathrm{sf}}(\tilde{u}q)$, we have $\kappa_h h \in U^-_{\mathrm{sf}}(\tilde{u})$.*

Proof. Write $h \in U^-_{\mathrm{sf}}(\tilde{u}q) = U^-_{\mathrm{sf}}(\tilde{u}) \cdot U^-_{\mathrm{sf}}(q)$. Using (3.5.8), we find

$$h \in \dot{\tilde{u}} \cdot \dot{\tilde{u}}^{-1} \cdot U^-_{\mathrm{sf}}(\tilde{u}) \cdot U^-_{\mathrm{sf}}(q) \subset \dot{\tilde{u}} \cdot B^{\mathrm{sf}}_- \cdot U_{\mathrm{sf}}(\tilde{u}^{-1}) \cdot U^-_{\mathrm{sf}}(q).$$

By (3.5.7), $B^{\mathrm{sf}}_- \cdot U_{\mathrm{sf}}(\tilde{u}^{-1}) \cdot U^-_{\mathrm{sf}}(q) = B^{\mathrm{sf}}_- \cdot U^-_{\mathrm{sf}}(q) \cdot U_{\mathrm{sf}}(\tilde{u}^{-1}) \subset B^{\mathrm{sf}}_- \cdot U_{\mathrm{sf}}(\tilde{u}^{-1})$. Writing $B^{\mathrm{sf}}_- \subset U_- \cdot T^{\mathrm{sf}}$, we get

$$h \in \dot{\tilde{u}} \cdot U_- \cdot T^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(\tilde{u}^{-1}) = T^{\mathrm{sf}} \cdot \dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} \cdot \dot{\tilde{u}} \cdot U_{\mathrm{sf}}(\tilde{u}^{-1}).$$

Applying (3.5.16), we find

$$h \in T^{\mathrm{sf}} \cdot \dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} \cdot T^{\mathrm{sf}} \cdot (U^{\diamond} \cap \dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1}) \cdot U^-_{\mathrm{sf}}(\tilde{u}) \subset \dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} \cdot T^{\mathrm{sf}} \cdot U^-_{\mathrm{sf}}(\tilde{u}).$$

Let $g \in \dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1}$ be such that $h \in g \cdot T^{\mathrm{sf}} \cdot U^-_{\mathrm{sf}}(\tilde{u})$. Recall from (3.6.5) that $U_2^{(J)} = \dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} \cap U_-$. By Lemma 3.4.1(i), there exists $h' \in U_2^{(J)}$ such that $h'g \in \dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} \cap U$. Thus

$$h'h \in (\dot{\tilde{u}}U_-\dot{\tilde{u}}^{-1} \cap U) \cdot T^{\mathrm{sf}} \cdot U^-_{\mathrm{sf}}(\tilde{u}) \subset U \cdot T^{\mathrm{sf}} \cdot U^-_{\mathrm{sf}}(\tilde{u}).$$

But observe that both h and h' belong to U_-. Since the factorization of $h'h$ as an element of $U \cdot T \cdot U_-$ is unique by Lemma 3.4.21(i), it follows that $h'h \in U^-_{\mathrm{sf}}(\tilde{u})$. By (3.4.20), $U^-_{\mathrm{sf}}(\tilde{u}) \subset B\dot{\tilde{u}}B$. By Lemma 3.6.7, $\kappa_{h'h} = 1$, so $\kappa_h = h'$, thus $\kappa_h h \in U^-_{\mathrm{sf}}(\tilde{u})$. $\square$

Corollary 3.6.12. *For $q \in W$ such that $\ell(\tilde{u}q) = \ell(\tilde{u}) + \ell(q)$ and $v \leq \tilde{u}$, we have $R^{\mathrm{sf}}_{\mathrm{id},\tilde{u}q} \subset G^{(J)}_{u,v}$.*

Proof. As we have already mentioned, Lemma 3.6.9(i) shows that $R^{\mathrm{sf}}_{\mathrm{id},\tilde{u}q} \subset \dot{u}G_0^{(J)}$. Let

$x \in R^{\mathrm{sf}}_{\mathrm{id},\tilde{u}q} = U^-_{\mathrm{sf}}(\tilde{u}q) \cdot B^{\mathrm{sf}}$, and let $b \in B^{\mathrm{sf}}$ and $h \in U^-_{\mathrm{sf}}(\tilde{u}q)$ be such that $x = hb$. By Lemma 3.6.2(iii), we have $\kappa_x = \kappa_h$. By Proposition 3.6.11, $\kappa_h h \in U^-_{\mathrm{sf}}(\tilde{u})$, therefore $\kappa_x x \in U^-_{\mathrm{sf}}(\tilde{u}) \cdot B^{\mathrm{sf}} = R^{\mathrm{sf}}_{\mathrm{id},\tilde{u}}$. By (3.5.20), we get $\kappa_x x \in \dot{v}B_-B$. □

Corollary 3.6.12 shows that the map $\zeta^{(J)}_{u,v}$ is defined on the whole $R^{\mathrm{sf}}_{\mathrm{id},\tilde{u}q}$.

Lemma 3.6.13. *Suppose that $u_0 \in W^J$ and $v_0 \leq \tilde{u}_0 := u_0 w_J$. Let $h \in U^-_{\mathrm{sf}}(\tilde{u}_0)$, and let $b_u, b_v \in U$ be such that $\dot{\tilde{u}}_0^{-1} h \in B_- \cdot b_u$ and $\dot{v}_0^{-1} h \in B_- \cdot b_v$. Then $[b_u b_v^{-1}]_J \in U_{\mathrm{sf}}(r)$ for some $r \in W_J$.*

Proof. First, recall from Lemma 3.4.21(i) and (3.5.11) that b_u and b_v are uniquely defined and satisfy $b_u \in U_{\mathrm{sf}}(\tilde{u}_0^{-1})$, $b_v \in U_{\mathrm{sf}}(v_0^{-1})$. Let $h = h_1 h_2$ for $h_1 \in U^-_{\mathrm{sf}}(u_0)$ and $h_2 \in U^-_{\mathrm{sf}}(w_J)$. Our first goal is to show that $[b_u]_J \in U_J$ satisfies (and is uniquely defined by) $\dot{w}_J^{-1} h_2 \in B_- \cdot [b_u]_J$. Letting $b'_u \in U_J$ be uniquely defined by $\dot{w}_J^{-1} h_2 \in B_- \cdot b'_u$, we thus need to show that $[b_u]_J = b'_u$.

By (3.5.9), there exists $d \in U_{\mathrm{sf}}(u_0^{-1})$ such that

$$\dot{w}_J^{-1} \dot{u}_0^{-1} h_1 \in B^{\mathrm{sf}}_- \cdot \dot{w}_J^{-1} \cdot d.$$

Since $d \in U$, we can use Lemma 3.4.22(iii) to factorize it as $d = [d]_J [d]^{(J)}_+$. Since $h_2 \in U^-_J \subset L_J$, Lemma 3.4.22(ii) shows that there exists $d' \in U^{(J)}$ such that $[d]^{(J)}_+ h_2 = h_2 d'$. Since $[d]_J \in U_J$ by Lemma 3.4.22(iv), (3.4.21) shows that $\dot{w}_J^{-1}[d]_J \in U_- \dot{w}_J^{-1}$. Combining the pieces together, we get

$$\dot{\tilde{u}}_0^{-1} h = \dot{w}_J^{-1} \dot{u}_0^{-1} h_1 h_2 \in B^{\mathrm{sf}}_- \cdot \dot{w}_J^{-1} \cdot [d]_J [d]^{(J)}_+ \cdot h_2 \subset B_- \cdot \dot{w}_J^{-1} h_2 d' = B_- \cdot b'_u d'.$$

On the other hand, $\dot{\tilde{u}}_0^{-1} h \in B_- \cdot b_u$, so $b_u = b'_u d'$, where $b'_u \in U_J$ and $d' \in U^{(J)}$. It follows that $[b_u]_J = b'_u$, and thus we have shown that $\dot{w}_J^{-1} h_2 \in B_- \cdot [b_u]_J$.

We now prove the result by induction on $\ell(u_0)$. When $\ell(u_0) = 0$, we have $\tilde{u}_0 = w_J$ and $v_0 \in W_J$. Thus there exists $v_1 \in W_J$ such that $w_J = v_0 \cdot v_1$ with $\ell(w_J) = \ell(v_0) + \ell(v_1)$. We have $b_u, b_v \in U_J$, so $[b_u b_v^{-1}]_J = b_u b_v^{-1}$ by Lemma 3.4.22(iv). By (3.5.10), there exist $b_0 \in U_{\mathrm{sf}}(v_0^{-1})$ and $b_1 \in U_{\mathrm{sf}}(v_1^{-1})$ such that

$$\dot{v}_0^{-1} h \in B^{\mathrm{sf}}_- \cdot b_0, \quad \dot{w}_J^{-1} h \in B^{\mathrm{sf}}_- \cdot b_1 b_0.$$

In particular, we have $b_v = b_0$ and $b_u = b_1 b_0$. Thus $[b_u b_v^{-1}]_J = b_1 \in U_{\mathrm{sf}}(v_1^{-1})$, and we are done with the base case.

Assume $\ell(u_0) > 0$, and let $i \in I$ be such that $u_1 := s_i u_0 < u_0$. By Lemma 3.4.4(i), $u_1 \in W^J$, so denote $\tilde{u}_1 := u_1 w_J \in W^J_{\max}$. Let $h \in U^-_{\mathrm{sf}}(\tilde{u}_0)$ be factorized as $h = h_i h'_1 h_2$ for $h_i = y_i(t) \in U^-_{\mathrm{sf}}(s_i)$, $h'_1 \in U^-_{\mathrm{sf}}(u_1)$, and $h_2 \in U^-_{\mathrm{sf}}(w_J)$.

Suppose that $s_i v_0 > v_0$, in which case we have $v_0 \leq \tilde{u}_1$. Let $h' := h'_1 h_2$ and $b'_u \in U$ be defined by $\dot{\tilde{u}}_1^{-1} h' \in B_- \cdot b'_u$. Since $s_i v_0 > v_0$, we see that $\dot{v}_0^{-1} h_i \in B_- \cdot \dot{v}_0^{-1}$, so $\dot{v}_0^{-1} h' \in B_- \cdot \dot{v}_0^{-1} h = B_- \cdot b_v$. By the induction hypothesis applied to $v_0 \leq \tilde{u}_1$ and $h' \in U^-_{\mathrm{sf}}(\tilde{u}_1)$, we have $[b'_u b_v^{-1}]_J \in U_{\mathrm{sf}}(r)$ for some $r \in W_J$. On the other hand, we have shown above that $[b_u]_J$ satisfies $\dot{w}_J^{-1} h_2 \in B_- \cdot [b_u]_J$. But since $h' = h'_1 h_2$ for $h_2 \in U^-_{\mathrm{sf}}(w_J)$, we get that $[b'_u]_J$ satisfies $\dot{w}_J^{-1} h_2 \in B_- \cdot [b'_u]_J$, thus $[b_u]_J = [b'_u]_J$. Therefore using Lemma 3.4.22(iv), we get

$$[b_u b_v^{-1}]_J = [b_u]_J [b_v^{-1}]_J = [b'_u]_J [b_v^{-1}]_J = [b'_u b_v^{-1}]_J \in U_{\mathrm{sf}}(r),$$

finishing the induction step in the case $s_i v_0 > v_0$.

Suppose now that $v_1 := s_i v_0 < v_0$. Let $h = h_i h'_1 h_2 \in U^-_{\mathrm{sf}}(\tilde{u}_0)$ be as above. By (3.5.8), $\dot{s}_i^{-1} h_i \in B^{\mathrm{sf}}_- \cdot U_{\mathrm{sf}}(s_i)$, so let $d_i \in U_{\mathrm{sf}}(s_i)$ be such that $\dot{s}_i^{-1} h_i \in B^{\mathrm{sf}}_- \cdot d_i$. By (3.5.7), $U_{\mathrm{sf}}(s_i) \cdot U^-_{\mathrm{sf}}(\tilde{u}_1) = U^-_{\mathrm{sf}}(\tilde{u}_1) \cdot U_{\mathrm{sf}}(s_i)$, so let $b_i \in U_{\mathrm{sf}}(s_i)$ and $h' \in U^-_{\mathrm{sf}}(\tilde{u}_1)$ be such that $d_i h'_1 h_2 = h' b_i$. We check using (3.5.9) that

$$\dot{\tilde{u}}_0^{-1} h \in B^{\mathrm{sf}}_- \cdot \dot{\tilde{u}}_1^{-1} h' \cdot b_i, \quad \dot{v}_0^{-1} h \in B^{\mathrm{sf}}_- \cdot \dot{v}_1^{-1} h' \cdot b_i. \tag{3.6.6}$$

Let $b'_u, b'_v \in U$ be defined by $\dot{\tilde{u}}_1^{-1} h' \in B_- \cdot b'_u$ and $\dot{v}_1^{-1} h' \in B_- \cdot b'_v$. Then by the induction hypothesis applied to $v_1 \leq \tilde{u}_1$ and $h' \in U^-_{\mathrm{sf}}(\tilde{u}_1)$, we find $[b'_u b'^{-1}_v]_J \in U_{\mathrm{sf}}(r)$ for some $r \in W_J$. But it is clear from (3.6.6) that $b_u = b'_u b_i$ and $b_v = b'_v b_i$. Therefore $[b_u b_v^{-1}]_J \in U_{\mathrm{sf}}(r)$. □

Theorem 3.6.14. *For all $v \leq \tilde{u}$, $w \in W^J$, $i \in I$, and $x \in R^{\mathrm{sf}}_{\mathrm{id},w_0}$, we have*

$$\Delta_i^{\pm}(\zeta^{(J)}_{u,v}(x)\dot{w}^{-1}) \in \mathcal{F}_{\mathrm{sf}}. \tag{3.6.7}$$

Proof. Let $q \in W$ be such that $w_0 = \tilde{u}q$, thus $\ell(\tilde{u}q) = \ell(\tilde{u}) + \ell(q)$. Let $x \in R^{\mathrm{sf}}_{\mathrm{id},w_0} = U^-_{\mathrm{sf}}(w_0) \cdot B^{\mathrm{sf}}$ be written as $x = h \cdot b$, where $h = h_1 h_2 h_3 \in U^-_{\mathrm{sf}}(w_0)$ for $h_1 \in U^-_{\mathrm{sf}}(u)$, $h_2 \in$

$U_{\mathrm{sf}}^-(w_J)$, $h_3 \in U_{\mathrm{sf}}^-(q)$, and $b \in B^{\mathrm{sf}}$. By (3.5.10), there exist $b_1 \in U_{\mathrm{sf}}(u^{-1})$, $b_2 \in U_{\mathrm{sf}}(w_J)$, and $b_3 \in U_{\mathrm{sf}}(q^{-1})$ such that

$$\dot{u}^{-1}h \in B_-^{\mathrm{sf}} \cdot b_1, \quad \dot{\tilde{u}}^{-1}h \in B_-^{\mathrm{sf}} \cdot b_2b_1, \quad \dot{w}_0^{-1}h \in B_-^{\mathrm{sf}} \cdot b_3b_2b_1. \tag{3.6.8}$$

Let $x' := hb_1^{-1}$. We have $x' = xb^{-1}b_1^{-1} \in xB \subset xP$, therefore $x' \in G_{u,v}^{(J)}$ and $\zeta_{u,v}^{(J)}(x') = \zeta_{u,v}^{(J)}(x)$ by Lemma 3.6.2(iv). On the other hand, by (3.6.8), $x' \in \dot{u}B_-^{\mathrm{sf}} \subset \dot{u}P_-$, so Lemma 3.6.3(ii) implies $\zeta_{u,v}^{(J)}(x') = x'\eta(x')^{-1}$.

Let us now compute $\eta(x') = [\dot{v}^{-1}\kappa_{x'}x']_J$. By Lemma 3.6.2(iii), $\kappa_x = \kappa_{x'} = \kappa_h$, and by Proposition 3.6.11, $\kappa_h h \in U_{\mathrm{sf}}^-(\tilde{u})$. Thus by (3.5.11), $\dot{v}^{-1}\kappa_h h \in B_-^{\mathrm{sf}} \cdot U_{\mathrm{sf}}(v^{-1})$, so let $d_0 \in B_-^{\mathrm{sf}}$ and $b_0 \in U_{\mathrm{sf}}(v^{-1})$ be such that $\dot{v}^{-1}\kappa_h h = d_0b_0$. By definition, $\kappa_h \in U_2^{(J)}$, so by (3.6.5), $\dot{\tilde{u}}^{-1}\kappa_h\dot{\tilde{u}} \in U_-$, and therefore using (3.6.8) we find

$$\dot{\tilde{u}}^{-1}\kappa_h h = \dot{\tilde{u}}^{-1}\kappa_h\dot{\tilde{u}} \cdot \dot{\tilde{u}}^{-1}h \in U_- \cdot \dot{\tilde{u}}^{-1}h \subset B_- \cdot b_2b_1.$$

We can now apply Lemma 3.6.13: we have $v \leq \tilde{u}$, $\kappa_h h \in U_{\mathrm{sf}}^-(\tilde{u})$, $\dot{\tilde{u}}^{-1}\kappa_h h \in B_- \cdot b_2b_1$, and $\dot{v}^{-1}\kappa_h h \in B_- \cdot b_0$. Let $b_u := b_2b_1 \in U$ and $b_v := b_0 \in U$. By Lemma 3.6.13, $[b_ub_v^{-1}]_J = [b_2b_1b_0^{-1}]_J \in U_{\mathrm{sf}}(r)$ for some $r \in W_J$.

Recall that $\dot{v}^{-1}\kappa_h h = d_0b_0$ for $d_0 \in B_-^{\mathrm{sf}}$ and $b_0 \in U_{\mathrm{sf}}(v^{-1})$. Thus

$$\eta(x') = [\dot{v}^{-1}\kappa_{x'}x']_J = [\dot{v}^{-1}\kappa_h x']_J = [\dot{v}^{-1}\kappa_h hb_1^{-1}]_J = [d_0b_0b_1^{-1}]_J.$$

By Lemma 3.4.22(iii), we get $[d_0b_0b_1^{-1}]_J = [d_0]_J[b_0b_1^{-1}]_J$. Thus

$$\zeta_{u,v}^{(J)}(x) = \zeta_{u,v}^{(J)}(x') = x'\eta(x')^{-1} = x'[b_0b_1^{-1}]_J^{-1}[d_0]_J^{-1}.$$

By (3.6.8), we have $\dot{w}_0^{-1}x' \in B_-^{\mathrm{sf}} \cdot b_3b_2$, so $x' \in B^{\mathrm{sf}}\dot{w}_0b_3b_2$. Using Lemma 3.4.22(iv), we thus get

$$\zeta_{u,v}^{(J)}(x) = x'[b_0b_1^{-1}]_J^{-1}[d_0]_J^{-1} \in B^{\mathrm{sf}} \cdot \dot{w}_0b_3[b_2b_1b_0^{-1}]_J[d_0]_J^{-1}.$$

We are interested in the element $\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}$. We know that $d_0 \in B_-^{\mathrm{sf}}$, thus $[d_0]_J \in T^{\mathrm{sf}}U_J^-$, and by Lemma 3.4.5, $\dot{w}[d_0]_J\dot{w}^{-1} \in T^{\mathrm{sf}} \cdot U_-$. Hence

$$\zeta_{u,v}^{(J)}(x)\dot{w}^{-1} \in B^{\mathrm{sf}} \cdot \dot{w}_0 b_3[b_2 b_1 b_0^{-1}]_J[d_0]_J^{-1}\dot{w}^{-1} \subset B^{\mathrm{sf}} \cdot \dot{w}_0 b_3[b_2 b_1 b_0^{-1}]_J\dot{w}^{-1} \cdot T^{\mathrm{sf}} \cdot U_-.$$

In particular, $\Delta_i^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) \in \mathcal{F}_{\mathrm{sf}}$ if and only if $\Delta_i^{\pm}(\dot{w}_0 b_3[b_2 b_1 b_0^{-1}]_J\dot{w}^{-1}) \in \mathcal{F}_{\mathrm{sf}}$. Recall that $b_3 \in U_{\mathrm{sf}}(q^{-1})$ and $[b_2 b_1 b_0^{-1}]_J \in U_{\mathrm{sf}}(r)$ for some $r \in W_J$. Thus $b_3[b_2 b_1 b_0^{-1}]_J \in U_{\mathrm{sf}}(q^{-1}r)$, so we are done by Proposition 3.5.13. $\square$

Proof of Theorem 3.6.4. Our strategy will be very similar to the one we used in the proof of Corollary 3.5.12.

Fix $(u,u) \preceq (v,w) \preceq (v',w') \in Q_J$. Let $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ with $|\mathbf{t}_1| = \ell(v')$, $|\mathbf{t}_2| = \ell(w_0) - \ell(w')$, and $|\mathbf{t}_3| := \ell(w') - \ell(v')$, and assume $\mathbb{K} = \overline{\mathcal{F}}$. Choose reduced words $\mathbf{i}$ for v'^{-1} and $\mathbf{j}$ for $w_0 w'^{-1}$, and let $(\mathbf{v}', \mathbf{w}') \in \mathrm{Red}(v', w')$. Suppose that $x \in \mathbf{g}_{\mathbf{v}',\mathbf{w}'}(\mathbf{t}_3) \cdot B^{\mathrm{sf}}$. Then

$$g(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) := \mathbf{x}_{\mathbf{i}}(\mathbf{t}_1) \cdot \mathbf{y}_{\mathbf{j}}(\mathbf{t}_2) \cdot \mathbf{g}_{\mathbf{v}',\mathbf{w}'}(\mathbf{t}_3) \in U_{\mathrm{sf}}(v'^{-1}) \cdot U_{\mathrm{sf}}^-(w_0 w'^{-1}) \cdot R_{v',w'}^{\mathrm{sf}}.$$

By Lemma 3.5.8, we have $g(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) \in R_{\mathrm{id},w_0}^{\mathrm{sf}}$. Thus by Theorem 3.6.14, for all $i \in I$ we have $\Delta_i^{\pm}(\zeta_{u,v}^{(J)}(g(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3))\dot{w}^{-1}) \in \mathcal{F}_{\mathrm{sf}}$. Denote by $f(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3) := \Delta_i^{\pm}(\zeta_{u,v}^{(J)}(g(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3))\dot{w}^{-1})$ the corresponding subtraction-free rational expression, which yields a continuous function $\mathbb{R}_{>0}^{|\mathbf{t}_1|} \times \mathbb{R}_{>0}^{|\mathbf{t}_2|} \times \mathbb{R}_{>0}^{|\mathbf{t}_3|} \to \mathbb{R}_{\geq 0}$. We claim that f extends to a continuous function $\mathbb{R}_{\geq 0}^{|\mathbf{t}_1|} \times \mathbb{R}_{\geq 0}^{|\mathbf{t}_2|} \times \mathbb{R}_{>0}^{|\mathbf{t}_3|} \to \mathbb{R}_{\geq 0}$. Indeed, fix some $(\mathbf{t}_1', \mathbf{t}_2', \mathbf{t}_3') \in \mathbb{R}_{\geq 0}^{|\mathbf{t}_1|} \times \mathbb{R}_{\geq 0}^{|\mathbf{t}_2|} \times \mathbb{R}_{>0}^{|\mathbf{t}_3|}$ and let $\mathbb{K} = \mathbb{C}$. The element $x' := g(\mathbf{t}_1', \mathbf{t}_2', \mathbf{t}_3')$ (obtained by evaluating at $(\mathbf{t}_1', \mathbf{t}_2', \mathbf{t}_3')$, see Section 3.5.4) belongs to $G_{\geq 0} \cdot R_{v',w'}^{>0}$, and by Lemma 3.4.17 there exist $v'', w'' \in W$ such that $v'' \leq v' \leq w' \leq w''$ and $x' \in R_{v'',w''}^{>0}$. Recall from Lemma 3.4.8(ii) that we have

$$v'' \leq v' \leq vr' \leq ur \leq wr' \leq w' \leq w''$$

for some $r', r \in W_J$ such that $\ell(vr') = \ell(v) + \ell(r')$. In particular, by Lemma 3.6.9(ii), $x' \in \dot{u}G_0^{(J)}$ and $\kappa_{x'}x' \in R_{v'',ur_w}^{>0}$ for some $r_w \in W_J$ such that $r_w \geq r$. By Corollary 3.5.12, $\kappa_{x'}x' \in \dot{v}\dot{r}'B_-B \subset \dot{v}G_0^{(J)}$, which shows that $x' \in G_{u,v}^{(J)}$. The map $\zeta_{u,v}^{(J)}$ is therefore regular at x' by Lemma 3.6.2(ii). The map $\Delta_i^{\pm}$ is regular on G by Lemma 3.4.21(ii), so in particular it is regular at $\zeta_{u,v}^{(J)}(x')\dot{w}^{-1}$. We have shown that the map $x'' \mapsto \Delta_i^{\pm}(\zeta_{u,v}^{(J)}(x'')\dot{w}^{-1})$ is regular at $x' = g(\mathbf{t}_1', \mathbf{t}_2', \mathbf{t}_3')$ for all $(\mathbf{t}_1', \mathbf{t}_2', \mathbf{t}_3') \in \mathbb{R}_{\geq 0}^{|\mathbf{t}_1|} \times \mathbb{R}_{\geq 0}^{|\mathbf{t}_2|} \times \mathbb{R}_{>0}^{|\mathbf{t}_3|}$. Thus the map $f(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ extends to a continuous function $\mathbb{R}_{\geq 0}^{|\mathbf{t}_1|} \times \mathbb{R}_{\geq 0}^{|\mathbf{t}_2|} \times \mathbb{R}_{>0}^{|\mathbf{t}_3|} \to \mathbb{R}_{\geq 0}$. By Lemma 3.5.9, we find that $f(0, 0, \mathbf{t}_3) := \lim_{\mathbf{t}_1, \mathbf{t}_2 \to 0} f(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$ belongs to $\mathcal{F}_{\mathrm{sf}}$, i.e., can be represented by a subtraction-free rational expression in the variables $\mathbf{t}_3$. On the other hand, it is clear that $f(0, 0, \mathbf{t}_3) = \Delta_i^{\pm}(\zeta_{u,v}^{(J)}(\mathbf{g}_{\mathbf{v}', \mathbf{w}'}(\mathbf{t}_3))\dot{w}^{-1})$.

Our next goal is to show that $f(0, 0, \mathbf{t}_3) \in \mathcal{F}_{\mathrm{sf}}^*$. Indeed, suppose otherwise that $f(0, 0, \mathbf{t}_3) = 0$ (as an element of $\mathcal{F}$). By Lemma 3.6.2(iv), $\zeta_{u,v}^{(J)}$ descends to a regular map $G_{u,v}^{(J)}/P \to G$ (still assuming $\mathbb{K} = \mathbb{C}$). Therefore the map $\bar{f} : G_{u,v}^{(J)}/P \to \mathbb{C}$ sending $x'P \mapsto \Delta_i^{\pm}(\zeta_{u,v}^{(J)}(x')\dot{w}^{-1})$ is also regular. If $f(0, 0, \mathbf{t}_3) = 0$ then $\bar{f}$ vanishes on $\pi_J(R_{v',w'}^{>0}) = \Pi_{v',w'}^{>0}$, and therefore it vanishes on its Zariski closure, which is $\Pi_{v',w'}$. We have $\pi_J(R_{v,w}^{>0}) = \Pi_{v,w}^{>0} \subset \Pi_{v',w'}$, thus $\bar{f}(x) = 0$ for any $x \in G_{u,v}^{(J)}$ such that $xB \in R_{v,w}^{>0}$. Let us show that this leads to a contradiction.

Let $x \in G$ be such that $xB \in R_{v,w}^{>0}$. By (3.4.27), there exists $x' \in xP$ such that $x'B \in R_{vr',wr'}^{>0}$. By Lemma 3.6.9(ii), we have $x' \in \dot{u}G_0^{(J)}$, and thus $x \in \dot{u}G_0^{(J)}$. Having $xB \in R_{v,w}^{>0}$ implies $x \in B_-\dot{v}B \cap B\dot{w}B$. Since $\kappa_x \in U_2^{(J)} \subset U_-$, we have $\kappa_x x \in B_-\dot{v}B$. By (3.4.34), $B_-\dot{v}B = (\dot{v}U_- \cap U_-\dot{v})B \subset \dot{v}B_-B$, thus $\kappa_x x \in \dot{v}B_-B$, and therefore $x \in G_{u,v}^{(J)}$. Moreover, $\dot{v}^{-1}\kappa_x x \in B_-B$, thus $\eta(x) = [\dot{v}^{-1}\kappa_x x]_J \in U_J^- T U_J$. On the other hand, $\pi_{\dot{u}P_-}(x) \in xU^{(J)} \subset xB \subset B\dot{w}B$, see Definition 3.6.1. Thus

$$\zeta_{u,v}^{(J)}(x) = \pi_{\dot{u}P_-}(x)\eta(x)^{-1} \in B\dot{w}B \cdot U_J T U_J^- = B\dot{w}B \cdot U_J^-.$$

Recall that because $w \in W^J$, we have $U_J^- \dot{w}^{-1} \subset \dot{w}^{-1}U_-$ by Lemma 3.4.5. Hence

$$\zeta_{u,v}^{(J)}(x)\dot{w}^{-1} \in B\dot{w}B \cdot U_J^- \cdot \dot{w}^{-1} \subset B\dot{w}B\dot{w}^{-1}B_-.$$

By (3.4.34) (after taking inverses of both sides), $B\dot{w}B = B \cdot (U_-\dot{w} \cap \dot{w}U)$, so

$$\zeta_{u,v}^{(J)}(x)\dot{w}^{-1} \in B \cdot (U_- \cap \dot{w}U\dot{w}^{-1}) \cdot B_- \subset B \cdot B_-.$$

In particular, $\Delta_i^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) \neq 0$ for all $i \in I$. This gives a contradiction, showing $f(0, 0, \mathbf{t}_3) \in \mathcal{F}_{\mathrm{sf}}^*$. But then evaluating f at any $\mathbf{t}_3' \in \mathbb{R}_{>0}^{\ell(w')-\ell(v')}$ yields a positive real number. We have shown that $\Delta_i^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) \neq 0$ for all $x \in G$ such that $xB \in R_{v',w'}^{>0}$. We are done by Lemma 3.4.21(iii). □

3.7 Affine Bruhat atlas for the projected Richardson stratification

In this section, we embed the stratification (3.4.23) of G/P inside the affine Richardson stratification of the affine flag variety. Throughout, we work over $\mathbb{K} = \mathbb{C}$.

3.7.1 Loop groups and affine flag varieties

Recall that G is a simple and simply connected algebraic group. Let $\mathcal{A} := \mathbb{C}[z, z^{-1}]$ and $\mathcal{A}_+, \mathcal{A}_- \subset \mathcal{A}$ denote the subrings given by $\mathcal{A}_+ := \mathbb{C}[z]$, $\mathcal{A}_- := \mathbb{C}[z^{-1}]$. Then we have ring homomorphisms $\bar{\mathrm{ev}}_0 : \mathcal{A}_+ \to \mathbb{C}$ (resp., $\bar{\mathrm{ev}}_\infty : \mathcal{A}_- \to \mathbb{C}$), sending a polynomial in z (resp., in z^{-1}) to its constant term. Let $\mathcal{G} := G(\mathcal{A})$ denote the polynomial loop group of G.

Remark 3.7.1. The group $\mathcal{G}$ is closely related to the *(minimal) affine Kac–Moody group* $\mathcal{G}^{\min}$ associated to G, introduced by Kac–Peterson [KP83, PK83]. Below we state many standard results on $\mathcal{G}$ without proof. We refer the reader unfamiliar with Kac–Moody groups to Section 3.A, where we give some background and explain how to derive these statements from Kumar's book [Kum02].

We introduce opposite Iwahori subgroups

$$\mathcal{B} := \{g(z) \in G(\mathcal{A}_+) \mid \bar{\mathrm{ev}}_0(g) \in B\}, \qquad \mathcal{B}_- := \{g(z^{-1}) \in G(\mathcal{A}_-) \mid \bar{\mathrm{ev}}_\infty(g) \in B_-\}$$

of $\mathcal{G}$, and denote by

$$\mathcal{U} := \{g(z) \in G(\mathcal{A}_+) \mid \bar{\mathrm{ev}}_0(g) \in U\}, \qquad \mathcal{U}_- := \{g(z^{-1}) \in G(\mathcal{A}_-) \mid \bar{\mathrm{ev}}_\infty(g) \in U_-\}$$

their unipotent radicals. There exists a tautological embedding $G \hookrightarrow \mathcal{G}$, and we treat G as a subset of $\mathcal{G}$.

We let $\mathcal{T} := \mathbb{C}^* \times T \subset \mathbb{C}^* \ltimes G$ be the affine torus, where $\mathbb{C}^*$ acts on $\mathcal{G}$ via loop rotation, see Section 3.8.2. The *affine root system* Δ of $\mathcal{G}$ is the subset of $X(\mathcal{T}) := \mathrm{Hom}(\mathcal{T}, \mathbb{C}^*) \cong X(T) \oplus \mathbb{Z}\delta$ given by

$$\Delta = \Delta_{\mathrm{re}} \sqcup \Delta_{\mathrm{im}}, \quad \text{where} \quad \Delta_{\mathrm{re}} := \{\beta + j\delta \mid \beta \in \Phi,\ j \in \mathbb{Z}\}, \quad \Delta_{\mathrm{im}} := \{j\delta \mid j \in \mathbb{Z} \setminus \{0\}\}$$

are the real and imaginary roots, and the set of *positive roots* $\Delta^+ \subset \Delta$ has the form

$$\Delta^+ = \{j\delta \mid j > 0\} \sqcup \{\beta + j\delta \mid \beta \in \Phi,\ j > 0\} \sqcup \{\beta \mid \beta \in \Phi^+\}. \tag{3.7.1}$$

We let $\Delta_{\mathrm{re}}^+ := \Delta^+ \cap \Delta_{\mathrm{re}}$ and $\Delta_{\mathrm{re}}^- := \Delta^- \cap \Delta_{\mathrm{re}}$. For each $\alpha \in \Delta_{\mathrm{re}}^+$ (resp., $\alpha \in \Delta_{\mathrm{re}}^-$) , we have a one-parameter subgroup $\mathcal{U}_\alpha \subset \mathcal{U}$ (resp., $\mathcal{U}_\alpha \subset \mathcal{U}_-$). The group $\mathcal{U}$ (resp., $\mathcal{U}_-$) is generated by $\{\mathcal{U}_\alpha\}_{\alpha \in \Delta_{\mathrm{re}}^+}$ (resp., $\{\mathcal{U}_\alpha\}_{\alpha \in \Delta_{\mathrm{re}}^-}$), and for each $\alpha \in \Delta_{\mathrm{re}}$, we fix a group isomorphism $x_\alpha : \mathbb{C} \xrightarrow{\sim} \mathcal{U}_\alpha$.

Let $Q_\Phi^\vee := \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$ denote the *coroot lattice* of Φ. The *affine Weyl group* $\tilde{W} = W \ltimes Q_\Phi^\vee$ is a semidirect product of W and $Q_\Phi^\vee$, i.e., as a set we have $\tilde{W} = W \times Q_\Phi^\vee$, and the product rule is given by $(w_1, \lambda_1) \cdot (w_2, \lambda_2) := (w_1 w_2, \lambda_1 + w_1 \lambda_2)$. For $\lambda \in Q_\Phi^\vee$, we denote the element $(\mathrm{id}, \lambda) \in \tilde{W}$ by τ_λ. The group $\tilde{W}$ is isomorphic to $N_{\mathbb{C}^* \ltimes \mathcal{G}}(\mathcal{T})/\mathcal{T}$, and for $f \in \tilde{W}$, we choose a representative $\dot{f} \in \mathcal{G}$ of f in $N_{\mathbb{C}^* \ltimes \mathcal{G}}(\mathcal{T})$, with the assumption that for $w \in W$, the representative $\dot{w} \in G \subset \mathcal{G}$ is given by (3.4.1). Thus $\tilde{W}$ is a Coxeter group with generators $s_0 \sqcup \{s_i\}_{i \in I}$, length function $\ell : \tilde{W} \to \mathbb{Z}_{\geq 0}$, and *affine Bruhat order* $\leq$. The group $\tilde{W}$ acts on Δ, and for $\alpha \in \Phi$, $\beta \in \Delta_{\mathrm{re}}$, $\lambda \in Q_\Phi^\vee$, and $w \in W$, we have

$$w\tau_\lambda w^{-1} = \tau_{w\lambda}, \quad \tau_\lambda \alpha = \alpha + \langle \lambda, \alpha \rangle \delta, \quad \tau_\lambda \delta = \delta, \quad \dot{\tau}_\lambda \mathcal{U}_\beta \dot{\tau}_\lambda^{-1} = \mathcal{U}_{\tau_\lambda \beta}. \tag{3.7.2}$$

Let $\mathcal{G}/\mathcal{B}$ denote the *affine flag variety* of G. This is an ind-variety that is isomorphic to the flag variety of the corresponding affine Kac-Moody group $\mathcal{G}^{\min}$, see Section 3.A.4. For each $h, f \in \tilde{W}$ we have Schubert cells $\mathring{\mathcal{X}}^f := \mathcal{B}\dot{f}\mathcal{B}/\mathcal{B}$ and opposite Schubert cells $\mathring{\mathcal{X}}_h := \mathcal{B}_-\dot{h}\mathcal{B}/\mathcal{B}$. If $h \not\leq f \in \tilde{W}$ then $\mathring{\mathcal{X}}^f \cap \mathring{\mathcal{X}}_h = \emptyset$. For $h \leq f$, we denote $\mathring{\mathcal{R}}_h^f := \mathring{\mathcal{X}}_h \cap \mathring{\mathcal{X}}^f$. For all $g \in \tilde{W}$, we have

$$\mathring{\mathcal{X}}^g = \bigsqcup_{h \le g} \mathring{\mathcal{R}}_h^g, \quad \mathring{\mathcal{X}}_g = \bigsqcup_{g \le f} \mathring{\mathcal{R}}_g^f, \quad \mathcal{X}^g := \bigsqcup_{h \le g} \mathring{\mathcal{X}}^h, \quad \mathcal{X}_g := \bigsqcup_{g \le f} \mathring{\mathcal{X}}_f. \tag{3.7.3}$$

For $g \in \tilde{W}$, let

$$\mathcal{C}_g := \dot{g}\mathcal{B}_-\mathcal{B}/\mathcal{B}, \quad \mathcal{U}_1(g) := \dot{g}\mathcal{U}_-\dot{g}^{-1} \cap \mathcal{U}, \quad \text{and} \quad \mathcal{U}_2(g) := \dot{g}\mathcal{U}_-\dot{g}^{-1} \cap \mathcal{U}_-. \tag{3.7.4}$$

As we explain in Section 3.A.5, the map $x \mapsto x\dot{g}\mathcal{B}$ gives biregular isomorphisms

$$\dot{g}\mathcal{U}_-\dot{g}^{-1} \xrightarrow{\sim} \mathcal{C}_g, \quad \mathcal{U}_1(g) \xrightarrow{\sim} \mathring{\mathcal{X}}^g, \quad \mathcal{U}_2(g) \xrightarrow{\sim} \mathring{\mathcal{X}}_g. \tag{3.7.5}$$

Let $\mathcal{U}^{(I)} \subset \mathcal{U}$ be the subgroup generated by $\{\mathcal{U}_\alpha\}_{\alpha \in \Delta_{\mathrm{re}}^+ \setminus \Phi^+}$. Similarly, let $\mathcal{U}_-^{(I)} \subset \mathcal{U}_-$ be the subgroup generated by $\{\mathcal{U}_\alpha\}_{\alpha \in \Delta_{\mathrm{re}}^- \setminus \Phi^-}$. For $x \in G \subset \mathcal{G}$, we have

$$x \cdot \mathcal{U}^{(I)} \cdot x^{-1} = \mathcal{U}^{(I)}, \quad x \cdot \mathcal{U}_-^{(I)} \cdot x^{-1} = \mathcal{U}_-^{(I)}. \tag{3.7.6}$$

3.7.2 Combinatorial Bruhat atlas for G/P

We fix an element $\lambda \in Q_\Phi^\vee$ such that $\langle \lambda, \alpha_i \rangle = 0$ for $i \in J$ and $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{<0}$ for $i \in I \setminus J$. Thus λ is anti-dominant and the stabilizer of λ in W is equal to W_J. Following [HL15], define a map

$$\psi : Q_J \to \tilde{W}, \quad (v, w) \mapsto v\tau_\lambda w^{-1}. \tag{3.7.7}$$

By [HL15, Thm. 2.2], the map ψ gives an order-reversing bijection between Q_J and a subposet of $\tilde{W}$. More precisely, let $\tau_\lambda^J := \tau_\lambda (w^J)^{-1}$, and recall from (3.7.2) that $u\tau_\lambda u^{-1} = \tau_{u\lambda}$. By [HL15, §2.3], for all $(v, w) \in Q_J$ we have

$$v\tau_\lambda w^{-1} = v \cdot \tau_\lambda^J \cdot w^J w^{-1}, \quad \ell(v\tau_\lambda w^{-1}) = \ell(v) + \ell(\tau_\lambda^J) + \ell(w^J w^{-1}), \tag{3.7.8}$$

see Figure 3-2 for an example. By [HL15, Thm. 2.2], for all $u \in W^J$ we have

$$\psi(Q_J^{\succeq(u,u)}) = \{g \in \tilde{W} \mid \tau_\lambda^J \leq g \leq \tau_{u\lambda}\}, \quad \text{and} \tag{3.7.9}$$

$$\psi(Q_J) = \{g \in \tilde{W} \mid \tau_\lambda^J \leq g \leq \tau_{w\lambda} \text{ for some } w \in W^J\}. \tag{3.7.10}$$

3.7.3 Bruhat atlas for the projected Richardson stratification of G/P

Let $u \in W^J$. Recall that $\lambda \in Q_\Phi^\vee$ has been fixed. We further assume that the representatives $\dot{\tau}_\lambda$ and $\dot{\tau}_{u\lambda}$ satisfy the identity $\dot{u}\dot{\tau}_\lambda\dot{u}^{-1} = \dot{\tau}_{u\lambda}$.

Our goal is to construct a geometric lifting of the map ψ. Recall the maps $x \mapsto g_1^{(J)}$ and $x \mapsto g_2^{(J)}$ from Definition 3.4.23. We define maps

$$\varphi_u : C_u^{(J)} \to \mathcal{G}, \qquad xP \mapsto g_1^{(J)}\dot{u} \cdot \dot{\tau}_\lambda \cdot (g_2^{(J)}\dot{u})^{-1} = g_1^{(J)} \cdot \dot{\tau}_{u\lambda} \cdot (g_2^{(J)})^{-1}, \quad \text{and} \tag{3.7.11}$$

$$\bar{\varphi}_u : C_u^{(J)} \to \mathcal{G}/\mathcal{B}, \qquad xP \mapsto \varphi_u(xP) \cdot \mathcal{B}. \tag{3.7.12}$$

The main result of this section is the following theorem.

Theorem 3.7.2.

(1) *The map $\bar{\varphi}_u$ is a biregular isomorphism*

$$\bar{\varphi}_u : C_u^{(J)} \xrightarrow{\sim} \mathcal{X}_{\tau_\lambda^J} \cap \mathring{\mathcal{X}}^{\tau_{u\lambda}} = \bigsqcup_{(v,w)\in Q_J^{\succeq(u,u)}} \mathring{\mathcal{R}}^{\tau_{u\lambda}}_{v\tau_\lambda w^{-1}},$$

and for all $(v, w) \succeq (u, u)$ in Q_J, $\bar{\varphi}_u$ restricts to a biregular isomorphism

$$\bar{\varphi}_u : C_u^{(J)} \cap \mathring{\Pi}_{v,w} \xrightarrow{\sim} \mathring{\mathcal{R}}^{\tau_{u\lambda}}_{v\tau_\lambda w^{-1}}.$$

(2) *Suppose that $(u, u) \preceq (v, w) \preceq (v', w')$ in Q_J. Then*

$$\bar{\varphi}_u\left(\Pi^{>0}_{v',w'}\right) \subset \mathcal{C}_{v\tau_\lambda w^{-1}}.$$

The remainder of this section will be devoted to the proof of Theorem 3.7.2.

3.7.4 An alternative definition of $\bar{\varphi}_u$

Recall the notation from Definition 3.4.23, and that we have fixed $u \in W^J$ and $\lambda \in Q_\Phi^\vee$ satisfying $\langle \lambda, \alpha_i \rangle = 0$ for $i \in J$ and $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}_{<0}$ for $i \in I \setminus J$. We list the rules for conjugating elements of $G \subset \mathcal{G}$ by $\dot{\tau}_\lambda$.

Lemma 3.7.3. *We have*

$$\dot{\tau}_\lambda \cdot p = p \cdot \dot{\tau}_\lambda \quad \text{for all } p \in L_J, \tag{3.7.13}$$

$$\dot{\tau}_\lambda \cdot U^{(J)} \cdot \dot{\tau}_\lambda^{-1} \subset \mathcal{U}_-^{(I)}, \quad \dot{\tau}_\lambda \cdot U_-^{(J)} \cdot \dot{\tau}_\lambda^{-1} \subset \mathcal{U}^{(I)}, \tag{3.7.14}$$

$$\dot{\tau}_\lambda^{-1} \cdot U^{(J)} \cdot \dot{\tau}_\lambda \subset \mathcal{U}^{(I)}, \quad \dot{\tau}_\lambda^{-1} \cdot U_-^{(J)} \cdot \dot{\tau}_\lambda \subset \mathcal{U}_-^{(I)}, \tag{3.7.15}$$

$$\dot{\tau}_{u\lambda} \cdot U_2^{(J)} \cdot \dot{\tau}_{u\lambda}^{-1} \subset \mathcal{U}^{(I)}, \quad \dot{\tau}_{u\lambda}^{-1} \cdot U_1^{(J)} \cdot \dot{\tau}_{u\lambda} \subset \mathcal{U}_-^{(I)}. \tag{3.7.16}$$

Proof. Recall that L_J is generated by T, U_J, and U_J^-, and since $\tau_\lambda \alpha = \alpha$ for all $\alpha \in \Phi_J$, we see that (3.7.13) follows from (3.7.2). By (3.7.2), we find $\tau_\lambda \alpha \in \Delta_{\mathrm{re}}^+ \setminus \Phi^+$ for $\alpha \in \Phi_-^{(J)}$ and $\tau_\lambda \alpha \in \Delta_{\mathrm{re}}^- \setminus \Phi^-$ for $\alpha \in \Phi_+^{(J)}$, which shows (3.7.14). Similarly, $\tau_\lambda^{-1} \alpha \in \Delta_{\mathrm{re}}^+ \setminus \Phi^+$ for $\alpha \in \Phi_+^{(J)}$ and $\tau_\lambda^{-1}\alpha \in \Delta_{\mathrm{re}}^- \setminus \Phi^-$ for $\alpha \in \Phi_-^{(J)}$, which shows (3.7.15).

To show (3.7.16), we use (3.7.6), (3.7.14), (3.7.15), and $U_1^{(J)}, U_2^{(J)} \subset \dot{u} U_-^{(J)} \dot{u}^{-1}$ to get

$$\dot{\tau}_{u\lambda} \cdot U_2^{(J)} \cdot \dot{\tau}_{u\lambda}^{-1} = \dot{u}\dot{\tau}_\lambda \dot{u}^{-1} \cdot U_2^{(J)} \cdot \dot{u}\dot{\tau}_\lambda^{-1}\dot{u}^{-1} \subset \dot{u}\dot{\tau}_\lambda \cdot U_-^{(J)} \cdot \dot{\tau}_\lambda^{-1}\dot{u}^{-1} \subset \dot{u}\mathcal{U}^{(I)}\dot{u}^{-1} = \mathcal{U}^{(I)},$$
$$\dot{\tau}_{u\lambda}^{-1} \cdot U_1^{(J)} \cdot \dot{\tau}_{u\lambda} = \dot{u}\dot{\tau}_\lambda^{-1} \dot{u}^{-1} \cdot U_1^{(J)} \cdot \dot{u}\dot{\tau}_\lambda\dot{u}^{-1} \subset \dot{u}\dot{\tau}_\lambda^{-1} \cdot U_-^{(J)} \cdot \dot{\tau}_\lambda\dot{u}^{-1} \subset \dot{u}\mathcal{U}_-^{(I)}\dot{u}^{-1} = \mathcal{U}_-^{(I)}. \qquad \square$$

The map $\bar{\varphi}_u$ can alternatively be characterized as follows. Recall from Definition 3.4.23 that we have a regular map $\kappa : \dot{u}G_0^{(J)} \to U_2^{(J)}$ that descends to a regular map $\kappa : C_u^{(J)} \to U_2^{(J)}$ by Lemma 3.6.2(iii). Recall also from Lemma 3.4.22(i) that $\dot{u}G_0^{(J)} = \dot{u}P_- \cdot B$.

Lemma 3.7.4. *Let $x \in \dot{u}P_-$. Then*

$$\bar{\varphi}_u(xP) = \kappa_x x \cdot \dot{\tau}_\lambda \cdot x^{-1} \cdot \mathcal{B}. \tag{3.7.17}$$

Proof. We continue using the notation of Definition 3.4.23. Let $p \in L_J$ and $g^{(J)} \in \dot{u}U_-^{(J)}\dot{u}^{-1}$ be such that $xp = g^{(J)}\dot{u}$. Note that $g_2^{(J)}\dot{u} = h_1^{(J)} g^{(J)}\dot{u} = h_1^{(J)} xp$, and since $h_1^{(J)} \in U_1^{(J)} \subset U \subset$

$\mathcal{B}$, we see that $(g_2^{(J)}\dot{u})^{-1}\cdot\mathcal{B} = (xp)^{-1}\cdot\mathcal{B}$. On the other hand, $\kappa_x xp = h_2^{(J)}g^{(J)}\dot{u} = g_1^{(J)}\dot{u}$. Since p commutes with $\dot{\tau}_\lambda$ by (3.7.13), we find

$$\bar{\varphi}_u(xP) = g_1^{(J)}\dot{u}\cdot\dot{\tau}_\lambda\cdot(g_2^{(J)}\dot{u})^{-1}\cdot\mathcal{B} = \kappa_x xp\cdot\dot{\tau}_\lambda\cdot(xp)^{-1}\cdot\mathcal{B} = \kappa_x x\cdot\dot{\tau}_\lambda\cdot x^{-1}\cdot\mathcal{B}. \qquad \square$$

3.7.5 The affine Richardson cell of $\bar{\varphi}_u$

Lemma 3.7.5. *We have*

$$C_u^{(J)} = \bigsqcup_{(v,w)\in Q_J^{\succeq(u,u)}} (C_u^{(J)}\cap\mathring{\Pi}_{v,w}). \tag{3.7.18}$$

Proof. The torus T acts on G/P by left multiplication and preserves the sets $C_u^{(J)}$ and $\mathring{\Pi}_{v,w}$ for all $(v,w)\in Q_J$. By (3.4.23), $\Pi_{v,w}$ contains $\dot{u}P$ if and only if $(u,u)\preceq(v,w)$. Suppose that $xP\in C_u^{(J)}\cap\mathring{\Pi}_{v,w}$ for some $(v,w)\in Q_J$. Then $TxP/P\subset C_u^{(J)}$, and by Lemma 3.6.8(iii), the closure of this set contains $\dot{u}P$. On the other hand, the closure of this set is contained inside $\Pi_{v,w}$, thus $(u,u)\preceq(v,w)$. $\square$

Lemma 3.7.6. *Let $(v,w)\in Q_J^{\succeq(u,u)}$. Then*

$$\bar{\varphi}_u(C_u^{(J)}\cap\mathring{\Pi}_{v,w})\subset\mathring{\mathcal{R}}^{\tau_{u\lambda}}_{v\tau_\lambda w^{-1}}. \tag{3.7.19}$$

Proof. Let $x\in\dot{u}G_0^{(J)}$ be such that $xP\in\mathring{\Pi}_{v,w}$. Let us first show that $\bar{\varphi}_u(xP)\in\mathring{\mathcal{X}}^{\tau_{u\lambda}}$. By (3.7.12), we have

$$\bar{\varphi}_u(xP) = g_1^{(J)}\cdot\dot{\tau}_{u\lambda}\cdot(g_2^{(J)})^{-1}\cdot\dot{\tau}_{u\lambda}^{-1}\cdot\dot{\tau}_{u\lambda}\cdot\mathcal{B}. \tag{3.7.20}$$

Observe that $g_1^{(J)}\in U_1^{(J)}\subset U$ and by (3.7.16), $\dot{\tau}_{u\lambda}\cdot(g_2^{(J)})^{-1}\cdot\dot{\tau}_{u\lambda}^{-1}\in\mathcal{U}^{(I)}$, so by (3.7.6), we have

$$\varphi_u(xP)\cdot\dot{\tau}_{u\lambda}^{-1}\in\mathcal{U},\quad\text{thus}\quad\varphi_u(xP)\in\mathcal{B}\cdot\dot{\tau}_{u\lambda}\cdot\mathcal{B}. \tag{3.7.21}$$

This proves that $\bar{\varphi}_u(xP)\in\mathring{\mathcal{X}}^{\tau_{u\lambda}}$.

We now show $\bar{\varphi}_u(xP)\in\mathring{\mathcal{X}}_{v\tau_\lambda w^{-1}}$. Recall that $\mathring{\Pi}_{v,w} = \pi_J(\mathring{R}_{v,w})$, so assume that $x\in B_-\dot{v}B\cap B\dot{w}B$. Since $\dot{u}G_0^{(J)} = \dot{u}P_-B$ by Lemma 3.4.22(i), we may assume that $x\in\dot{u}P_-$, in which case $\bar{\varphi}_u(xP)$ is given by (3.7.17). We have $\kappa_x x\in B_-\dot{v}B$ and $x^{-1}\in B\dot{w}^{-1}B$, so it suffices to show

$$B_-\dot{v}B\cdot\dot{\tau}_\lambda\cdot B\dot{w}^{-1}B\subset\mathcal{B}_-\cdot\dot{v}\dot{\tau}_\lambda\dot{w}^{-1}\cdot\mathcal{B}. \tag{3.7.22}$$

Clearly we have

$$B_-\dot{v}B\cdot\dot{\tau}_\lambda\cdot B\dot{w}^{-1}B\subset\mathcal{B}_-\cdot\dot{v}\cdot U^{(J)}\cdot U_J\cdot\dot{\tau}_\lambda\cdot U^{(J)}\cdot U_J\cdot\dot{w}^{-1}\cdot\mathcal{B}.$$

By (3.7.13) and Lemma 3.4.22(ii), U_J can be moved to the right past $\dot{\tau}_\lambda$ and $U^{(J)}$. We can then move $U^{(J)}$ to the left past $\dot{\tau}_\lambda$ using (3.7.14), which gives

$$B_-\dot{v}B\cdot\dot{\tau}_\lambda\cdot B\dot{w}^{-1}B\subset\mathcal{B}_-\cdot\dot{v}\cdot U^{(J)}\cdot\mathcal{U}_-^{(I)}\cdot\dot{\tau}_\lambda\cdot U_J\cdot\dot{w}^{-1}\cdot\mathcal{B}.$$

By (3.7.6), $\mathcal{U}_-^{(I)}$ can be moved to the left past $\dot{v}\cdot U^{(J)}$, and then $U^{(J)}$ can be moved to the right past $\dot{\tau}_\lambda$ using (3.7.15), yielding

$$B_-\dot{v}B\cdot\dot{\tau}_\lambda\cdot B\dot{w}^{-1}B\subset\mathcal{B}_-\cdot\dot{v}\cdot\dot{\tau}_\lambda\cdot\mathcal{U}^{(I)}\cdot U_J\cdot\dot{w}^{-1}\cdot\mathcal{B}.$$

By (3.7.6), $\mathcal{U}^{(I)}$ can be moved to the right past $U_J\cdot\dot{w}^{-1}$. Since $w\in W^J$, Lemma 3.4.5 shows that $U_J\cdot\dot{w}^{-1}\subset\dot{w}^{-1}U$, so (3.7.22) follows. □

3.7.6 Proof of Theorem 3.7.2(1)

Observe that $\mathcal{X}_{\tau_\lambda^J}\cap\mathring{\mathcal{X}}^{\tau_{u\lambda}}=\bigsqcup_{(v,w)\in Q_J^{\succeq(u,u)}}\mathring{\mathcal{R}}^{\tau_{u\lambda}}_{v\tau_\lambda w^{-1}}$ by (3.7.3) and (3.7.9). By (3.7.19), $\bar{\varphi}_u(C_u^{(J)})\subset\mathcal{X}_{\tau_\lambda^J}\cap\mathring{\mathcal{X}}^{\tau_{u\lambda}}$. Let us identify $\mathring{\mathcal{X}}^{\tau_{u\lambda}}$ with the affine variety $\mathcal{U}_1(\tau_{u\lambda})$ via (3.7.5), and denote by $\bar{\varphi}_u^\dagger : C_u^{(J)}\to\mathcal{U}_1(\tau_{u\lambda})$ the composition of (3.7.5) and $\bar{\varphi}_u$.

We claim that $\bar{\varphi}_u^\dagger$ gives a biregular isomorphism between $C_u^{(J)}$ and a closed subvariety of $\mathcal{U}_1(\tau_{u\lambda})$. Let $x\in\dot{u}G_0^{(J)}$ and let $g^{(J)}$, $g_1^{(J)}$, $g_2^{(J)}$ be as in Definition 3.4.23. Let $y:=\varphi_u(xP)\cdot\dot{\tau}_{u\lambda}^{-1}$, so $\bar{\varphi}_u(xP)=y\cdot\dot{\tau}_{u\lambda}\cdot\mathcal{B}$. Thus $\bar{\varphi}_u^\dagger(xP)=y$ if and only if $y\in\mathcal{U}_1(\tau_{u\lambda})$. By (3.7.21), we have $y\in\mathcal{U}$. Hence in order to prove $y\in\mathcal{U}_1(\tau_{u\lambda})$, we need to show $y\in\dot{\tau}_{u\lambda}\mathcal{U}_-\dot{\tau}_{u\lambda}^{-1}$. Conjugating both sides by $\dot{\tau}_{u\lambda}$, we get

$$\dot{\tau}_{u\lambda}^{-1}\cdot y\cdot\dot{\tau}_{u\lambda}=\dot{\tau}_{u\lambda}^{-1}g_1^{(J)}\dot{\tau}_{u\lambda}\cdot(g_2^{(J)})^{-1},$$

which belongs to $\mathcal{U}_-$ since $(g_2^{(J)})^{-1} \in U_2^{(J)} \subset U_-$ by definition and $\dot{\tau}_{u\lambda}^{-1} g_1^{(J)} \dot{\tau}_{u\lambda} \in \mathcal{U}_-^{(I)}$ by (3.7.16). Thus $y \in \mathcal{U}_1(\tau_{u\lambda})$ and $\bar{\varphi}_u^\dagger(xP) = y$. By Lemma 3.4.2, we may identify $C_u^{(J)}$ with $U_1^{(J)} \times U_2^{(J)}$, so let $\bar{\varphi}_u^\ddagger : U_1^{(J)} \times U_2^{(J)} \to \mathcal{U}_1(\tau_{u\lambda})$ be the map sending $(g_1^{(J)}, g_2^{(J)}) \mapsto y := g_1^{(J)} \cdot \dot{\tau}_{u\lambda} (g_2^{(J)})^{-1} \dot{\tau}_{u\lambda}^{-1}$.

Let $\Theta_1 := u\Phi_-^{(J)} \cap \Phi^+$ and $\Theta_2 := u\Phi_-^{(J)} \cap \Phi^-$, thus $U_1^{(J)} = U(\Theta_1)$, $U_2^{(J)} = U_-(\Theta_2)$, and $\Theta_1 \sqcup \Theta_2 = u\Phi_-^{(J)}$. By the proof of (3.7.16), $\tau_{u\lambda}\Theta_2 \subset \Delta_{\mathrm{re}}^+ \setminus \Phi^+$ and $\tau_{u\lambda}^{-1}\Theta_1 \subset \Delta_{\mathrm{re}}^-$, thus $\Theta_1 \sqcup \tau_{u\lambda}\Theta_2 \subset \mathrm{Inv}(\tau_{u\lambda}^{-1})$. Let $\Theta_3 \subset \Delta_{\mathrm{re}}^+$ be defined by $\Theta_3 := \mathrm{Inv}(\tau_{u\lambda}^{-1}) \setminus (\Theta_1 \sqcup \Theta_2)$. By Lemma 3.A.1, the multiplication map gives a biregular isomorphism

$$\mathcal{U}(\Theta_1) \times \mathcal{U}(\tau_{u\lambda}\Theta_2) \times \prod_{\alpha \in \Theta_3} \mathcal{U}_\alpha \xrightarrow{\sim} \mathcal{U}(\mathrm{Inv}(\tau_{u\lambda}^{-1})) = \mathcal{U}_1(\tau_{u\lambda}), \tag{3.7.23}$$

where $\mathcal{U}(\Theta)$ denotes the subgroup generated by $\{\mathcal{U}_\alpha\}_{\alpha \in \Theta}$. In particular, $\mathcal{U}(\Theta_1) \cdot \mathcal{U}(\tau_{u\lambda}\Theta_2)$ is a closed subvariety of $\mathcal{U}_1(\tau_{u\lambda})$ isomorphic to $\mathbb{C}^{|\Theta_1|+|\Theta_2|} = \mathbb{C}^{\ell(w^J)}$. Observe that $\mathcal{U}(\tau_{u\lambda}\Theta_2) = \dot{\tau}_{u\lambda} U_2^{(J)} \dot{\tau}_{u\lambda}^{-1}$, thus $\bar{\varphi}_u^\ddagger$ essentially coincides with the restriction of the map (3.7.23) to $\mathcal{U}(\Theta_1) \times \mathcal{U}(\tau_{u\lambda}\Theta_2) \times \{1\}$. We have thus shown that $\bar{\varphi}_u^\ddagger$ gives a biregular isomorphism between $U_1^{(J)} \times U_2^{(J)}$ and a closed $\ell(w^J)$-dimensional subvariety of $\mathcal{U}_1(\tau_{u\lambda})$. Therefore $\bar{\varphi}_u$ gives a biregular isomorphism between $C_u^{(J)}$ and a closed $\ell(w^J)$-dimensional subvariety $\bar{\varphi}_u(C_u^{(J)})$ of $\mathring{\mathcal{X}}^{\tau_{u\lambda}}$. By Proposition 3.A.2, $\mathcal{X}_{\tau_\lambda^J} \cap \mathring{\mathcal{X}}^{\tau_{u\lambda}}$ is a closed irreducible subvariety of $\mathring{\mathcal{X}}^{\tau_{u\lambda}}$, and by (3.7.8) and Proposition 3.A.2, it has dimension $\ell(w^J)$. Since $\bar{\varphi}_u(C_u^{(J)}) \subset \mathcal{X}_{\tau_\lambda^J} \cap \mathring{\mathcal{X}}^{\tau_{u\lambda}}$, it follows that $\bar{\varphi}_u(C_u^{(J)}) = \mathcal{X}_{\tau_\lambda^J} \cap \mathring{\mathcal{X}}^{\tau_{u\lambda}}$. We are done with the proof of Theorem 3.7.2(1).

Remark 3.7.7. Alternatively, the proof of Theorem 3.7.2(1) could be deduced from *Deodhar-type parametrizations* [Had84, Had85, BD94] of $\mathring{\mathcal{R}}_{v\tau_\lambda w^{-1}}^{\tau_{u\lambda}}$, by observing that any reduced word for $\tau_{u\lambda}$ that is compatible with the length-additive factorization $\tau_{u\lambda} = u \cdot \tau_\lambda^J \cdot w^J u^{-1}$ in (3.7.8) contains a unique reduced subword for τ_λ^J.

3.7.7 Proof of Theorem 3.7.2(2)

We use the notation and results from Section 3.6. Let $x \in G$ be such that $xP \in \Pi_{v',w'}^{>0}$. Since $\Pi_{v',w'}^{>0} = \pi_J(R_{v',w'}^{>0})$, we may assume that $xB \in R_{v',w'}^{>0}$. Then $x \in \dot{u}G_0^{(J)}$ by Lemma 3.6.9(ii), so $\bar{\varphi}_u(xP)$ is defined. In addition, by Lemma 3.4.22(i) we may assume that $x \in \dot{u}P_-$. By definition, $\bar{\varphi}_u(xP) \in \mathcal{C}_{v\tau_\lambda w^{-1}}$ if and only if $\dot{w}\dot{\tau}_\lambda^{-1}\dot{v}^{-1}\bar{\varphi}_u(xP) \in \mathcal{B}_-\mathcal{B}/\mathcal{B}$. By (3.7.17), this is equivalent to

$$\dot{w}\dot{\tau}_\lambda^{-1}\dot{v}^{-1} \cdot \kappa_x x \cdot \dot{\tau}_\lambda \cdot x^{-1} \in \mathcal{B}_-\mathcal{B}. \tag{3.7.24}$$

By Theorem 3.6.4, $x \in G_{u,v}^{(J)}$, so $\dot{v}^{-1}\kappa_x x \in G_0^{(J)}$. Let us factorize $y := \dot{v}^{-1}\kappa_x x$ as $y = [y]_-^{(J)}[y]_0[y]_+^{(J)}$ using Lemma 3.4.22(iii). By (3.7.13) and (3.7.15), we get

$$\dot{w}\dot{\tau}_\lambda^{-1}\dot{v}^{-1} \cdot \kappa_x x \cdot \dot{\tau}_\lambda \cdot x^{-1} = \dot{w} \cdot \dot{\tau}_\lambda^{-1}[y]_-^{(J)}\dot{\tau}_\lambda \cdot \dot{\tau}_\lambda^{-1}[y]_0\dot{\tau}_\lambda \cdot \dot{\tau}_\lambda^{-1}[y]_+^{(J)}\dot{\tau}_\lambda \cdot x^{-1} \in \dot{w} \cdot \mathcal{U}_-^{(I)} \cdot [y]_0 \cdot \mathcal{U}^{(I)} \cdot x^{-1}.$$

Using (3.7.6), we can move $\mathcal{U}_-^{(I)}$ to the left and $\mathcal{U}^{(I)}$ to the right, so we see that (3.7.24) is equivalent to $\dot{w}[y]_0 x^{-1} \in \mathcal{B}_-\mathcal{B}$. By Definition 3.6.1, we have $[y]_0 = \eta(x)$, and by Lemma 3.6.3(ii), we have $\zeta_{u,v}^{(J)}(x) = x\eta(x)^{-1} = x[y]_0^{-1}$. By Theorem 3.6.4, $\zeta_{u,v}^{(J)}(x) \in BB_-\dot{w}$, and after taking inverses, we obtain $\dot{w}[y]_0 x^{-1} \in B_-B \subset \mathcal{B}_-\mathcal{B}$, finishing the proof. $\square$

3.8 From Bruhat atlas to Fomin–Shapiro atlas

We use Theorem 3.7.2 to prove Theorem 3.2.5.

3.8.1 Affine Bruhat projections

We first define the affine flag variety version of the map $\bar{\nu}_g$ from (3.2.1). We will need some results on Gaussian decomposition inside $\mathcal{G}$, see Section 3.A.5 for a proof.

Lemma 3.8.1. *Let $\mathcal{G}_0 := \mathcal{B}_- \cdot \mathcal{B}$.*

(i) *The multiplication map gives a biregular isomorphism of ind-varieties:*

$$\mathcal{U}_- \times \mathcal{T} \times \mathcal{U} \xrightarrow{\sim} \mathcal{G}_0. \tag{3.8.1}$$

For $x \in \mathcal{G}_0$, we denote by $[x]_- \in \mathcal{U}_-$, $[x]_0 \in \mathcal{T}$, and $[x]_+ \in \mathcal{U}$ the unique elements such that $x = [x]_-[x]_0[x]_+$.

(ii) *For $g \in \tilde{W}$, the multiplication map gives biregular isomorphisms of ind-varieties:*

$$\mu_{12} : \mathcal{U}_1(g) \times \mathcal{U}_2(g) \xrightarrow{\sim} \dot{g}\mathcal{U}_-\dot{g}^{-1}, \quad \mu_{21} : \mathcal{U}_2(g) \times \mathcal{U}_1(g) \xrightarrow{\sim} \dot{g}\mathcal{U}_-\dot{g}^{-1}. \tag{3.8.2}$$

The group $\dot{g}\mathcal{U}_-\dot{g}^{-1}$, as well as its subgroups $\mathcal{U}_1(g)$ and $\mathcal{U}_2(g)$, act on $\mathcal{C}_g$. The following result, which we state for the polynomial loop group $\mathcal{G}$, holds in Kac-Moody generality.

Proposition 3.8.2. *Let $g \in \tilde{W}$.*

(i) *For $x \in \mathcal{G}$ such that $x\mathcal{B} \in \mathcal{C}_g$, there exist unique elements $y_1 \in \mathcal{U}_1(g)$ and $y_2 \in \mathcal{U}_2(g)$ such that $y_1 x\mathcal{B} \in \mathring{\mathcal{X}}_g$ and $y_2 x\mathcal{B} \in \mathring{\mathcal{X}}^g$.*

(ii) *The map $\tilde{\nu}_g : \mathcal{C}_g \xrightarrow{\sim} \mathring{\mathcal{X}}_g \times \mathring{\mathcal{X}}^g$ sending $x\mathcal{B} \mapsto (y_1 x\mathcal{B}, y_2 x\mathcal{B})$ is a biregular isomorphism of ind-varieties.*

(iii) *For all $h, f \in \tilde{W}$ satisfying $h \leq g \leq f$, the map $\tilde{\nu}_g$ restricts to a biregular isomorphism $\mathcal{C}_g \cap \mathring{\mathcal{R}}_h^f \xrightarrow{\sim} \mathring{\mathcal{R}}_g^f \times \mathring{\mathcal{R}}_h^g$ of finite-dimensional varieties.*

Proof. Let us first prove an affine analog of Lemma 3.4.2. Let $\nu_1 : \dot{g}\mathcal{U}_-\dot{g}^{-1} \to \mathcal{U}_2(g)$, $\nu_2 : \dot{g}\mathcal{U}_-\dot{g}^{-1} \to \mathcal{U}_1(g)$ denote the second component of μ_{12}^{-1} and μ_{21}^{-1} (cf. (3.8.2)), respectively, and let $\nu := (\nu_1, \nu_2) : \dot{g}\mathcal{U}_-\dot{g}^{-1} \to \mathcal{U}_2(g) \times \mathcal{U}_1(g)$. We claim that ν is a biregular isomorphism. By Lemma 3.8.1(ii), ν is a regular morphism. Let us now compute the inverse of ν. Given $x_1 \in \mathcal{U}_1(g)$ and $x_2 \in \mathcal{U}_2(g)$, we claim that there exist unique $y_1 \in \mathcal{U}_1(g)$ and $y_2 \in \mathcal{U}_2(g)$ such that $y_1 x_2 = y_2 x_1$. Indeed, this equation is equivalent to $y_2^{-1} y_1 = x_1 x_2^{-1}$, so we must have $y_2 = [x_1 x_2^{-1}]_-^{-1}$ and $y_1 = [x_1 x_2^{-1}]_+$. Clearly, $\nu^{-1}(x_2, x_1) = y_1 x_2 = y_2 x_1$, and by Lemma 3.8.1(i), the map ν^{-1} is regular. Applying (3.7.5) finishes the proof of (i) and (ii).

We now prove (iii). Observe that if $x\mathcal{B} \in \mathcal{C}_g \cap \mathring{\mathcal{R}}_h^f$ for some $h \leq f \in \tilde{W}$ then $x \in \mathcal{B}_-\dot{h}\mathcal{B} \cap \mathcal{B}\dot{f}\mathcal{B}$. Let y_1, y_2 be as in (ii). Then $y_1 \in \mathcal{U}_1(g) \subset \mathcal{U}$, so $y_1 x \in \mathcal{B}\dot{f}\mathcal{B}$. Similarly, $y_2 \in \mathcal{U}_2(g) \subset \mathcal{U}_-$, so $y_2 x \in \mathcal{B}_-\dot{h}\mathcal{B}$. It follows that if $x\mathcal{B} \in \mathcal{C}_g \cap \mathring{\mathcal{R}}_h^f$ then $\tilde{\nu}_g(x\mathcal{B}) \in \mathring{\mathcal{R}}_h^g \times \mathring{\mathcal{R}}_g^f$. In particular, we must have $h \leq g \leq f$, and we are done by (3.7.3). □

3.8.2 Torus action

Recall that $\mathcal{T} = \mathbb{C}^* \times T$ is the affine torus. The group $\mathbb{C}^*$ acts on $\mathcal{G}$ via loop rotation as follows. For $t \in \mathbb{C}^*$, we have $t \cdot g(z) = g(tz)$. We form the semidirect product $\mathbb{C}^* \ltimes \mathcal{G}$ with multiplication given by $(t_1, x_1(z)) \cdot (t_2, x_2(z)) := (t_1 t_2, x_1(z) x_2(t_1 z))$, for $(t_1, x_1(z)), (t_2, x_2(z)) \in \mathbb{C}^* \times \mathcal{G}$. Let $Y(\mathcal{T}) := \mathrm{Hom}(\mathbb{C}^*, \mathcal{T}) \cong \mathbb{Z}d \oplus Y(T)$. For $\lambda \in Y(\mathcal{T})$, $t \in \mathbb{C}^*$, $t' \in \mathbb{C}$, and $\alpha \in \Delta_{\mathrm{re}}$, we have

$$\lambda(t) x_\alpha(t') \lambda(t)^{-1} = x_\alpha(t^{\langle \lambda, \alpha \rangle} t'). \tag{3.8.3}$$

where $x_\alpha : \mathbb{C} \xrightarrow{\sim} \mathcal{U}_\alpha$ is as in Section 3.7.1, and $\langle \cdot, \cdot \rangle : Y(\mathcal{T}) \times X(\mathcal{T}) \to \mathbb{Z}$ extends the pairing from Section 3.4.1 in such a way that $\langle d, \delta \rangle = 1$ and $\langle d, \alpha_i \rangle = \langle \alpha_i^\vee, \delta \rangle = 0$ for $i \in I$.

Let $g \in \tilde{W}$ and denote $N := \ell(g)$. If $\mathrm{Inv}(g) = \{\alpha^{(1)}, \dots, \alpha^{(N)}\}$ then by Lemma 3.A.1, the map $\mathbf{x}_g : \mathbb{C}^N \to \mathcal{U}_1(g)$ given by

$$\mathbf{x}_g(t_1, \dots, t_N) := x_{\alpha^{(1)}}(t_1) \cdots x_{\alpha^{(N)}}(t_N) \tag{3.8.4}$$

is a biregular isomorphism. For $\mathbf{t} = (t_1, \dots, t_N) \in \mathbb{C}^N$, define $\|\mathbf{t}\| := (|t_1|^2 + \cdots + |t_N|^2)^{\frac{1}{2}} \in \mathbb{R}_{\geq 0}$, and let $\| \cdot \| : \mathcal{U}_1(g) \to \mathbb{R}_{\geq 0}$ be defined by $\|y\| := \|\mathbf{x}_g^{-1}(y)\|$. Identifying $\mathcal{U}_1(g)$ with $\mathring{\mathcal{X}}^g$ via (3.7.5), we get a function $\| \cdot \| : \mathring{\mathcal{X}}^g \to \mathbb{R}_{\geq 0}$.

We say that $\tilde{\rho} \in Y(\mathcal{T})$ is a *regular dominant integral coweight* if $\langle \tilde{\rho}, \delta \rangle \in \mathbb{Z}_{>0}$ and $\langle \tilde{\rho}, \alpha_i \rangle \in \mathbb{Z}_{>0}$ for all $i \in I$. In this case, we have $\langle \tilde{\rho}, \alpha \rangle \in \mathbb{Z}_{>0}$ for any $\alpha \in \Delta_{\mathrm{re}}^+$. Let us choose such a coweight $\tilde{\rho}$, and define $\vartheta_g : \mathbb{R}_{>0} \times \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{B}$ by $\vartheta_g(t, x\mathcal{B}) := \tilde{\rho}(t)x\mathcal{B}$.

It follows from (3.8.3) that if $g \in \tilde{W}$ and $y \in \mathcal{U}_1(g)$ is such that $\mathbf{x}_g^{-1}(y) = (t_1, \dots, t_N)$ then there exist $k_1, \dots, k_N \in \mathbb{Z}_{>0}$ satisfying

$$\|\vartheta_g(t, y\dot{g}\mathcal{B})\| = \left(t^{k_1}|t_1|^2 + \cdots + t^{k_N}|t_N|^2\right)^{\frac{1}{2}} \quad \text{for all } t \in \mathbb{R}_{>0}. \tag{3.8.5}$$

3.8.3 Proof of Theorem 3.2.5

By Corollary 3.4.20, $((G/P)_{\mathbb{R}}, (G/P)_{\geq 0}, Q_J)$ is a TNN space in the sense of Definition 3.2.1. Thus it suffices to construct a Fomin–Shapiro atlas.

Let $(u, u) \preceq (v, w) \in Q_J$, and denote $f := (u, u)$, $g := (v, w)$. Thus we have $\psi(f) = \tau_{u\lambda}$ and $\psi(g) = v\tau_\lambda w^{-1}$. Moreover, for the maximal element $\hat{1} = (\mathrm{id}, w^J) \in Q_J$, we have $\psi(\hat{1}) = \tau_\lambda^J$. By Theorem 3.7.2(1), the map $\bar{\varphi}_u$ gives an isomorphism $C_u^{(J)} \xrightarrow{\sim} \mathcal{X}_{\psi(\hat{1})} \cap \mathring{\mathcal{X}}^{\psi(f)}$. Let $\mathcal{O}_g^{\mathbb{C}} \subset C_u^{(J)}$ be the preimage of $\mathcal{C}_{\psi(g)} \cap \mathcal{X}_{\psi(\hat{1})} \cap \mathring{\mathcal{X}}^{\psi(f)}$ under $\bar{\varphi}_u$, and denote by $\mathcal{O}_g := \mathcal{O}_g^{\mathbb{C}} \cap (G/P)_{\mathbb{R}}$. Since $\mathcal{C}_{\psi(g)}$ is open in $\mathcal{G}/\mathcal{B}$, we see that $\mathcal{O}_g^{\mathbb{C}}$ is open in $C_u^{(J)}$ which is open in G/P, so $\mathcal{O}_g$ is an open subset of $(G/P)_{\mathbb{R}}$. By Theorem 3.7.2(2), $\mathcal{O}_g$ contains $\mathrm{Star}_g^{\geq 0}$, which shows (FS5). Moreover, we claim that $\mathcal{O}_g \subset \mathrm{Star}_g$. Indeed, if $h \succeq f$ but $h \not\succeq g$ then $\psi(h) \not\leq \psi(g)$. The map $\bar{\varphi}_u$ sends $\mathring{\Pi}_h \cap C_u^{(J)}$ to $\mathring{\mathcal{R}}_{\psi(h)}^{\psi(f)}$, which does not intersect $\mathcal{C}_{\psi(g)}$ by (3.A.3).

We now define the smooth cone (Z_g, ϑ_g). Throughout, we identify $\mathring{\mathcal{X}}^{\psi(g)}$ with $\mathbb{C}^{N_g}$ for

$N_g := \ell(\psi(g))$ via (3.8.4). We set $Z_g^{\mathbb{C}} := \mathcal{X}_{\psi(\hat{1})} \cap \mathring{\mathcal{X}}^{\psi(g)}$ and $\mathring{Z}_{g,h}^{\mathbb{C}} := \mathring{\mathcal{R}}_{\psi(h)}^{\psi(g)}$ for $g \preceq h \in Q_J$. We let $Z_g := Z_g^{\mathbb{C}} \cap \mathbb{R}^{N_g}$ and $\mathring{Z}_{g,h} := \mathring{Z}_{g,h}^{\mathbb{C}} \cap \mathbb{R}^{N_g}$ denote the corresponding sets of real points. Thus (FS1) follows trivially. The action ϑ_g restricts to $\mathbb{R}^{N_g}$, and by (3.8.5), it satisfies (SC2). As we discussed in Section 3.8.2, the action of ϑ_g also preserves Z_g (showing (SC1)) and $\mathring{Z}_{g,h}$ (showing (FS2)).

Finally, we define a map $\bar{\nu}_g : \mathcal{O}_g^{\mathbb{C}} \to (\mathring{\Pi}_g \cap \mathcal{O}_g^{\mathbb{C}}) \times \mathbb{C}^{N_g}$ as follows. Let $\tilde{\nu}_g = (\tilde{\nu}_{g,1}, \tilde{\nu}_{g,2}) : \mathcal{C}_g \xrightarrow{\sim} \mathring{\mathcal{X}}_g \times \mathring{\mathcal{X}}^g$ be the map from Proposition 3.8.2. We let $\bar{\nu}_{g,2} := \tilde{\nu}_{g,2} \circ \bar{\varphi}_u$, so it sends $\mathcal{O}_g^{\mathbb{C}} \to \mathcal{C}_{\psi(g)} \to \mathring{\mathcal{X}}^{\psi(g)} \cong \mathbb{C}^{N_g}$. By Proposition 3.8.2(iii), the image of $\bar{\nu}_{g,2}$ is precisely $Z_g^{\mathbb{C}}$. We also let $\bar{\nu}_{g,1} := \bar{\varphi}_u^{-1} \circ \tilde{\nu}_{g,1} \circ \bar{\varphi}_u$, thus it sends

$$\mathcal{O}_g^{\mathbb{C}} \xrightarrow{\sim} \mathcal{C}_{\psi(g)} \cap \mathcal{X}_{\psi(\hat{1})} \cap \mathring{\mathcal{X}}^{\psi(f)} \to \mathring{\mathcal{R}}_{\psi(g)}^{\psi(f)} \xrightarrow{\sim} \mathring{\Pi}_g \cap \mathcal{O}_g^{\mathbb{C}}.$$

It follows from Theorem 3.7.2(1) and Proposition 3.8.2 that $\bar{\nu}_g := (\bar{\nu}_{g,1}, \bar{\nu}_{g,2})$ gives a biregular isomorphism $\mathcal{O}_g^{\mathbb{C}} \xrightarrow{\sim} (\mathring{\Pi}_g \cap \mathcal{O}_g^{\mathbb{C}}) \times Z_g^{\mathbb{C}}$. All maps in the definition of $Z_g^{\mathbb{C}}$ are defined over $\mathbb{R}$, thus $\bar{\nu}_g$ gives a smooth embedding $\mathcal{O}_g \to (\mathring{\Pi}_g^{\mathbb{R}} \cap \mathcal{O}_g) \times \mathbb{R}^{N_g}$ with image $(\mathring{\Pi}_g^{\mathbb{R}} \cap \mathcal{O}_g) \times Z_g$. By Lemma 3.3.3, we find that Z_g is an embedded submanifold of $\mathbb{R}^{N_g}$, so we get a diffeomorphism

$$\bar{\nu}_g : \mathcal{O}_g \xrightarrow{\sim} (\mathring{\Pi}_g^{\mathbb{R}} \cap \mathcal{O}_g) \times Z_g.$$

By Theorem 3.7.2(1) and Proposition 3.8.2(iii), we find that for $h \succeq g$, $\bar{\nu}_g$ sends $\mathring{\Pi}_h \cap \mathcal{O}_g$ to $(\mathring{\Pi}_g \cap \mathcal{O}_g) \times \mathring{Z}_{g,h}$, showing (FS3). When $xP \in \mathring{\Pi}_g \cap \mathcal{O}_g$, we have $\bar{\varphi}_u(xP) \in \mathring{\mathcal{R}}_{\psi(g)}^{\psi(f)}$, so clearly $\tilde{\nu}_{g,1}(\bar{\varphi}_u(xP)) = \bar{\varphi}_u(xP)$ and $\tilde{\nu}_{g,2}(\bar{\varphi}_u(xP)) \in \mathring{\mathcal{R}}_{\psi(g)}^{\psi(g)}$. Thus $\bar{\nu}_{g,1}(xP) = x$ and $\bar{\nu}_{g,2}(xP) = 0$, showing (FS4). We have thus completed all requirements of Definitions 3.2.1, 3.2.2, and 3.2.3. □

3.9 The case $G = \mathrm{SL}_n$

In this section, we illustrate our construction in type A. We mostly focus on the case when G/P is the Grassmannian $\mathrm{Gr}(k, n)$ so that $(G/P)_{\geq 0}$ is the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k, n)$. Throughout, we assume $\mathbb{K} = \mathbb{C}$.

3.9.1 Preliminaries

Fix an integer $n \geq 1$ and recall the notation $[n] := \{1, 2, \ldots, n\}$, $\binom{[n]}{k} := \{S \subset [n] : |S| = k\}$.

Let $G = \mathrm{SL}_n$ be the group of $n \times n$ matrices over $\mathbb{C}$ of determinant 1. We have subgroups $B, B_-, T, U, U_- \subset G$ consisting of upper triangular, lower triangular, diagonal, upper unitriangular, and lower unitriangular matrices of determinant 1, respectively. The Weyl group W is the group S_n of permutations of $[n]$, and for $i \in I = [n-1]$, $s_i \in W$ is the simple transposition of elements i and $i+1$. If $w \in W$ is written as a product $w = s_{i_1} \ldots s_{i_l}$ then the action of w on $[n]$ is given by $w(j) := s_{i_1}(\ldots(s_{i_l}(j))\ldots)$ for $j \in [n]$. For $S \subset [n]$, we set $wS := \{w(j) \mid j \in S\}$. For example, if $n = 3$ and $w = s_2 s_1$ then $w(1) = 3$, $w(2) = 1$, $w(3) = 2$, and $w\{1,3\} = \{2,3\}$.

For $i \in [n-1]$, the homomorphism $\phi_i : \mathrm{SL}_2 \to G$ just sends a matrix $A \in \mathrm{SL}_2$ to the $n \times n$ matrix $\phi_i(A) \in \mathrm{SL}_n$ which has a 2×2 block equal to A in rows and columns $i, i+1$. Thus if $n = 3$ then $\dot{s}_1 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\dot{s}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$, and for $w = s_2 s_1$, $\dot{w} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$. In general, given $w \in S_n$, $\dot{w}$ contains a ± 1 in row $w(j)$ and column j for each $j \in [n]$, and the sign of this entry is -1 if and only if the number of ± 1's strictly below and to the left of it is odd. In other words, the $(w(j), j)$-th entry of $\dot{w}$ equals $(-1)^{\#\{i<j \mid w(i)>w(j)\}}$.

For $x \in \mathrm{SL}_n$, x^T is just the matrix transpose of x, and x^ι defined in (3.4.4) has (i,j)-th entry equal to the determinant of the submatrix obtained from x by deleting row j and column i.

For $i \in [n-1]$, the function $\Delta_i^{\mp} : \mathrm{SL}_n \to \mathbb{C}$ is the top-left $i \times i$ principal minor, while $\Delta_i^{\pm} : \mathrm{SL}_n \to \mathbb{C}$ is the bottom-right $i \times i$ principal minor. The subset $G_0^{\mp} = B_- B$ consists precisely of matrices $x \in \mathrm{SL}_n$ all of whose top-left principal minors are nonzero, in agreement with Lemma 3.4.21(iii). We denote $\Delta_n^{\mp}(x) = \Delta_n^{\pm}(x) := \det x = 1$.

3.9.2 Flag variety

The group B acts on $G = \mathrm{SL}_n$ by right multiplication, and G/B is the *complete flag variety in* $\mathbb{C}^n$. It consists of *flags* $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$ in $\mathbb{C}^n$ such that $\dim V_i = i$ for $i \in [n]$. For a matrix $x \in \mathrm{SL}_n$, the element $xB \in G/B$ gives rise to a flag $V_0 \subset V_1 \subset \cdots \subset V_n$ such that V_i is the span of columns $1, \ldots i$ of x. For $k \in [n]$, $S \in \binom{[n]}{k}$, and $x \in \mathrm{SL}_n$, we denote by Δ_S^{flag} the determinant of the $k \times k$ submatrix of x with row set S and column set $[k]$. Thus for each $k \in [n]$, we have a map $\Delta_k^{\mathrm{flag}} : G/B \to \mathbb{CP}^{\binom{n}{k}-1}$ sending xB to $(\Delta_S^{\mathrm{flag}}(x))_{S \in \binom{[n]}{k}}$. Here $\binom{[n]}{k}$ is identified with the set $W\omega_k$ from Lemma 3.4.21(iv).

3.9.3 Partial flag variety

For $J \subset [n]$, we have a parabolic subgroup $P \subset G$, and the partial flag variety G/P consists of *partial flags* $\{0\} = V_0 \subset V_{j_1} \subset \cdots \subset V_{j_l} \subset V_n = \mathbb{C}^n$, where $\{j_1 < \cdots < j_l\} := ([n-1] \setminus J)$ and $\dim V_{j_i} = j_i$ for $i \in [l]$. The projection $\pi_J : G/B \to G/P$ sends a flag $(V_0, V_1, \ldots, V_n)$ to $(V_0, V_{j_1}, \ldots, V_{j_l}, V_n)$. When $J = \emptyset$, we have $P = B$ and $G/P = G/B$. We will focus on the "opposite" special case.

Unless otherwise stated, we assume that $J = [n-1] \setminus \{k\}$ for some fixed $k \in [n-1]$.

In this case, G/P is the *(complex) Grassmannian* $\mathrm{Gr}(k,n)$, which we will identify with the space of $n \times k$ full rank matrices modulo column operations. Let us write matrices in SL_n in block form $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, where A is of size $k \times k$ and D is of size $(n-k) \times (n-k)$. For a matrix $x = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in \mathrm{SL}_n$, we denote by $[x| := \left[\begin{array}{c} A \\ \hline C \end{array}\right]$ the $n \times k$ submatrix consisting of the first k columns of x. Thus every $x \in \mathrm{SL}_n$ gives rise to an element xP of G/P which is a k-dimensional subspace $V_k \subset \mathbb{C}^n$ equal to the column span of $[x|$. The map Δ_k^{flag} in this case is the classical *Plücker embedding* $\Delta_k^{\mathrm{flag}} : \mathrm{Gr}(k,n) \hookrightarrow \mathbb{CP}^{\binom{n}{k}-1}$, cf. Section 2.2.

The set W^J from Section 3.4.6 consists of *Grassmannian permutations*: we have $w \in W^J$ if and only if $w = \mathrm{id}$ or every reduced word for w ends with s_k. Equivalently, $w \in W^J$ if and only if $w(1) < \cdots < w(k)$ and $w(k+1) < \cdots < w(n)$, so the map $w \mapsto w[k]$ gives a bijection $W^J \to \binom{[n]}{k}$. The parabolic subgroup W_J (generated by $\{s_j\}_{j \in J}$) consists of permutations $w \in S_n$ such that $w[k] = [k]$, and the longest element $w_J \in W_J$ is given by $(w_J(1), \ldots, w_J(n)) = (k, \ldots, 1, n, \ldots, k+1)$. The maximal element w^J of W^J is given by $(w^J(1), \ldots, w^J(n)) = (n-k+1, \ldots, n, 1, \ldots, n-k)$. We have

$$U_J = \left\{ \left[\begin{array}{c|c} U_k & 0 \\ \hline 0 & U_{n-k} \end{array}\right] \right\}, \quad U_-^{(J)} = \left\{ \left[\begin{array}{c|c} I_k & 0 \\ \hline C & I_{n-k} \end{array}\right] \right\}, \quad L_J = \left\{ \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right] \right\}, \quad P = \left\{ \left[\begin{array}{c|c} A & B \\ \hline 0 & D \end{array}\right] \right\},$$

where U_r is an $r \times r$ upper unitriangular matrix, I_r is an $r \times r$ identity matrix, $A \in \mathrm{SL}_k$, $D \in \mathrm{SL}_{n-k}$, and B, C are arbitrary $k \times (n-k)$ and $(n-k) \times k$ matrices, respectively.

3.9.4 Affine charts

We have $G_0^{(J)} := \{x \in G \mid \Delta_{[k]}^{\mathrm{flag}}(x) \neq 0\}$, and for $x = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in G_0^{(J)}$ (such that $\Delta_{[k]}^{\mathrm{flag}}(x) = \det A \neq 0$), the factorization $x = [x]_-^{(J)}[x]_0^{(J)}[x]_+^{(J)}$ from Lemma 3.4.22(iii) is given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} I_k & 0 \\ \hline CA^{-1} & I_{n-k} \end{array}\right] \cdot \left[\begin{array}{c|c} A & 0 \\ \hline 0 & D - CA^{-1}B \end{array}\right] \cdot \left[\begin{array}{c|c} I_k & A^{-1}B \\ \hline 0 & I_{n-k} \end{array}\right]. \tag{3.9.1}$$

The matrix $D - CA^{-1}B$ is called the *Schur complement of D in x*.

For $u \in W^J$, the set $C_u^{(J)} \subset G/P$ consists of elements xP such that $\Delta_{u[k]}^{\mathrm{flag}}(x) \neq 0$. The (inverse of the) isomorphism (3.4.31) essentially amounts to computing the reduced column echelon form of an $n \times k$ matrix: if $x \in G$ is such that $xP \in C_u^{(J)}$ is sent to $g^{(J)} \in \dot{u}U_-^{(J)}\dot{u}^{-1}$ via (3.4.31) then the $n \times k$ matrices $[x|$ and $\left[g^{(J)}\dot{u}\right|$ have the same column span, and the submatrix of $\left[g^{(J)}\dot{u}\right|$ with row set $u[k]$ is the $k \times k$ identity matrix. Let us say that an $n \times k$ matrix M is in *$u[k]$-echelon form* if its submatrix with row set $u[k]$ is the $k \times k$ identity matrix.

The matrices $g_1^{(J)}\dot{u}$ and $g_2^{(J)}\dot{u}$ from Definition 3.4.23 are obtained from $g^{(J)}\dot{u}$ simply by replacing some entries with 0. Explicitly, let $(M_{i,j}) := \left[g^{(J)}\dot{u}\right|$, $(M'_{i,j}) := \left[g_1^{(J)}\dot{u}\right|$, and $(M''_{i,j}) := \left[g_2^{(J)}\dot{u}\right|$ be the corresponding $n \times k$ matrices. Thus $M_{i,j} = \delta_{i,u(j)}$ for all $i \in u[k]$ and $j \in [k]$, and we have

$$M'_{i,j} = \begin{cases} M_{i,j}, & \text{if } i \leq u(j), \\ 0, & \text{otherwise,} \end{cases} \qquad M''_{i,j} = \begin{cases} M_{i,j}, & \text{if } i \geq u(j), \\ 0, & \text{otherwise,} \end{cases} \qquad \text{for all } i \in [n] \text{ and } j \in [k].$$

The operation $M \mapsto M'$, which we call *u-truncation*, will play an important role.

Example 3.9.1. Let $G/P = \mathrm{Gr}(2,4)$ and $u = s_3 s_2 \in W^J$, so $u[k] = \{1,4\}$. We have

$$x = g^{(J)}\dot{u} = \begin{bmatrix} 1 & & & \\ x_1 & x_2 & -1 & \\ x_3 & x_4 & & -1 \\ & 1 & & \end{bmatrix}, \quad [x| = \begin{bmatrix} 1 & \\ x_1 & x_2 \\ x_3 & x_4 \\ & 1 \end{bmatrix}, \quad \left[g_1^{(J)}\dot{u}\right| = \begin{bmatrix} 1 & \\ & x_2 \\ & x_4 \\ & 1 \end{bmatrix}, \quad \left[g_2^{(J)}\dot{u}\right| = \begin{bmatrix} 1 & \\ x_1 & \\ x_3 & \\ & 1 \end{bmatrix}.$$

3.9.5 Positroid varieties

We review the background on *positroid varieties* inside $\mathrm{Gr}(k,n)$, which were introduced in [KLS13], building on the work of Postnikov [Pos07]. Let $\tilde{S}_n$ be the group of *affine permutations*, i.e., bijections $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(i+n)=f(i)+n$ for all $i\in\mathbb{Z}$. We have a function $\mathrm{av}:\tilde{S}_n\to\mathbb{Z}$ sending f to $\mathrm{av}(f):=\frac{1}{n}\sum_{i=1}^n(f(i)-i)$, which is an integer for all $f\in\tilde{S}_n$. For $j\in\mathbb{Z}$, denote $\tilde{S}_{j,n}:=\{f\in\tilde{S}_n\mid \mathrm{av}(f)=j\}$. Every $f\in\tilde{S}_n$ is determined by the sequence $f(1),\dots,f(n)$, thus we write f in *window notation* as $f=[f(1),\dots,f(n)]$. For $\lambda\in\mathbb{Z}^n$, define $\tau_\lambda\in\tilde{S}_n$ by $\tau_\lambda:=[d_1,\dots,d_n]$, where $d_i=i+n\lambda_i$ for all $i\in[n]$. Let $\mathrm{Bound}(k,n)\subset\tilde{S}_{k,n}$ be the set of *bounded affine permutations*, which consists of all $f\in\tilde{S}_n$ satisfying $\mathrm{av}(f)=k$ and $i\leq f(i)\leq i+n$ for all $i\in\mathbb{Z}$. The subset $\tilde{S}_{0,n}$ is a Coxeter group with generators $s_1,\dots,s_{n-1},s_n=s_0$, where for $i\in[n]$, $s_i:\mathbb{Z}\to\mathbb{Z}$ sends $i\mapsto i+1$, $i+1\mapsto i$, and $j\mapsto j$ for all $j\not\equiv i,i+1 \pmod n$. We let $\leq$ denote the Bruhat order on $\tilde{S}_{0,n}$, and $\ell:\tilde{S}_{0,n}\to\mathbb{Z}_{\geq 0}$ denote the length function. The sets $\tilde{S}_{0,n}$ and $\tilde{S}_{k,n}$ are in bijection sending $(i\mapsto f(i))$ to $(i\mapsto f(i)+k)$, and thus we get a poset structure and a length function on $\tilde{S}_{k,n}$. When $f\leq g$, we write $g\leq^{\mathrm{op}} f$, and we will be interested in the poset $(\mathrm{Bound}(k,n),\leq^{\mathrm{op}})$, which has a unique maximal element $\tau_k:=[1+k,2+k,\dots,n+k]$. It is known that $\mathrm{Bound}(k,n)$ is a lower order ideal of $(\tilde{S}_{k,n},\leq^{\mathrm{op}})$. We fix $\lambda=1^k0^{n-k}:=(1,\dots,1,0,\dots,0)\in\mathbb{Z}^n$ (with k 1's). Then $\tau_\lambda=[1+n,\dots,k+n,k+1,\dots,n]$ is one of the $\binom{[n]}{k}$ minimal elements of $(\mathrm{Bound}(k,n),\leq^{\mathrm{op}})$. The group S_n is naturally a subset of $\tilde{S}_{0,n}$, and we have $\tau_k=\tau_\lambda(w^J)^{-1}=\tau_\lambda^J$, where τ_λ^J was introduced in Section 3.7.2.

Given an $n\times k$ matrix M and $i\in[n]$, we let M_i denote the i-th row of M. We extend this to all $i\in\mathbb{Z}$ in such a way that $M_{i+n}=(-1)^{k-1}M_i$ for all $i\in\mathbb{Z}$. Thus we view M as a periodic $\mathbb{Z}\times k$ matrix. (The sign $(-1)^{k-1}$ is chosen so that if $M\in\mathrm{Gr}_{\geq 0}(k,n)$, then the matrix with rows $M_i,\dots,M_{i+n-1}$ belongs to $\mathrm{Gr}_{\geq 0}(k,n)$ for all $i\in\mathbb{Z}$, see Section 3.9.11.) Every $n\times k$ matrix M of rank k gives rise to a map $f_M:\mathbb{Z}\to\mathbb{Z}$ sending $i\in\mathbb{Z}$ to the minimal $j\geq i$ such that M_i belongs to the linear span of $M_{i+1},\dots,M_j$. It is easy to see that $f_M\in\mathrm{Bound}(k,n)$ and f_M depends only on the column span of M. For $h\in\mathrm{Bound}(k,n)$, the *(open) positroid variety* $\mathring{\Pi}_h\subset\mathrm{Gr}(k,n)$ is the subset $\mathring{\Pi}_h:=\{M\in\mathrm{Gr}(k,n)\mid f_M=h\}$. Its Zariski closure inside $\mathrm{Gr}(k,n)$ is $\Pi_h=\bigsqcup_{g\leq^{\mathrm{op}}h}\mathring{\Pi}_g$.

For $h \in \mathrm{Bound}(k,n)$, define the *Grassmann necklace* $\mathcal{I}_h = (I_a)_{a\in\mathbb{Z}}$ of h by

$$I_a := \{h(i) \mid i < a,\ h(i) \geq a\} \quad \text{for } a \in \mathbb{Z}. \tag{3.9.2}$$

Then I_a is a k-element subset of $[a, a+n)$, where for $a \leq b \in \mathbb{Z}$ we set $[a,b) := \{a, a+1, \dots, b-1\}$. For $a \leq b \in \mathbb{Z}$ and $M \in \mathrm{Gr}(k,n)$, define $\mathrm{rk}(M; a, b)$ to be the rank of the submatrix of M with row set $[a,b)$. For $a, b \in \mathbb{Z}$ and $h \in \tilde{S}_n$, define $r_{a,b}(h) := \#\{i < a \mid h(i) \geq b\}$. We describe two well known alternative characterizations of open positroid varieties, see [KLS13].

Proposition 3.9.2. *Let $h \in \mathrm{Bound}(k,n)$ and let $\mathcal{I}_h = (I_a)_{a\in\mathbb{Z}}$ be its Grassmann necklace.*

(i) *The set $\mathring{\Pi}_h$ consists of all $M \in \mathrm{Gr}(k,n)$ such that for each $a \in \mathbb{Z}$, I_a is the lexicographically minimal k-element subset S of $[a, a+n)$ such that the rows $(M_i)_{i\in S}$ are linearly independent.*

(ii) *For $M \in \mathrm{Gr}(k,n)$, we have $M \in \mathring{\Pi}_h$ if and only if*

$$k - \mathrm{rk}(M; a, b) = r_{a,b}(h) \quad \textit{for all } a \leq b \in \mathbb{Z}. \tag{3.9.3}$$

We use *window notation* for Grassmann necklaces as well, i.e., we write $\mathcal{I}_h = [I_1, \dots, I_n]$.

Recall that we have fixed $\lambda = 1^k 0^{n-k} \in \mathbb{Z}^n$. For $(v,w) \in Q_J$, define $f_{v,w} \in \tilde{S}_n$ by

$$f_{v,w} := v\tau_\lambda w^{-1}. \tag{3.9.4}$$

Theorem 3.9.3 ([KLS13])**.** *The map $(v,w) \mapsto f_{v,w}$ gives a poset isomorphism*

$$(Q_J, \preceq) \xrightarrow{\sim} (\mathrm{Bound}(k,n), \leq^{\mathrm{op}}).$$

For $(v,w) \in Q_J$, we have $\mathring{\Pi}_{v,w} = \mathring{\Pi}_{f_{v,w}}$ and $\Pi_{v,w} = \Pi_{f_{v,w}}$ as subsets of $G/P = \mathrm{Gr}(k,n)$.

Example 3.9.4. There are n positroid varieties of codimension 1, each given by the condition $\Delta^{\mathrm{flag}}_{\{i-k+1,\dots,i\}} = 0$ for some $i \in [n]$. The top element $(\mathrm{id}, w^J) \in Q_J$ covers n elements, namely, (s_i, w^J) for $i \in [n-1]$ together with $(\mathrm{id}, s_{n-k}w^J)$. For $i \in [n-1]$, $v = s_i$, and $w = w^J$, we have $f_{v,w} = s_i\tau^J_\lambda$, which corresponds to the variety $\Delta^{\mathrm{flag}}_{\{i-k+1,\dots,i\}} = 0$. For the remaining pair $v = \mathrm{id}$, $w = s_{n-k}w^J$, we have $f_{v,w} = \tau^J_\lambda s_{n-k}$, which corresponds to the cell $\Delta^{\mathrm{flag}}_{\{n-k+1,\dots,n\}} = 0$.

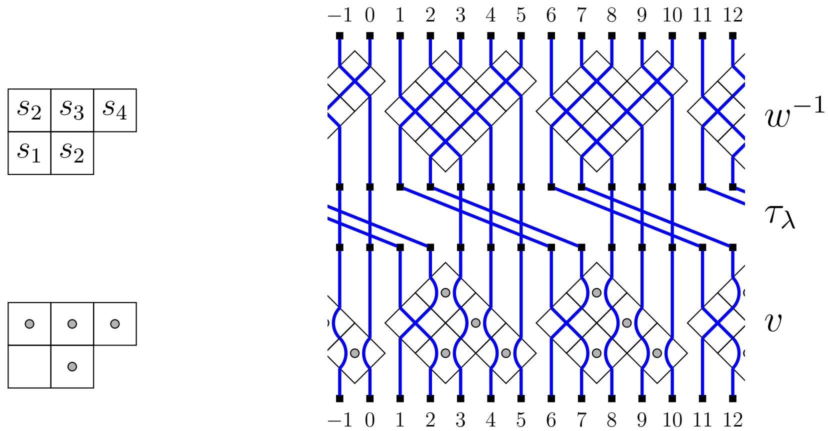


Figure 3-2: A Le diagram (bottom left) and the labeling of its squares by simple transpositions (top left). The result of applying the bijection of Theorem 3.9.3 (right). See Example 3.9.6 for details.

Example 3.9.5. It is easy to see directly from (3.9.4) and (3.9.2) that the first element of the Grassmann necklace of $f_{v,w}$ is $I_1 = v[k]$. Similarly, $w[k] = \{i \in [n] \mid f_{v,w}(i) > n\}$.

Example 3.9.6. Elements of $\mathrm{Bound}(k,n)$ and Q_J are in bijection with *Le diagrams* of [Pos07]. The bijection between Q_J and the set of Le diagrams is described in [Pos07, §19]: a pair $(v,w) \in Q_J$ gives rise to a Le diagram whose *shape* is a Young diagram inside a $k \times (n-k)$ rectangle, corresponding to the set $w[k]$. The squares of the Le diagram correspond to the terms in a reduced expression for w, as shown in Figure 3-2 (top left): the box with coordinates (i,j) in matrix notation is labeled by s_{k+j-i}. The terms in the positive subexpression for v inside w correspond to the squares of the Le diagram that are *not* filled with dots, see Figure 3-2 (bottom left). Thus the bijection of Theorem 3.9.3 can be pictorially represented as in Figure 3-2 (right). We refer to [Pos07, §19] or [Wil07, Appendix A] for the precise description. For the example in Figure 3-2, we have $v = s_1$, $w = s_2s_1s_4s_3s_2$, and $f_{v,w} = [3,4,7,5,6]$ in window notation, which is obtained by following the strands in Figure 3-2 (right) starting from the top.

3.9.6 Polynomial loop group

We explain how the construction in Section 3.7 applies to the case $G/P = \mathrm{Gr}(k,n)$. Recall that $\mathcal{A} := \mathbb{C}[z,z^{-1}]$. Let $\mathrm{GL}_n(\mathcal{A})$ denote the *polynomial loop group of* GL_n, consisting of

$n \times n$ matrices with entries in $\mathcal{A}$ whose determinant is a nonzero Laurent monomial in z, i.e., an invertible element of $\mathcal{A}$. (We use $\mathrm{GL}_n(\mathcal{A})$ instead of $\mathrm{SL}_n(\mathcal{A})$ as the constructions are combinatorially more elegant.) We have a group homomorphism $\mathrm{val} : \mathrm{GL}_n(\mathcal{A}) \to \mathbb{Z}$ sending $x \in \mathrm{GL}_n(\mathcal{A})$ to $j \in \mathbb{Z}$ such that $\det x = cz^{-j}$ for some $c \in \mathbb{C}^*$, and we let $\mathrm{GL}_n^{(j)}(\mathcal{A}) := \{x \in \mathrm{GL}_n(\mathcal{A}) \mid \mathrm{val}\, x = j\}$. The subgroups $\mathrm{GL}_n(\mathcal{A}_+)$ and $\mathrm{GL}_n(\mathcal{A}_-)$ are contained inside the group $\mathrm{GL}_n^{(0)}(\mathcal{A})$ of matrices whose determinant belongs to $\mathbb{C}^*$. We have subgroups $U(\mathcal{A}_+) := \bar{\mathrm{ev}}_0^{-1}(U)$, $U_-(\mathcal{A}_-) := \bar{\mathrm{ev}}_\infty^{-1}(U_-)$, $B(\mathcal{A}_+) := \bar{\mathrm{ev}}_0^{-1}(B)$ and $B_-(\mathcal{A}_-) := \bar{\mathrm{ev}}_\infty^{-1}(B_-)$ of $\mathrm{GL}_n^{(0)}(\mathcal{A})$. Thus in the notation of Section 3.7 for $G = \mathrm{SL}_n$, we have $\mathcal{G} = \mathrm{SL}_n(\mathcal{A}) \subsetneq \mathrm{GL}_n^{(0)}(\mathcal{A})$, $\mathcal{B} = \mathrm{SL}_n(\mathcal{A}) \cap B(\mathcal{A}_+) \subsetneq B(\mathcal{A}_+)$, $\mathcal{U} = U(\mathcal{A}_+)$, and $\mathcal{U}_- = U_-(\mathcal{A}_-)$.

To each matrix $x \in \mathrm{GL}_n(\mathcal{A})$, we associate a $\mathbb{Z} \times \mathbb{Z}$ matrix $\tilde{x} = (\tilde{x}_{i,j})_{i,j\in\mathbb{Z}}$ that is uniquely defined by the conditions

1. $\tilde{x}_{i,j} = \tilde{x}_{i+n,j+n}$ for all $i, j \in \mathbb{Z}$, and

2. the entry $x_{i,j}(z)$ equals the finite sum $\sum_{d\in\mathbb{Z}} \tilde{x}_{i,j+dn} z^d$ for all $i, j \in [n]$.

One can check that if $x = x_1x_2$, then $\tilde{x} = \tilde{x}_1\tilde{x}_2$. With this identification, the subgroups $\mathcal{U}$, $\mathcal{U}_-$, $B(\mathcal{A}_+)$, and $B_-(\mathcal{A}_-)$ have a very natural meaning. For example, $x \in \mathrm{GL}_n(\mathcal{A})$ belongs to $\mathcal{U}$ if and only if $\tilde{x}_{i,j} = 0$ for $i > j$ and $\tilde{x}_{i,i} = 1$ for all $i \in \mathbb{Z}$. Similarly, $B(\mathcal{A}_+)$ consists of all elements $x \in \mathrm{GL}_n(\mathcal{A})$ such that $\tilde{x}_{i,j} = 0$ for $i > j$ and $\tilde{x}_{i,i} \neq 0$ for all $i \in \mathbb{Z}$.

To each affine permutation $f \in \tilde{S}_{k,n}$, we associate an element $\dot{f} \in \mathrm{GL}_n(\mathcal{A})$ so that the corresponding $\mathbb{Z}\times\mathbb{Z}$ matrix $\tilde{f}$ satisfies $\tilde{f}_{i,j} = 1$ if $i = f(j)$ and $\tilde{f}_{i,j} = 0$ otherwise, for all $i, j \in \mathbb{Z}$. In other words, if for $i, j \in [n]$ there exists $d \in \mathbb{Z}$ such that $f(j) = i+dn$ then $\dot{f}_{i,j}(z) := z^{-d}$, and otherwise $\dot{f}_{i,j}(z) := 0$. Observe that $\mathrm{val}\, \dot{f} = k$ for all $f \in \tilde{S}_{k,n}$, thus $\dot{f} \in \mathrm{GL}_n^{(k)}(\mathcal{A})$. Recall that we have fixed $\lambda = 1^k 0^{n-k} \in \mathbb{Z}^n$. We obtain $\dot{\tau}_\lambda = \mathrm{diag}\left(\frac{1}{z}, \dots, \frac{1}{z}, 1, \dots, 1\right)$ with k entries equal to $\frac{1}{z}$, and for $u \in W^J$, we therefore get $\dot{\tau}_{u\lambda} = \mathrm{diag}(c_1, \dots, c_n)$, where $c_i = \frac{1}{z}$ for $i \in u[k]$ and $c_i = 1$ for $i \notin u[k]$.

3.9.7 Affine flag variety

The quotient $\mathrm{GL}_n^{(k)}(\mathcal{A})/B(\mathcal{A}_+)$ is isomorphic to the affine flag variety $\mathcal{G}/\mathcal{B}$ of Section 3.7 for the case $G = \mathrm{SL}_n$. Indeed, $\mathrm{GL}_n^{(0)}(\mathcal{A})$ acts simply transitively on $\mathrm{GL}_n^{(k)}(\mathcal{A})$ and we clearly have $\mathrm{GL}_n^{(0)}(\mathcal{A})/B(\mathcal{A}_+) \cong \mathcal{G}/\mathcal{B}$. For $f \leq^{\mathrm{op}} h \in \tilde{S}_{k,n}$ and $g \in \tilde{S}_{k,n}$, we have subsets $\mathring{\mathcal{X}}^f, \mathring{\mathcal{X}}_h, \mathring{\mathcal{R}}_h^f, \mathcal{C}_g \subset \mathrm{GL}_n^{(k)}(\mathcal{A})/B(\mathcal{A}_+)$ defined by

$$\mathring{\mathcal{X}}^f := B(\mathcal{A}_+) \cdot \dot{f} \cdot B(\mathcal{A}_+)/B(\mathcal{A}_+), \qquad \mathring{\mathcal{X}}_h := B_-(\mathcal{A}_-) \cdot \dot{h} \cdot B(\mathcal{A}_+)/B(\mathcal{A}_+),$$
$$\mathring{\mathcal{R}}_h^f := \mathring{\mathcal{X}}_h \cap \mathring{\mathcal{X}}^f, \qquad \mathcal{C}_g := \dot{g} \cdot B_-(\mathcal{A}_-) \cdot B(\mathcal{A}_+)/B(\mathcal{A}_+).$$

Let us now calculate the map φ_u from (3.7.11). Recall that it sends $xP \in C_u^{(J)}$ to $g_1^{(J)} \cdot \dot{\tau}_{u\lambda} \cdot (g_2^{(J)})^{-1}$. Assuming as before that $x = g^{(J)}\dot{u} \in \dot{u}U_-^{(J)}$, consider the corresponding $n \times k$ matrix $(M_{i,j}) := [x|$ in $u[k]$-echelon form.

Proposition 3.9.7. *The matrix $y := \varphi_u(xP) \in \mathrm{GL}_n^{(k)}(\mathcal{A})$ is given for all $i, j \in [n]$ by*

$$y_{i,j}(z) = \begin{cases} \delta_{i,j}, & \text{if } j \notin u[k], \\ -M_{i,s}, & \text{if } i > j \text{ and } j = u(s) \text{ for some } s \in [k], \\ \frac{M_{i,s}}{z}, & \text{if } i \leq j \text{ and } j = u(s) \text{ for some } s \in [k]. \end{cases} \tag{3.9.5}$$

Proof. This follows by directly computing the product $g_1^{(J)} \cdot \dot{\tau}_{u\lambda} \cdot (g_2^{(J)})^{-1}$. $\square$

Example 3.9.8. In the notation of Example 3.9.1, we have

$$y = g_1^{(J)} \cdot \dot{\tau}_{u\lambda} \cdot (g_2^{(J)})^{-1} = \begin{bmatrix} 1 & & & \\ & 1 & x_2 & \\ & & 1 & x_4 \\ & & & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{z} & & & \\ & 1 & & \\ & & 1 & \\ & & & \frac{1}{z} \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ -x_1 & 1 & & \\ -x_3 & & 1 & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{z} & & & \\ -x_1 & 1 & & \frac{x_2}{z} \\ -x_3 & & 1 & \frac{x_4}{z} \\ & & & \frac{1}{z} \end{bmatrix}. \tag{3.9.6}$$

Remark 3.9.9. The map $\bar{\varphi}_u : xP \mapsto g_1^{(J)} \cdot \dot{\tau}_{u\lambda} \cdot (g_2^{(J)})^{-1} \cdot B(\mathcal{A}_+)$ is a slight variation of a similar embedding of [Sni10] which we denote $\bar{\varphi}'_u$. We have $\bar{\varphi}'_u(xP) = g_1^{(J)} \cdot \dot{\tau}_{u\lambda} \cdot g_2^{(J)} \cdot B(\mathcal{A}_+)$, and the corresponding matrix $y' = \varphi'_u(xP) := g_1^{(J)} \cdot \dot{\tau}_{u\lambda} \cdot g_2^{(J)}$ is given by (3.9.5) except that $-M_{i,s}$ should be replaced by $M_{i,s}$. Thus y' is obtained from y by substituting $z \mapsto -z$ and then changing the signs of all columns in $u[k]$. In particular, y' and y are related by an element of the affine torus from Section 3.8.2. Proposition 3.9.14 below is due to Snider [Sni10]. Theorem 3.7.2(1) generalizes Snider's result to arbitrary G/P.

We give a standard convenient characterization of $\mathring{\mathcal{X}}_h$ using *lattices*. For each $x \in \mathrm{GL}_n(\mathcal{A})$ and column $a \in \mathbb{Z}$, we introduce a Laurent polynomial $x_a(t) \in \mathbb{C}[t, t^{-1}]$ defined by $x_a(t) :=$

$\sum_{i\in\mathbb{Z}} \tilde{x}_{i,a}t^i$, and an infinite-dimensional linear subspace $L_a(x) \subset \mathbb{C}[t,t^{-1}]$ given by $L_a(x) := \mathrm{Span}\{x_j(t) \mid j < a\}$, where Span denotes the space of all finite linear combinations. For $b \in \mathbb{Z}$, define another linear subspace $E_b \subset \mathbb{C}[t,t^{-1}]$ by $E_b := \mathrm{Span}\{t^i \mid i \geq b\}$. Finally, for $a, b \in \mathbb{Z}$, define $r_{a,b}(x) \in \mathbb{Z}$ to be the dimension of $L_a(x) \cap E_b$. In other words, $r_{a,b}(x)$ is the dimension of the space of $\mathbb{Z} \times 1$ vectors that have zeros in rows $b-1, b-2, \dots$ and can be obtained as finite linear combinations of columns $a-1, a-2, \dots$ of $\tilde{x}$. Recall from Section 3.9.5 that for $a, b \in \mathbb{Z}$ and $h \in \tilde{S}_n$, we denote $r_{a,b}(h) := \#\{i < a \mid h(i) \geq b\}$.

Lemma 3.9.10. *Let $x \in \mathrm{GL}_n^{(d)}(\mathcal{A})$ and $h \in \tilde{S}_{d,n}$ for some $d \in \mathbb{Z}$. Then*

$$x \cdot B(\mathcal{A}_+) \in \mathring{\mathcal{X}}_h \quad \text{if and only if} \quad r_{a,b}(x) = r_{a,b}(h) \quad \text{for all } a, b \in \mathbb{Z}. \tag{3.9.7}$$

Proof. It is clear that $r_{a,b}(x) = r_{a,b}(h)$ when $x = \dot{h}$. One can check that $r_{a,b}(y_-xy_+) = r_{a,b}(x)$ for all $x \in \mathrm{GL}_n^{(d)}(\mathcal{A})$, $y_- \in B_-(\mathcal{A}_-)$, $y_+ \in B(\mathcal{A}_+)$, and $a, b \in \mathbb{Z}$. This proves (3.9.7) since $\mathrm{GL}_n^{(d)}(\mathcal{A})/B(\mathcal{A}_+) = \bigsqcup_{h\in\tilde{S}_{d,n}} \mathring{\mathcal{X}}_h$ by (3.A.2). $\square$

Remark 3.9.11. A *lattice* $\mathcal{L}$ is usually defined (see e.g. [Kum02, §13.2.13]) to be a free $\mathbb{C}[[z]]$-submodule of $\mathbb{C}((t)) \cong \mathbb{C}((z))^n$ (where $z = t^n$) satisfying $\mathcal{L} \otimes_{\mathbb{C}[[z]]} \mathbb{C}((z)) \cong \mathbb{C}((z))^n$. The $\mathbb{C}[[z]]$-submodule generated by our $L_a(x)$ gives a lattice $\mathcal{L}_a(x)$ in the usual sense.

Definition 3.9.12. Suppose we are given an $n \times k$ matrix M in $u[k]$-echelon form. Recall that we have defined the row M_a for all $a \in \mathbb{Z}$ in such a way that $M_{a+n} = (-1)^{k-1}M_a$. For $a \in \mathbb{Z}$ and $j \in [k]$, denote by $\theta^u_{a,j} \in [a, a+n)$ the unique integer that is equal to $u(j)$ modulo n. Define the *u-truncation* $M^{\mathrm{tr}^a_u}$ of M to be the $[a, a+n) \times k$ matrix $M^{\mathrm{tr}^a_u} = (M^{\mathrm{tr}^a_u}_{i,j})$ such that for $i \in [a, a+n)$ and $j \in [k]$, the entry $M^{\mathrm{tr}^a_u}_{i,j}$ is equal to $M_{i,j}$ if $i \leq \theta^u_{a,j}$ and to 0 otherwise, see Example 3.9.18. Thus $M^{\mathrm{tr}^a_u}$ is obtained from the matrix with rows $M_a, \dots, M_{a+n-1}$ by setting an entry to 0 if it is below the corresponding ± 1 in the same column, and we label its rows by $a, \dots, a+n-1$ rather than by $1, \dots, n$. For example, if $x = g^{(J)}\dot{u}$ and $M = [x|$ then $M^{\mathrm{tr}^1_u} = \left[g^{(J)}_1\dot{u}\right|$, cf. Example 3.9.1.

Lemma 3.9.13. *Let $x = g^{(J)}\dot{u} \in \dot{u}U^{(J)}_-$, $M := [x|$, and $y := \varphi_u(xP)$. Then for all $a \in \mathbb{Z}$,* *the space* $L_a(y)$ *has a basis*

$$\{t^i \mid i < a\} \sqcup \{P_1(t), \dots, P_k(t)\}, \quad \textit{where} \quad P_s(t) := \sum_{i=a}^{a+n-1} M^{\mathrm{tr}^a_u}_{i,s} t^i \quad \textit{for } s \in [k]. \tag{3.9.8}$$

Proof. For a subset $S \subset \mathbb{Z}$, define $S + n\mathbb{Z} := \{j + in \mid j \in S,\ i \in \mathbb{Z}\}$. The space $L_a(y)$ is the span of $y_j(t)$ for all $j < a$. If $j \notin u[k] + n\mathbb{Z}$ then $y_j(t) = t^j$ by definition. If $j \in u[k] + n\mathbb{Z}$ then $y_{j-n}(t) = t^j + \sum_{j-n<i<j} c_i t^i$, where c_i is zero for $i \in u[k] + n\mathbb{Z}$. It follows that $L_a(y)$ contains t^i for all $i < a$. Moreover, the only indices $j < a$ such that $y_j(t) \notin \mathrm{Span}\{t^i \mid i < a\}$ are those that belong to $[a-n, a) \cap (u[k] + n\mathbb{Z})$. Let $j \in [a-n, a) \cap (u[k] + n\mathbb{Z})$ be such an index, and let $s \in [k]$ be the unique index such that $u(s) \in j + n\mathbb{Z}$. Then clearly $y_j(t) \pm P_s(t) \in \mathrm{Span}\{t^i \mid i < a\}$, where the sign depends on the parity of $\frac{j-u(s)}{n} \in \mathbb{Z}$. Thus $P_s(t) \in L_a(y)$ for all $s \in [k]$, and $L_a(y)$ is the span of $\{t^i \mid i < a\} \sqcup \{P_1(t), \dots, P_k(t)\}$. Since the Laurent polynomials $P_s(t)$ have different degrees, they must be linearly independent. □

We give an alternative proof of Theorem 3.7.2(1) for the case $G/P = \mathrm{Gr}(k,n)$.

Proposition 3.9.14. *For* $h \in \mathrm{Bound}(k,n)$ *such that* $\tau_{u\lambda} \leq^{\mathrm{op}} h$, *the map* $\bar{\varphi}_u$ *gives isomorphisms*

$$\bar{\varphi}_u : C_u^{(J)} \xrightarrow{\sim} \mathring{\mathcal{X}}^{\tau_{u\lambda}}, \quad \bar{\varphi}_u : C_u^{(J)} \cap \mathring{\Pi}_h \xrightarrow{\sim} \mathring{\mathcal{R}}_h^{\tau_{u\lambda}}.$$

Proof. It is clear from (3.9.5) that we have a biregular isomorphism $U_1^{(J)} \times U_2^{(J)} \xrightarrow{\sim} \mathcal{U}_1(\tau_{u\lambda})$ sending $(g_1^{(J)}, g_2^{(J)}) \mapsto g_1^{(J)} \cdot \dot{\tau}_{u\lambda}(g_2^{(J)})^{-1}\dot{\tau}_{u\lambda}^{-1}$. Thus the map $(g_1^{(J)}, g_2^{(J)}) \mapsto g_1^{(J)} \cdot \dot{\tau}_{u\lambda} \cdot (g_2^{(J)})^{-1} \cdot B(\mathcal{A}_+)$ gives a parametrization of $\mathring{\mathcal{X}}^{\tau_{u\lambda}}$, see (3.7.5). Since $C_u^{(J)} = \bigsqcup_{h \in \mathrm{Bound}(k,n)} (C_u^{(J)} \cap \mathring{\Pi}_h)$, let us fix $h \in \mathrm{Bound}(k,n)$ and $x = g^{(J)}\dot{u} \in \dot{u}U_-^{(J)}$. Denote $M := [x|$ and $y := \varphi_u(xP)$. By (3.9.3), we have $M \in \mathring{\Pi}_h$ if and only if $k - \mathrm{rk}(M; a, b) = r_{a,b}(h)$ for all $a \leq b \in \mathbb{Z}$. By (3.9.7), we have $y \cdot B(\mathcal{A}_+) \in \mathring{\mathcal{X}}_h$ if and only if $r_{a,b}(y) = r_{a,b}(h)$ for all $a, b \in \mathbb{Z}$. If $a > b$ then $r_{a,b}(y) = r_{a,b+1}(y) + 1$ by (3.9.8) and $r_{a,b}(h) = r_{a,b+1}(h) + 1$ since $h \in \mathrm{Bound}(k,n)$ satisfies $h^{-1}(b) \leq b$, so $h^{-1}(b) < a$. We have shown that $y \cdot B(\mathcal{A}_+) \in \mathring{\mathcal{X}}_h$ if and only if $r_{a,b}(y) = r_{a,b}(h)$ for all $a \leq b \in \mathbb{Z}$. Thus it suffices to show

$$r_{a,b}(y) + \mathrm{rk}(M; a, b) = k \quad \text{for all } a \leq b \in \mathbb{Z}. \tag{3.9.9}$$

By (3.9.8), $r_{a,b}(y)$ is the dimension of $\operatorname{Span}\{P_1(t), \dots, P_k(t)\} \cap E_b$. By the rank-nullity theorem, $k - r_{a,b}(y)$ is the rank of the submatrix of $M^{\mathrm{tr}_u^a}$ with row set $[a, b)$, which is obtained by downward row operations from the submatrix of M with row set $[a, b)$. This shows (3.9.9). $\square$

Remark 3.9.15. By Theorem 3.7.2(1), the image of $\bar{\varphi}_u$ is $\mathcal{X}_{\tau_\lambda^J} \cap \mathring{\mathcal{X}}^{\tau_u\lambda}$, where $\tau_\lambda^J = \tau_\lambda(w^J)^{-1}$. But recall from Section 3.9.5 that $\tau_\lambda(w^J)^{-1} = \tau_k$, and since $\mathring{\mathcal{X}}_{\tau_k}$ is dense in $\mathrm{GL}_n^{(k)}(\mathcal{A})/B(\mathcal{A}_+)$, we find that $\mathcal{X}_{\tau_\lambda^J} \cap \mathring{\mathcal{X}}^{\tau_u\lambda} = \mathring{\mathcal{X}}^{\tau_u\lambda}$.

Example 3.9.16. Suppose that $x = g^{(J)}\dot{u}$ is given in Example 3.9.1, then $y = \varphi_u(xP)$ is the matrix from Example 3.9.8. It is clear that $y \in B(\mathcal{A}_+) \cdot \dot{\tau}_{u\lambda}$ regardless of the values of x_1, x_2, x_3, x_4, and therefore $y \cdot B(\mathcal{A}_+)$ belongs to $\mathring{\mathcal{X}}^{\tau_u\lambda}$. We can try to factorize y as an element of $B_-(\mathcal{A}_-) \cdot \dot{\tau}_k \cdot B(\mathcal{A}_+)$:

$$y = \begin{bmatrix} 1 & & \frac{x_2}{(x_1x_4 - x_2x_3)z} & \frac{x_4}{x_3 z} \\ & 1 & & \\ & \frac{x_4}{x_2} & 1 & \\ & \frac{1}{x_2} & \frac{x_1}{x_1x_4 - x_2x_3} & 1 \end{bmatrix} \cdot \begin{bmatrix} & & \frac{1}{z} & \\ & & & \frac{1}{z} \\ 1 & & & \\ & 1 & & \end{bmatrix} \cdot \begin{bmatrix} \frac{x_1x_4 - x_2x_3}{x_2} & -\frac{x_4}{x_2} & 1 & \\ & \frac{x_3}{x_1x_4 - x_2x_3} & -\frac{x_1}{x_1x_4 - x_2x_3} & \\ & & \frac{1}{x_3} & \\ -x_1 z & z & & x_2 \end{bmatrix}.$$

This factorization makes sense only when all denominators on the right-hand side are nonzero, which shows that $y \cdot B(\mathcal{A}_+) \in \mathring{\mathcal{R}}_{\tau_k}^{\tau_u\lambda}$ whenever the minors $\Delta_{12}^{\mathrm{flag}}(x) = x_2$, $\Delta_{23}^{\mathrm{flag}} = x_1x_4 - x_2x_3$, and $\Delta_{34}^{\mathrm{flag}} = x_3$ are nonzero. Observe also that $\Delta_{14}^{\mathrm{flag}}(x) = 1$. Thus $y \cdot B(\mathcal{A}_+) \in \mathring{\mathcal{R}}_{\tau_k}^{\tau_u\lambda}$ precisely when $xP \in \mathring{\Pi}_{\tau_k}$, where $\tau_k = [3, 4, 5, 6]$ in window notation. If $x_2 = 0$ then $xP \in \mathring{\Pi}_h$ for $h = [2, 4, 5, 7]$. In this case, we have

$$\dot{h} = \begin{bmatrix} & & \frac{1}{z} & \\ 1 & & & \\ & & & \frac{1}{z} \\ & 1 & & \end{bmatrix}, \quad y|_{x_2=0} = \begin{bmatrix} 1 & -\frac{1}{x_1 z} & & \frac{x_4}{x_3 z} \\ & 1 & & \\ & & 1 & \\ & -\frac{x_3}{x_1x_4} & \frac{1}{x_4} & 1 \end{bmatrix} \cdot \begin{bmatrix} & & \frac{1}{z} & \\ 1 & & & \\ & & & \frac{1}{z} \\ & 1 & & \end{bmatrix} \cdot \begin{bmatrix} -x_1 & 1 & & \\ & \frac{x_3}{x_1x_4} & -\frac{1}{x_4} & \\ & & \frac{1}{x_3} & \\ -x_3 z & & z & x_4 \end{bmatrix}.$$

Therefore $y|_{x_2=0}$ belongs to $\mathring{\mathcal{R}}_h^{\tau_u\lambda}$ whenever $x_1, x_3, x_4 \neq 0$. Observe that the Grassmann necklace of h is given by $\mathcal{I}_h = [\{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}]$ in window notation, and the corresponding flag minors of $x|_{x_2=0}$ are given by $\Delta_{13}^{\mathrm{flag}} = x_4$, $\Delta_{23}^{\mathrm{flag}} = x_1x_4$, $\Delta_{34}^{\mathrm{flag}} = x_3$, $\Delta_{14}^{\mathrm{flag}} = 1$, in agreement with Proposition 3.9.14.

3.9.8 Preimage of $\mathcal{C}_g$

For this section, we fix $\tau_{u\lambda} \leq^{\mathrm{op}} g \in \mathrm{Bound}(k,n)$. We would like to understand the preimage of $\left(\mathring{\mathcal{X}}^{\tau_{u\lambda}} \cap \mathcal{C}_g\right) \subset \mathrm{GL}_n^{(k)}(\mathcal{A})/B(\mathcal{A}_+)$ under the map $\bar{\varphi}_u$. For a set $S \subset [a, a+n)$ of size k, define $\Delta_S^{\mathrm{tr}_u^a}(M)$ to be the determinant of the $k \times k$ submatrix of $M^{\mathrm{tr}_u^a}$ with row set S. Let $\mathcal{I}_g = (I_a)_{a\in\mathbb{Z}}$ be the Grassmann necklace of g.

Proposition 3.9.17. *Suppose that $xP \in C_u^{(J)}$ and let $M := \left[g^{(J)}\dot{u}\right|$. Then $\bar{\varphi}_u(xP) \in \mathcal{C}_g$ if and only if $\Delta_{I_a}^{\mathrm{tr}_u^a}(M) \neq 0$ for all $a \in [n]$.*

Proof. Let $h \in \tilde{S}_n$ be the unique element such that $\dot{g}^{-1}\bar{\varphi}_u(xP)$ belongs to $\mathring{\mathcal{X}}_h$, thus $\bar{\varphi}_u(xP) \in \mathcal{C}_g$ if and only if $h = \mathrm{id}$. Since $\mathrm{val}\,\varphi_u(xP) = k$ and $\mathrm{val}\,\dot{g}^{-1} = -k$, we get $h \in \tilde{S}_{0,n}$. Hence $h = \mathrm{id}$ if and only if $r_{a,a}(h) = 0$ for all $a \in \mathbb{Z}$. Let $y := \varphi_u(xP)$ and $y' := \dot{g}^{-1}y$. Then for $a \in \mathbb{Z}$, we get $L_a(y') = g^{-1}L_a(y)$, where g^{-1} acts on $\mathbb{C}[t,t^{-1}]$ by a linear map sending $t^j \mapsto t^{g^{-1}(j)}$. In particular, $L_a(y') \cap E_a = (g^{-1}L_a(y)) \cap E_a$ has the same dimension as $L_a(y) \cap gE_a$. Let us denote $H_a := \{t^i \mid i \geq a\}$, so $E_a = \mathrm{Span}(H_a)$ and $gE_a = \mathrm{Span}(gH_a)$. Since $g(i) \geq i$ for all $i \in \mathbb{Z}$, it follows from (3.9.2) that $gH_a = H_a \setminus \{t^j\}_{j\in I_a}$. Therefore by (3.9.8), $L_a(y) \cap gE_a = \{0\}$ if and only if $\mathrm{Span}\{P_j(t)\}_{j\in[k]} \cap \mathrm{Span}\left(H_a \setminus \{t^j\}_{j\in I_a}\right) = \{0\}$, which happens precisely when the submatrix of $M^{\mathrm{tr}_u^a}$ with row set I_a is nonsingular, i.e., $\Delta_{I_a}^{\mathrm{tr}_u^a}(M) \neq 0$. $\square$

Example 3.9.18. Suppose that x is the matrix from Example 3.9.1, then $y := \varphi_u(xP)$ is given in Example 3.9.8. We have

$$M = \begin{bmatrix} 1 & \\ x_1 & x_2 \\ x_3 & x_4 \\ & 1 \end{bmatrix}, \quad M^{\mathrm{tr}_u^1} = \begin{bmatrix} 1 & \\ & x_2 \\ & x_4 \\ & 1 \end{bmatrix}, \quad M^{\mathrm{tr}_u^2} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \\ & 1 \\ -1 & \end{bmatrix}, \quad M^{\mathrm{tr}_u^3} = \begin{bmatrix} x_3 & x_4 \\ & 1 \\ -1 & \\ & \end{bmatrix}, \quad M^{\mathrm{tr}_u^4} = \begin{bmatrix} & 1 \\ -1 & \\ & \\ & \end{bmatrix}.$$

Suppose that $g = [2,4,5,7]$ as in Example 3.9.16, then its Grassmann necklace is $\mathcal{I}_g = [\{1,3\},\{2,3\},\{3,4\},\{4,5\}]$ in window notation. This gives

$$\Delta_{13}^{\mathrm{tr}_u^1}(M) = x_4, \quad \Delta_{23}^{\mathrm{tr}_u^2}(M) = x_1x_4 - x_2x_3, \quad \Delta_{34}^{\mathrm{tr}_u^3}(M) = x_3, \quad \Delta_{45}^{\mathrm{tr}_u^4}(M) = 1. \tag{3.9.10}$$

On the other hand, recall from Example 3.9.16 that $\dot{g} = \begin{bmatrix} & & \frac{1}{z} & \\ 1 & & & \\ & & & \frac{1}{z} \\ & 1 & & \end{bmatrix}$. Since $y \in \mathcal{C}_g$ if and only if $\dot g^{-1}y \in B_-(\mathcal{A}_-)\cdot B(\mathcal{A}_+)$, we can factorize it as

$$\dot g^{-1}y = \begin{bmatrix} -x_1 & 1 & & \frac{x_2}{z} \\ & & & \frac{1}{z} \\ 1 & & & \\ -x_3z & & z & x_4 \end{bmatrix} = \begin{bmatrix} 1 & & & \frac{x_2}{x_4z} \\ -\frac{x_3}{x_1x_4-x_2x_3} & 1 & & \frac{1}{x_4z} \\ -\frac{x_4}{x_1x_4-x_2x_3} & \frac{x_4}{x_3} & 1 & \\ & & & 1 \end{bmatrix}\cdot\begin{bmatrix} -\frac{x_1x_4-x_2x_3}{x_4} & 1 & -\frac{x_2}{x_4} & \\ & \frac{x_3}{x_1x_4-x_2x_3} & -\frac{x_1}{x_1x_4-x_2x_3} & \\ & & \frac{1}{x_3} & \\ -x_3z & & z & x_4 \end{bmatrix}. \tag{3.9.11}$$

Again, this is valid only when the denominators in the right-hand side are nonzero. We thus see that $\dot g^{-1}y$ belongs to $B_-(\mathcal{A}_-)\cdot B(\mathcal{A}_+)$ precisely when all minors in (3.9.10) are nonzero, in agreement with Proposition 3.9.17.

3.9.9 Fomin–Shapiro atlas

The computation in (3.9.11) can now be used to find the maps $\bar\nu_g$ and ϑ_g. As in Section 3.8.3, denote by $\mathcal{O}_g \subset C_u^{(J)}$ the preimage of $\mathcal{C}_g \cap \mathring{\mathcal{X}}^{\tau_u\lambda}$ under $\bar\varphi_u$. Thus for our running example, $\mathcal{O}_g$ is the subset of $C_u^{(J)}$ where all minors in (3.9.10) are nonzero. We are interested in the map $\bar\nu_g = (\bar\nu_{g,1}, \bar\nu_{g,2}) : \mathcal{O}_g \to (\mathring{\Pi}_g \cap \mathcal{O}_g) \times Z_g$ from (3.2.1), defined in Section 3.8.3. The first component is $\bar\nu_{g,1} = \bar\varphi_u^{-1} \circ \tilde\nu_{g,1} \circ \bar\varphi_u$, where $\tilde\nu_g : \mathcal{C}_g \cap \mathring{\mathcal{X}}^{\tau_u\lambda} \xrightarrow{\sim} \mathring{\mathcal{R}}_g^{\tau_u\lambda} \times \mathring{\mathcal{X}}^g$ is the map from Proposition 3.8.2(ii). In order to compute it, we consider the factorization $\dot g^{-1}y = y_-\cdot y_+ \in \mathcal{U}_-\cdot B(\mathcal{A}_+)$ from (3.9.11). The group $\mathcal{U}_1(g)$ is 1-dimensional since $\ell(g) = 1$, and the corresponding element $y_1 \in \mathcal{U}_1(g)$ from Proposition 3.8.2(ii) can be computed by factorizing $\dot g y_- \dot g^{-1}$ as an element of $\mathcal{U}_1(g)\cdot\mathcal{U}_2(g)$:

$$\dot g y_- \dot g^{-1} = \begin{bmatrix} 1 & -\frac{x_4}{(x_1x_4-x_2x_3)z} & & \frac{x_4}{x_3z} \\ & 1 & \frac{x_2}{x_4} & \\ & & 1 & \\ & -\frac{x_3}{x_1x_4-x_2x_3} & \frac{1}{x_4} & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & \frac{x_2}{x_4} & \\ & & 1 & \\ & & & 1 \end{bmatrix}\cdot\begin{bmatrix} 1 & -\frac{x_4}{(x_1x_4-x_2x_3)z} & & \frac{x_4}{x_3z} \\ & 1 & & \\ & & 1 & \\ & -\frac{x_3}{x_1x_4-x_2x_3} & \frac{1}{x_4} & 1 \end{bmatrix}, \quad y_1 = \begin{bmatrix} 1 & & & \\ & 1 & -\frac{x_2}{x_4} & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

Therefore the map $\tilde\nu_{g,1}$ sends $y \cdot B(\mathcal{A}_+)$ from (3.9.6) to

$$y_1y \cdot B(\mathcal{A}_+) = \begin{bmatrix} \frac{1}{z} & & & \\ -\frac{x_1x_4-x_2x_3}{x_4} & 1 & -\frac{x_2}{x_4} & \\ -x_3 & & 1 & \frac{x_4}{z} \\ & & & \frac{1}{z} \end{bmatrix}\cdot B(\mathcal{A}_+) = \begin{bmatrix} \frac{1}{z} & & & \\ -\frac{x_1x_4-x_2x_3}{x_4} & 1 & & \\ -x_3 & & 1 & \frac{x_4}{z} \\ & & & \frac{1}{z} \end{bmatrix}\cdot B(\mathcal{A}_+).$$

Applying $\bar{\varphi}_u^{-1}$ to the right-hand side, we see that the map $\bar{\nu}_{g,1}$ is given by

$$\bar{\nu}_{g,1} : \mathcal{O}_g \to \mathring{\Pi}_g \cap \mathcal{O}_g, \qquad \begin{bmatrix} 1 & \\ x_1 & x_2 \\ x_3 & x_4 \\ & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \\ \frac{x_1x_4 - x_2x_3}{x_4} & \\ x_3 & x_4 \\ & 1 \end{bmatrix}.$$

Similarly, factorizing $\dot{g}y_-\dot{g}^{-1}$ as an element of $\mathcal{U}_2(g) \cdot \mathcal{U}_1(g)$, we find that

$$\tilde{\nu}_{g,2}(y \cdot B(\mathcal{A}_+)) = y_2 y \cdot B(\mathcal{A}_+) = \begin{bmatrix} 1 & & & \\ & 1 & \frac{x_2}{x_4} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \dot{g} \cdot B(\mathcal{A}_+).$$

We have $N_g = \ell(g) = 1$, and the map $\bar{\nu}_{g,2} : \mathcal{O}_g \to Z_g = \mathbb{R}$ sends $\begin{bmatrix} 1 & \\ x_1 & x_2 \\ x_3 & x_4 \\ & 1 \end{bmatrix} \mapsto \frac{x_2}{x_4}$.

Torus action

We compute the maps from Section 3.8.2. Let $\tilde{\rho} \in Y(\mathcal{T})$ denote the group homomorphism $\tilde{\rho} : \mathbb{C}^* \to \mathbb{C}^* \times T$ sending $t \mapsto \tilde{\rho}(t) := (t^n, \operatorname{diag}(t^{n-1}, \ldots, t, 1))$. If $x \in \mathrm{GL}_n(\mathcal{A})$ is represented by a $\mathbb{Z} \times \mathbb{Z}$ matrix $(\tilde{x}_{i,j})$ then the element $y := \tilde{\rho}(t) x \tilde{\rho}(t)^{-1} \in \mathrm{GL}_n(\mathcal{A})$ satisfies $\tilde{y}_{i,j} = t^{j-i}\tilde{x}_{i,j}$ for all $i, j \in \mathbb{Z}$.

Example 3.9.19. Continuing the above example, we find that

$$\tilde{\rho}(t) \cdot y_2 y \cdot \tilde{\rho}(t)^{-1} \cdot B(\mathcal{A}_+) = \begin{bmatrix} 1 & & & \\ & 1 & \frac{tx_2}{x_4} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \cdot \dot{g} \cdot B(\mathcal{A}_+), \quad \text{and} \quad \|y_2 y \cdot B(\mathcal{A}_+)\| = \frac{|x_2|}{|x_4|}.$$

Thus the action of ϑ_g on Z_g is given by $\vartheta_g\left(t, \frac{x_2}{x_4}\right) = \frac{tx_2}{x_4}$. The pullback of this action to $\mathcal{O}_g \subset C_u^{(J)}$ via $\bar{\nu}_g^{-1}$ preserves x_3, x_4, and $x_1x_4 - x_2x_3$ (since it preserves $\bar{\nu}_{g,1}(x)$), but multiplies $\frac{x_2}{x_4}$ by t. Therefore it is given by

$$\bar{\nu}_g^{-1} \circ (\mathrm{id} \times \vartheta_g(t, \cdot)) \circ \bar{\nu}_g : \mathcal{O}_g \to \mathcal{O}_g, \qquad \begin{bmatrix} 1 & \\ x_1 & x_2 \\ x_3 & x_4 \\ & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & \\ x_1 + (t-1)\frac{x_2x_3}{x_4} & tx_2 \\ x_3 & x_4 \\ & 1 \end{bmatrix}.$$

3.9.10 The maps κ and $\zeta_{u,v}^{(J)}$

The subset $\dot{u}G_0^{(J)}$ consists of matrices $x \in G$ such that $\Delta_{u[k]}^{\mathrm{flag}}(x) \neq 0$. Suppose that $x = g^{(J)}\dot{u} \in \dot{u}U_-^{(J)}$. Then the elements $g_1^{(J)}\dot{u}$ and $g_2^{(J)}\dot{u}$ are obtained from x by setting some entries to zero, see Section 3.9.4. The map $x \mapsto \kappa_x x$ from Definition 3.4.23 sends $x = g^{(J)}\dot{u}$

to $g_1^{(J)}\dot{u}$, e.g., if $[x| = \begin{bmatrix} 1 & \\ x_1 & x_2 \\ x_3 & x_4 \\ & 1 \end{bmatrix}$ then $[\kappa_x x| = \begin{bmatrix} 1 & \\ & x_2 \\ & x_4 \\ & 1 \end{bmatrix}$ as in Example 3.9.1. Comparing this to Section 3.9.8, we see that if $M = [x|$ is in $u[k]$-echelon form then $[\kappa_x x|$ is the u-truncation $M^{\mathrm{tr}_u^1}$.

Let now $(v, w) \in Q_J^{\succeq(u,u)}$, thus $\tau_{u\lambda} \leq^{\mathrm{op}} g := f_{v,w}$, and denote $\mathcal{I}_g := (I_a)_{a\in\mathbb{Z}}$. The set $G_{u,v}^{(J)}$ from (3.6.1) consists of $x \in G$ such that $\Delta_{u[k]}^{\mathrm{flag}}(x) \neq 0$ and $\Delta_{v[k]}^{\mathrm{flag}}(\kappa_x x) \neq 0$. But recall from Example 3.9.5 that $v[k] = I_1$. Thus

$$G_{u,v}^{(J)} = \left\{x \in G \mid \Delta_{u[k]}^{\mathrm{flag}}(x) \neq 0 \text{ and } \Delta_{I_1}^{\mathrm{tr}_u^1}(M) \neq 0\right\}, \quad \text{where } M := \left[g^{(J)}\dot{u}\right|. \tag{3.9.12}$$

Example 3.9.20. We compute the maps κ and $\zeta_{u,v}^{(J)}$ for our running example. Suppose that $x = g^{(J)}\dot{u}$ is given in Example 3.9.1, and let $g = [2, 4, 5, 7]$ as in Example 3.9.18. Then $g = s_2\tau_k$, so under the correspondence (3.9.4), we have $g = f_{v,w}$ for $v = s_2$ and $w = w^J = s_2s_1s_3s_2$, see also Example 3.9.4. Since $v[k] = I_1 = \{1, 3\}$, we see that $x \in G_{u,v}^{(J)}$ whenever $x_4 \neq 0$. We have just computed that $[\kappa_x x| = \begin{bmatrix} 1 & \\ & x_2 \\ & x_4 \\ & 1 \end{bmatrix}$, thus $\dot{v}^{-1}\kappa_x x = \begin{bmatrix} 1 & & & \\ & x_4 & & -1 \\ & -x_2 & 1 & \\ & 1 & & \end{bmatrix}$. Factorizing it as an element of $U_-^{(J)} \cdot L_J \cdot U^{(J)}$ via (3.9.1), we get

$$\dot{v}^{-1}\kappa_x x = \begin{bmatrix} 1 & & & \\ & x_4 & & -1 \\ & -x_2 & 1 & \\ & 1 & & \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -\frac{x_2}{x_4} & 1 & \\ & \frac{1}{x_4} & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & x_4 & & \\ & & 1 & -\frac{x_2}{x_4} \\ & & & \frac{1}{x_4} \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & 1 & & -\frac{1}{x_4} \\ & & 1 & \\ & & & 1 \end{bmatrix}, \quad [\dot{v}^{-1}\kappa_x x]_J = \begin{bmatrix} 1 & & & \\ & x_4 & & \\ & & 1 & -\frac{x_2}{x_4} \\ & & & \frac{1}{x_4} \end{bmatrix}.$$

Thus we have computed $\eta(x) = [\dot{v}^{-1}\kappa_x x]_J$ from Definition 3.6.1. Since $x \in \dot{u}U_-^{(J)}$, we use Lemma 3.6.3(ii) to find

$$\zeta_{u,v}^{(J)}(x) = x\eta(x)^{-1} = \begin{bmatrix} 1 & & & \\ x_1 & \frac{x_2}{x_4} & -1 & -x_2 \\ x_3 & 1 & & -x_4 \\ & \frac{1}{x_4} & & \end{bmatrix}, \quad \text{thus} \quad \zeta_{u,v}^{(J)}(x)\dot{w}^{-1} = \begin{bmatrix} & & 1 & \\ -1 & -x_2 & x_1 & \frac{x_2}{x_4} \\ & -x_4 & x_3 & 1 \\ & & & \frac{1}{x_4} \end{bmatrix}.$$

Therefore the bottom-right principal minors of $\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}$ are

$$\Delta_1^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) = \frac{1}{x_4}, \quad \Delta_2^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) = \frac{x_3}{x_4}, \quad \Delta_3^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) = \frac{x_1x_4 - x_2x_3}{x_4}. \tag{3.9.13}$$

By Proposition 3.9.17, the preimage of $\mathcal{C}_g$ under $\bar{\varphi}_u$ is described by $\Delta_{I_a}^{\mathrm{tr}_u^a}(M) \neq 0$ for all $a \in [n]$. Alternatively, as we showed in Section 3.7.7, the preimage of $\mathcal{C}_g$ under $\bar{\varphi}_u$ is described by $\Delta_i^\pm(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) \neq 0$ for all $i \in [n-1]$. The following result has been computationally checked for all $n \leq 5$, $k \in [n]$, and $(u,u) \preceq (v,w) \in Q_J$:

Conjecture 3.9.21. Let $(u,u) \preceq (v,w) \in Q_J$. Denote $g := f_{v,w}$, and let $\mathcal{I}_g := (I_a)_{a\in\mathbb{Z}}$ be the Grassmann necklace of g. Suppose that $x = g^{(J)}\dot{u} \in G_{u,v}^{(J)}$ and let $M := [x|$. Then

$$\Delta_{n+1-i}^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) = \frac{\Delta_{I_i}^{\mathrm{tr}_u^i}(M)}{\Delta_{I_1}^{\mathrm{tr}_u^1}(M)} \quad \text{for all } i \in [n]. \tag{3.9.14}$$

For example, compare (3.9.13) with (3.9.10). Recall also that when $i = 1$, $\Delta_n^\pm(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) := 1$, so in this case (3.9.14) holds trivially.

3.9.11 Total positivity

We recall the background on the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k,n)$ of [Pos07]. By a result of Whitney [Whi52], $G_{\geq 0}$ is the set of matrices in $\mathrm{SL}_n(\mathbb{R})$ all of whose minors (of arbitrary sizes) are nonnegative. The following characterizations are well known:

$$(G/B)_{\geq 0} = \left\{xB \in (G/B)_{\mathbb{R}} \mid \Delta_S^{\mathrm{flag}}(x) \geq 0 \text{ for all } S \subset [n]\right\}, \tag{3.9.15}$$

$$\mathrm{Gr}_{\geq 0}(k,n) = (G/P)_{\geq 0} = \left\{xP \in (G/P)_{\mathbb{R}} \mid \Delta_S^{\mathrm{flag}}(x) \geq 0 \text{ for all } S \in \binom{[n]}{k}\right\}. \tag{3.9.16}$$

Eq. (3.9.16) is due to Rietsch, see [Lam16, Rmk. 3.8] for a proof. Eq. (3.9.15) can be easily deduced from the proof of Lemma 3.4.17 combined with the results of [Whi52]. We warn the reader that the analogous statement is false for other choices of J. For instance, when $G = \mathrm{SL}_4$ and $J = \{2\}$, $(G/P)_{\geq 0}$ does *not* contain all $xP \in (G/P)_{\mathbb{R}}$ such that $\Delta_S^{\mathrm{flag}}(x) \geq 0$ for all $S \in \binom{[n]}{1} \cup \binom{[n]}{3}$, see [Che11, §10.1].

For $f \in \mathrm{Bound}(k,n)$, we let $\Pi_f^{>0} := \mathring{\Pi}_f \cap \mathrm{Gr}_{\geq 0}(k,n)$ and $\Pi_f^{\geq 0} := \Pi_f \cap \mathrm{Gr}_{\geq 0}(k,n)$, thus for $(v,w) \in Q_J$, we have $\Pi_{f_{v,w}}^{>0} = \Pi_{v,w}^{>0}$ and $\Pi_{f_{v,w}}^{\geq 0} = \Pi_{v,w}^{\geq 0}$ by Theorem 3.9.3.

Recall that if $M = [x|$ is in $u[k]$-echelon form then $M' := [\kappa_x x|$ equals the u-truncation $M^{\mathrm{tr}_u^1}$ of M.

Proposition 3.9.22. *Let $\tau_{u\lambda} \leq^{\mathrm{op}} g \leq^{\mathrm{op}} h \in \mathrm{Bound}(k,n)$, and denote by $\mathcal{I}_g := (I_a)_{a\in\mathbb{Z}}$ the Grassmann necklace of g. Suppose that a matrix M in $u[k]$-echelon form belongs to*

$\mathrm{Gr}_{\geq 0}(k,n)$. *Then*

$$M^{\mathrm{tr}_u^a} \in \mathrm{Gr}_{\geq 0}(k,n) \quad \textit{and} \quad \Delta_{I_a}^{\mathrm{tr}_u^a}(M) > 0 \quad \textit{for all } a \in \mathbb{Z}. \tag{3.9.17}$$

Proof. Applying Theorem 3.9.3, we have $(u,u) \preceq (v,w) \preceq (v',w') \in Q_J$, where $g = f_{v,w}$ and $h = f_{v',w'}$. By (3.4.22), we get $v' \leq vr' \leq ur \leq wr' \leq w'$ for some $r, r' \in W_J$.

Suppose first that $a = 1$. Let $x \in G$ be such that $M = \left[g^{(J)}\dot{u}\right|$ and $xP \in \Pi_h^{>0}$, and denote $M' := M^{\mathrm{tr}_u^1}$. We may assume that $xB \in R_{v',w'}^{>0}$. By Corollary 3.6.10, we find that $\kappa_x xP \in \Pi_{\bar{v}',u}^{>0}$, where $\bar{v}' := v' \triangleleft r_w^{-1}$ for some $r_w \in W_J$ satisfying $r_w \geq r$, see Lemma 3.6.9(ii). This shows that $M' \in \mathrm{Gr}_{\geq 0}(k,n)$. Since $ur \leq ur_w$, we find that $ur \triangleleft r_w^{-1} \leq u$ by Lemma 3.4.6, therefore $ur \triangleleft r_w^{-1} = u$. Applying $\triangleleft r_w^{-1}$ to $v' \leq vr' \leq ur$ via Lemma 3.4.6(iii), we see that $\bar{v}' \leq (vr' \triangleleft r_w^{-1}) \leq u$. Let $v = v_1 v_2$ for $v_1 \in W^J$ and $v_2 \in W_J$ be the parabolic factorization of v. Then $vr' \triangleleft r_w^{-1} \in v_1 W_J$, thus $(v_1, v_1) \preceq (\bar{v}', u) \in Q_J$, which is equivalent to $\Delta_{v_1[k]}^{\mathrm{flag}}(\kappa_x x) > 0$. From Example 3.9.5 we have that $v[k] = I_1$, and $v_1[k] = v[k]$ since $v \in v_1 W_J$, so $\Delta_{I_1}^{\mathrm{tr}_u^1}(M) = \Delta_{I_1}^{\mathrm{flag}}(\kappa_x x) > 0$. We have shown (3.9.17) for $a = 1$. Applying the cyclic shift $\chi : \mathrm{Gr}_{\geq 0}(k,n) \to \mathrm{Gr}_{\geq 0}(k,n)$ (which takes M to the matrix with rows $(M_{a+1})_{a \in [n]}$), we obtain (3.9.17) for all $a \in \mathbb{Z}$. $\square$

Note that our proof of Proposition 3.9.22 involves a lifting from G/P to G/B, so it does not stay completely inside $\mathrm{Gr}(k,n)$.

Problem 3.9.23. Give a self-contained proof of Proposition 3.9.22.

Example 3.9.24. We now consider an example for the case $G/P = \mathrm{Gr}(2,5)$. Let $u := s_2 \in W^J$, so $u[k] = \{1,3\}$. Consider $(v',w') \in Q_J$ given by $v' = s_1$, $w' = s_2 s_1 s_4 s_3 s_2$ as in Figure 3-2, thus $h := f_{v',w'} = [3,4,7,5,6]$. We use Marsh–Rietsch parametrizations[1] from Section 3.4.9 to compute $x \in G$ such that $xB \in R_{v',w'}^{>0}$ and $xP \in \Pi_h^{>0}$:

$$x := y_2(t_1)\dot{s}_1 y_4(t_3) y_3(t_4) y_2(t_5) = \begin{bmatrix} & -1 & & & \\ 1 & & & & \\ t_1 & t_5 & 1 & & \\ & t_4t_5 & t_4 & 1 & \\ & t_3t_4t_5 & t_3t_4 & t_3 & 1 \end{bmatrix}, \quad M := \left[g^{(J)}\dot{u}\right| = \begin{bmatrix} 1 & \\ \frac{t_5}{t_1} & \frac{1}{t_1} \\ & 1 \\ -t_4t_5 & \\ -t_3t_4t_5 & \end{bmatrix}.$$

[1]For the Grassmannian case, Marsh–Rietsch parametrizations are closely related to *BCFW bridge parametrizations*, see [BCFW05, AHBC[+]16, Kar16].

Observe that $xB \in (G/B)_{\geq 0}$ since all flag minors of x are nonnegative. (For instance, the first column of x consists of nonnegative entries.) In fact, flag minors of x are subtraction-free rational expressions in $\mathbf{t} = (t_1, t_3, t_4, t_5)$, cf. (3.5.19). The $n \times k$ matrix $[x|$ is *not* in $u[k]$-echelon form, but the matrix $M := \left[g^{(J)}\dot{u}\right|$ is. Up to a common scalar, the 2×2 flag minors of M are the same as the corresponding flag minors of x, however, other (i.e., 1×1) flag minors of M are not necessarily nonnegative. The Grassmann necklace of h is $\mathcal{I}_h = [\{1,2\}, \{2,3\}, \{3,4\}, \{4,7\}, \{5,7\}]$. Using Proposition 3.9.2(i), we check that indeed $xP \in \Pi_h^{>0}$.

Let us choose $(v, w) \in Q_J$ for $v = s_2 s_1$, $w = s_2 s_1 s_4 s_3 s_2$, so that $g := f_{v,w} = [2, 4, 8, 5, 6]$. The corresponding Le diagram is obtained from the one in Figure 3-2 (bottom left) by removing the dot in the bottom row. We have $(u, u) \preceq (v, w) \preceq (v', w')$ and $\tau_{u\lambda} \leq^{\mathrm{op}} g \leq^{\mathrm{op}} h$. We compute the elements $\kappa_x = h_2^{(J)} \in U_2^{(J)}$, $\pi_{\dot{u}P_-}(x)$, $\eta(x)$, and $\zeta_{u,v}^{(J)}(x) = \pi_{\dot{u}P_-}(x) \cdot \eta(x)^{-1}$ from Definition 3.6.1:

$$g^{(J)}\dot{u} = \begin{bmatrix} 1 & & & & \\ \frac{t_5}{t_1} & \frac{1}{t_1} & -1 & & \\ & 1 & & & \\ -t_4 t_5 & & & 1 & \\ -t_3 t_4 t_5 & & & & 1 \end{bmatrix}, \qquad \kappa_x = \begin{bmatrix} 1 & & & & \\ -\frac{t_5}{t_1} & 1 & & & \\ & & 1 & & \\ t_4 t_5 & & & 1 & \\ t_3 t_4 t_5 & & & & 1 \end{bmatrix}, \qquad \kappa_x x = \begin{bmatrix} & -1 & & & \\ 1 & \frac{t_5}{t_1} & & & \\ t_1 & t_5 & 1 & & \\ & & t_4 & 1 & \\ & & t_3 t_4 & t_3 & 1 \end{bmatrix},$$

$$\pi_{\dot{u}P_-}(x) = \begin{bmatrix} & -1 & & & \\ 1 & & -\frac{1}{t_1} & & \\ t_1 & t_5 & & & \\ & t_4 t_5 & t_4 & 1 & \\ & t_3 t_4 t_5 & t_3 t_4 & t_3 & 1 \end{bmatrix}, \qquad \eta(x) = \begin{bmatrix} t_1 & t_5 & & & \\ & 1 & & & \\ & & \frac{1}{t_1} & & \\ & & t_4 & 1 & \\ & & t_3 t_4 & t_3 & 1 \end{bmatrix}, \qquad \zeta_{u,v}^{(J)}(x) = \begin{bmatrix} & -1 & & & \\ \frac{1}{t_1} & -\frac{t_5}{t_1} & -1 & & \\ 1 & & & & \\ & t_4 t_5 & & 1 & \\ & t_3 t_4 t_5 & & & 1 \end{bmatrix}.$$

We see that all flag minors of $\kappa_x x$ are nonnegative, cf. Lemma 3.6.9(ii). Observe that $\kappa_{g^{(J)}\dot{u}} = \kappa_x$ by Lemma 3.6.2(iii), so by Lemma 3.6.3(ii), we could alternatively compute $\zeta_{u,v}^{(J)}(x)$ as the product $g^{(J)}\dot{u} \cdot \eta(g^{(J)}\dot{u})^{-1}$:

$$\eta(g^{(J)}\dot{u}) = \begin{bmatrix} & 1 & & & \\ -1 & & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}, \quad \zeta_{u,v}^{(J)}(x) = g^{(J)}\dot{u} \cdot \eta(g^{(J)}\dot{u})^{-1} = \begin{bmatrix} 1 & & & & \\ \frac{t_5}{t_1} & \frac{1}{t_1} & -1 & & \\ & 1 & & & \\ -t_4 t_5 & & & 1 & \\ -t_3 t_4 t_5 & & & & 1 \end{bmatrix} \cdot \begin{bmatrix} & -1 & & & \\ 1 & & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{bmatrix}.$$

Finally, we compute the bottom-right $i \times i$ principal minors of $\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}$ and observe that they are all nonzero subtraction-free expressions in $\mathbf{t}$, agreeing with Theorems 3.6.4

and 3.6.14:

$$\zeta_{u,v}^{(J)}(x)\dot{w}^{-1} = \begin{bmatrix} & & & & -1 \\ -1 & & \frac{1}{t_1} & & -\frac{t_5}{t_1} \\ & & 1 & & \\ & 1 & & & t_4t_5 \\ & & & -1 & t_3t_4t_5 \end{bmatrix}, \qquad \begin{aligned} &\Delta_1^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) = t_3t_4t_5, && \Delta_2^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) = t_4t_5, \\ &\Delta_3^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) = t_4t_5, && \Delta_4^{\pm}(\zeta_{u,v}^{(J)}(x)\dot{w}^{-1}) = \frac{t_5}{t_1}. \end{aligned}$$

Let us check that this agrees with Conjecture 3.9.21. The Grassmann necklace of g is $\mathcal{I}_g = [\{1,3\},\{2,3\},\{3,4\},\{4,8\},\{5,8\}]$ in window notation. We see that the corresponding u-truncated minors of $M = \left[g^{(J)}\dot{u}\right|$ are indeed given by

$$\Delta_{13}^{\mathrm{tr}_u^1}(M) = 1, \quad \Delta_{23}^{\mathrm{tr}_u^2}(M) = \frac{t_5}{t_1}, \quad \Delta_{34}^{\mathrm{tr}_u^3}(M) = t_4t_5, \quad \Delta_{48}^{\mathrm{tr}_u^4}(M) = t_4t_5, \quad \Delta_{58}^{\mathrm{tr}_u^5}(M) = t_3t_4t_5.$$

3.10 Further directions

In addition to Theorem 1.2.1 and [Her14], we expect the regularity theorem to hold for many other spaces occurring in total positivity. The most natural immediate direction is total positivity for Kac–Moody flag varieties.

Let $\mathcal{G}^{\min}$ be a minimal Kac–Moody group, $\mathcal{U}^{\min}$,$\mathcal{U}_-^{\min}$, $\mathcal{B}^{\min}$, $\mathcal{B}_-^{\min}$ be unipotent and Borel subgroups, and $\tilde{W}$ be the Weyl group as in Section 3.A. Furthermore, let $\mathcal{P}^{\min} \supset \mathcal{B}^{\min}$ denote a standard parabolic subgroup of $\mathcal{G}^{\min}$ (a group of the form $\mathcal{G}^{\min} \cap \mathcal{P}_Y$ in the notation of [Kum02]).

Definition 3.10.1. Define the *totally nonnegative part* $\mathcal{U}_{\geq 0}^-$ of $\mathcal{U}_-^{\min}$ to be the subsemigroup generated by $\{x_{\alpha_i}(t) \mid t \in \mathbb{R}_{>0},\ 1 \leq i \leq r\}$. Define the *totally nonnegative part* of the flag variety $\mathcal{G}^{\min}/\mathcal{P}^{\min}$ to be the closure $(\mathcal{G}^{\min}/\mathcal{P}^{\min})_{\geq 0} := \overline{\mathcal{U}_{\geq 0}^- \mathcal{P}^{\min}/\mathcal{P}^{\min}}$.

When $\mathcal{G}^{\min}$ is an affine Kac–Moody group of type A, Definition 3.10.1 agrees with the definition of Lam and Pylyavskyy (cf. [LP12, Theorem 2.6]) for the polynomial loop group.

Conjecture 3.10.2 (Regularity conjecture for Kac–Moody groups and flag varieties)**.**

1. The intersection of $\mathcal{U}_{\geq 0}^-$ with the Bruhat stratification $\{\mathcal{B}^{\min}\dot{w}\mathcal{B}^{\min} \mid w \in \tilde{W}\}$ of $\mathcal{G}^{\min}$ endows $\mathcal{U}_{\geq 0}^-$ with an (infinite) cell decomposition with closure partial order equal to the Bruhat order of $\tilde{W}$. Furthermore, the link of the identity in any (closed) cell is a regular CW complex homeomorphic to a closed ball.

2. The intersection of $(\mathcal{G}^{\min}/\mathcal{B}^{\min})_{\geq 0}$ with the open Richardson stratification $\mathring{\mathcal{R}}_u^v$ of $\mathcal{G}^{\min}/\mathcal{B}^{\min}$ endows $(\mathcal{G}^{\min}/\mathcal{B}^{\min})_{\geq 0}$ with the structure of a regular CW complex. The closure partial order is the interval order of the Bruhat order of $\tilde{W}$, and after adding a minimum, every interval of the closure partial order is thin and shellable.

3. The intersection of $(\mathcal{G}^{\min}/\mathcal{P}^{\min})_{\geq 0}$ with the open projected Richardson stratification $\Pi^\circ_{v,w}$ of $\mathcal{G}^{\min}/\mathcal{P}^{\min}$ endows $(\mathcal{G}^{\min}/\mathcal{P}^{\min})_{\geq 0}$ with the structure of a regular CW complex. The closure partial order is the natural partial order on $\mathcal{P}$-Bruhat intervals of $\tilde{W}$, and after adding a minimum, every interval of the closure partial order is thin and shellable.

Note that every interval in the Bruhat order of $\tilde{W}$ is known to be thin and shellable [BW82]. The stratification $\Pi^\circ_{v,w}$ and the $\mathcal{P}$-Bruhat order can be defined analogously to [KLS14].

We include a list of some other spaces occurring in total positivity which we expect to have a natural regular CW complex structure.

1. The *totally nonnegative part of double Bruhat cells* [FZ99]. It has been expected that a link of a double Bruhat cell inside another double Bruhat cell is a regular CW complex homeomorphic to a closed ball. Our Theorem 3.3.11 confirms this in type A, since double Bruhat cells for GL_n embed in the Grassmannian $\mathrm{Gr}(n, 2n)$, see [Pos07, Remark 3.11].

2. The space of planar electrical networks from Section 1.5 and the space of planar Ising models from Section 1.6. These spaces are homeomorphic to closed balls by Theorems 1.5.1 and 4.1.3, and have cell decompositions whose face poset is graded, thin, and shellable [HK18].

3. *Amplituhedra* [AHT14] and, more generally, *Grassmann polytopes* [Lam16]. Grassmann polytopes generalize convex polytopes into the Grassmannian $\mathrm{Gr}(k, n)$. The former are well known to be regular CW complexes homeomorphic to closed balls. We caution that not all Grassmann polytopes are balls.

4. The *totally nonnegative part of the wonderful compactification* [He07]. A cell decomposition of this space was constructed in [He07].

We expect that most spaces in this list are TNN spaces that admit a Fomin–Shapiro atlas.

3.A Appendix: Kac–Moody flag varieties

We recall some background on Kac–Moody groups, and refer to [Kum02] for all missing definitions. We start by introducing the minimal Kac–Moody group $\mathcal{G}^{\min}$ and its flag variety $\mathcal{G}^{\min}/\mathcal{B}^{\min}$, and then explain how they relate to the polynomial loop group $\mathcal{G}$ and its flag variety $\mathcal{G}/\mathcal{B}$ from Section 3.7.

3.A.1 Kac–Moody Lie algebras

Suppose that $\tilde{A}$ is a *generalized Cartan matrix* [Kum02, Dfn. 1.1.1]. Thus $\tilde{A}$ is an $r \times r$ integer matrix for some $r \geq 1$. We assume $\tilde{A}$ is *symmetrizable*, that is, there exists a diagonal matrix $D \in \mathrm{GL}_r(\mathbb{Q})$ such that $D\tilde{A}$ is a symmetric matrix. As in [Kum02, §1.1], denote by $\mathfrak{g}$ the *Kac–Moody Lie algebra* associated with $\tilde{A}$, and let $\mathfrak{h} \subset \mathfrak{g}$ be its *Cartan subalgebra*, whose dual is denoted by $\mathfrak{h}^*$. Thus $\mathfrak{h}$ and $\mathfrak{h}^*$ are vector spaces over $\mathbb{C}$ of dimension $\tilde{r} := 2r - \mathrm{rank}(\tilde{A})$, and we let $\langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$ denote the natural pairing.

We let $\Delta \subset \mathfrak{h}^*$ denote the *root system* of $\mathfrak{g}$, as defined in [Kum02, §1.2]. Let $\{\alpha_i\}_{i=1}^r \subset \mathfrak{h}^*$ be the *simple roots* and $\{\alpha_i^\vee\}_{i=1}^r \in \mathfrak{h}$ be the *simple coroots*.

Let $\Delta_{\mathrm{re}} \subset \Delta$ denote the set of *real roots* and $\Delta_{\mathrm{im}} \subset \Delta$ denote the set of *imaginary roots*, so $\Delta = \Delta_{\mathrm{re}} \sqcup \Delta_{\mathrm{im}}$. Also let $\Delta = \Delta^+ \sqcup \Delta^-$ denote the decomposition of Δ into *positive* and *negative* roots, and denote $\Delta_{\mathrm{re}}^+ := \Delta^+ \cap \Delta_{\mathrm{re}}$ and $\Delta_{\mathrm{re}}^- := \Delta_{\mathrm{re}} \cap \Delta^-$. Denote by $\tilde{W}$ the *Weyl group* associated with $\tilde{A}$ as in [Kum02, §1.3]. Thus $\tilde{W}$ acts on Δ, and preserves the subset Δ_{re}. Moreover, $\tilde{W}$ is generated by *simple reflections* $s_1, \dots, s_r \in \tilde{W}$, and $(\tilde{W}, \{s_i\}_{i=1}^r)$ is a Coxeter group by [Kum02, Prop. 1.3.21]. We let $(\tilde{W}, \leq)$ denote the *Bruhat order* on $\tilde{W}$ and $\ell : \tilde{W} \to \mathbb{Z}_{\geq 0}$ denote the length function.

3.A.2 Kac–Moody groups

Let $\mathcal{G}^{\min}$ be the *minimal Kac–Moody group* associated to $\tilde{A}$ by Kac and Peterson [PK83, KP83], see [Kum02, §7.4]. For each real root $\alpha \in \Delta_{\mathrm{re}}$, there is a one-parameter subgroup $\mathcal{U}_\alpha \subset \mathcal{G}^{\min}$ by [Kum02, Dfn. 6.2.7].[2] For each $\alpha \in \Delta_{\mathrm{re}}$, we fix an isomorphism $x_\alpha : \mathbb{C} \xrightarrow{\sim} \mathcal{U}_\alpha$

[2]The results in [Kum02] are usually stated for the *maximal Kac–Moody group* which he denotes by $\mathcal{G}$. However, these results apply to $\mathcal{G}^{\min}$ as well, see Remark 3.A.3.

of algebraic groups. Similarly to the subgroups U, U_-, T, B, B_- of G, we have subgroups $\mathcal{U}^{\min}, \mathcal{U}_-^{\min}, \mathcal{T}^{\min}, \mathcal{B}^{\min}, \mathcal{B}_-^{\min}$ of $\mathcal{G}^{\min}$. The subgroup $\mathcal{U}^{\min}$ is generated by $\{\mathcal{U}_\alpha\}_{\alpha\in\Delta_{\mathrm{re}}^+}$, and $\mathcal{U}_-^{\min}$ is generated by $\{\mathcal{U}_\alpha\}_{\alpha\in\Delta_{\mathrm{re}}^-}$. Next, $\mathcal{T}^{\min}$ is an $\tilde{r}$-dimensional algebraic torus defined in [Kum02, §6.1.6], $\mathcal{B}^{\min} = \mathcal{T}^{\min} \ltimes \mathcal{U}^{\min}$ is the *standard positive Borel subgroup* and $\mathcal{B}_-^{\min} = \mathcal{T}^{\min} \ltimes \mathcal{U}_-^{\min}$ is the *standard negative Borel subgroup*.

We define a *bracket closed subset* $\Theta \subset \Delta_{\mathrm{re}}$ in the same way as in Section 3.4.2, and for a bracket closed subset $\Theta \subset \Delta_{\mathrm{re}}^+$ (resp., $\Theta \subset \Delta_{\mathrm{re}}^-$), we have a subgroup $\mathcal{U}(\Theta) \subset \mathcal{U}^{\min}$ (resp., $\mathcal{U}_-(\Theta) \subset \mathcal{U}_-^{\min}$), generated by $\mathcal{U}_\alpha$ for $\alpha \in \Theta$, see [Kum02, Eq. 6.1.1(6)] (resp., [Kum02, §6.2.7]). For $w \in \tilde{W}$, $\operatorname{Inv}(w) := \Delta^+ \cap w^{-1}\Delta^- \subset \Delta_{\mathrm{re}}^+$ is a bracket closed subset of size $\ell(w)$, cf. [Kum02, Ex. 6.1.5(b)]. We state the Kac–Moody analog of Lemma 3.4.1(i).

Lemma 3.A.1 ([Kum02, Lemma 6.1.4]). *Suppose that $\Theta = \bigsqcup_{i=1}^n \Theta_i$ and $\Theta, \Theta_1, \dots, \Theta_n \subset \Delta_{\mathrm{re}}^+$ are finite bracket closed subsets. Then $\mathcal{U}(\Theta), \mathcal{U}(\Theta_1), \dots, \mathcal{U}(\Theta_n)$ are finite-dimensional unipotent algebraic groups, and the multiplication map gives a biregular isomorphism*

$$\mathcal{U}(\Theta_1) \times \cdots \times \mathcal{U}(\Theta_n) \xrightarrow{\sim} \mathcal{U}(\Theta). \tag{3.A.1}$$

3.A.3 Kac–Moody flag varieties

The Weyl group $\tilde{W}$ equals $N_{\mathcal{G}^{\min}}(\mathcal{T}^{\min})/\mathcal{T}^{\min}$, where $N_{\mathcal{G}^{\min}}(\mathcal{T}^{\min})$ is the normalizer of $\mathcal{T}^{\min}$ in $\mathcal{G}^{\min}$, cf. [Kum02, Lemma 7.4.2]. For $f \in \tilde{W}$, we denote by $\dot{f} \in \mathcal{G}^{\min}$ an arbitrary representative of f in $N_{\mathcal{G}^{\min}}(\mathcal{T}^{\min})$.

By [Kum02, Lemma 7.4.2, Ex. 7.4.E(9), and Thm. 5.2.3(g)], we have Bruhat and Birkhoff decompositions of $\mathcal{G}^{\min}$:

$$\mathcal{G}^{\min} = \bigsqcup_{f\in\tilde{W}} \mathcal{B}^{\min}\dot{f}\mathcal{B}^{\min}, \quad \mathcal{G}^{\min} = \bigsqcup_{h\in\tilde{W}} \mathcal{B}_-^{\min}\dot{h}\mathcal{B}^{\min}. \tag{3.A.2}$$

We let $\mathcal{G}^{\min}/\mathcal{B}^{\min}$ denote the *Kac–Moody flag variety* of $\mathcal{G}^{\min}$. For each $h, f \in \tilde{W}$, we have Schubert cells $\mathring{\mathcal{X}}^f := \mathcal{B}^{\min}\dot{f}\mathcal{B}^{\min}/\mathcal{B}^{\min}$ and opposite Schubert cells $\mathring{\mathcal{X}}_h := \mathcal{B}_-^{\min}\dot{h}\mathcal{B}^{\min}/\mathcal{B}^{\min}$. If $h \not\leq f \in \tilde{W}$ then by [Kum02, Lemma 7.1.22(b)], $\mathring{\mathcal{X}}^f \cap \mathring{\mathcal{X}}_h = \emptyset$. For $h \leq f$, we denote by $\mathring{\mathcal{R}}_h^f := \mathring{\mathcal{X}}_h \cap \mathring{\mathcal{X}}^f$. Therefore (3.7.3) follows from (3.A.2). The flag variety $\mathcal{G}^{\min}/\mathcal{B}^{\min}$ is a *projective ind-variety* by [Kum02, §7.1], the Schubert cell $\mathring{\mathcal{X}}^f$ and Schubert variety $\mathcal{X}^f$ are finite-dimensional subvarieties, while the opposite Schubert cell $\mathring{\mathcal{X}}_h$ and opposite Schubert variety $\mathcal{X}_h$ are ind-subvarieties.

Proposition 3.A.2. *For $h \leq f \in \tilde{W}$, $\mathcal{X}_h \cap \mathcal{X}^f$ is a closed irreducible $(\ell(f) - \ell(h))$-dimensional subvariety of $\mathcal{X}^f$, and $\mathring{\mathcal{R}}_h^f$ is an open dense subset of $\mathcal{X}_h \cap \mathcal{X}^f$.*

Proof. By (3.7.5), $\mathring{\mathcal{X}}^f$ is $\ell(f)$-dimensional, and by [Kum02, Lemma 7.3.10], $\mathring{\mathcal{X}}_h \cap \mathcal{X}^f$ has codimension $\ell(h)$ in $\mathcal{X}^f$. The rest follows by [Kum17, Prop. 6.6]. □

For $g \in \tilde{W}$, let $\mathcal{C}_g := \dot{g}\mathcal{B}_-^{\min}\mathcal{B}^{\min}/\mathcal{B}^{\min}$. We have

$$\mathcal{G}^{\min}/\mathcal{B}^{\min} = \bigsqcup_{h \leq f} \mathring{\mathcal{R}}_h^f \quad \text{and} \quad \mathcal{C}_g = \bigsqcup_{h \leq g \leq f} (\mathcal{C}_g \cap \mathring{\mathcal{R}}_h^f), \tag{3.A.3}$$

where the unions are taken over $h, f \in \tilde{W}$. The first part of (3.A.3) follows from (3.A.2), and for the second part, see the proof of Proposition 3.8.2(iii).

Remark 3.A.3. Let $\hat{\mathcal{G}} \supset \mathcal{G}^{\min}$ be the "maximal" Kac–Moody group (denoted $\mathcal{G}$ in [Kum02]) associated to $\tilde{A}$, and let $\hat{\mathcal{B}} \supset \mathcal{B}^{\min}$ be its standard positive Borel subgroup. Then the standard negative Borel subgroup of $\hat{\mathcal{G}}$ is still $\mathcal{B}_-^{\min}$. By [Kum02, Eq. 7.4.5(2)], we may identify $\mathcal{G}^{\min}/\mathcal{B}^{\min} \xrightarrow{\sim} \hat{\mathcal{G}}/\hat{\mathcal{B}}$. By [Kum02, Eq. 7.4.2(3)], $\mathring{\mathcal{X}}^f$ coincides with the variety $\hat{\mathcal{B}} f \hat{\mathcal{B}}/\hat{\mathcal{B}}$ in [Kum02, Dfn. 7.1.13] for $f \in \tilde{W}$. Similarly, for $h \in \tilde{W}$, $\mathring{\mathcal{X}}_h = \mathcal{B}_-^{\min} \cdot \dot{h}\mathcal{B}^{\min}/\mathcal{B}^{\min}$ coincides with the variety $\mathcal{B}_\emptyset^h := \mathcal{B}_-^{\min} h\hat{\mathcal{B}}/\hat{\mathcal{B}}$ defined in the last paragraph of [Kum02, §7.1.20].

3.A.4 Affine Kac–Moody groups and polynomial loop groups

Suppose that $\tilde{A}$ is the affine Cartan matrix associated to a simple and simply-connected algebraic group G. Thus we have $r = |I| + 1$, $\tilde{r} = |I| + 2$, and $\tilde{A}$ is defined by [Kum02, Eq. 13.1.1(7)]. Let $\mathcal{G}$ denote the polynomial loop group from Section 3.7. Our goal is to explain that the flag varieties $\mathcal{G}/\mathcal{B}$ and $\mathcal{G}^{\min}/\mathcal{B}^{\min}$ are isomorphic.

Let $C \subset T \subset G$ be the center of G, and let $\tilde{C} \subset \mathcal{T}^{\min} \subset \mathcal{G}^{\min}$ be the center of $\mathcal{G}^{\min}$, see [Kum02, Lemma 6.2.9(c)]. By [Kum02, Cor. 13.2.9], there exists a surjective group homomorphism $\psi : \mathcal{G}^{\min} \to (\mathbb{C}^* \ltimes \mathcal{G})/C$ with kernel $\tilde{C}$, where $\mathbb{C}^*$ acts on $\mathcal{G}$ as in Section 3.8.2, see also [Kum02, Dfn. 13.2.1]. The groups $\mathcal{U}, \mathcal{U}_- \subset \mathcal{G}$ are identified with the groups $\mathcal{U}^{\min}, \mathcal{U}_-^{\min} \subset \mathcal{G}^{\min}$, and we have $\mathcal{T}/C \cong \mathcal{T}^{\min}/\tilde{C}$. Thus ψ induces an isomorphism $\mathcal{G}^{\min}/\mathcal{B}^{\min} \xrightarrow{\sim} \mathcal{G}/\mathcal{B}$ between the affine Kac–Moody flag variety and the affine flag variety. The Weyl groups $\tilde{W}$ of $\mathcal{G}$ and $\mathcal{G}^{\min}$ are isomorphic by [Kum02, Prop. 13.1.7], and the root systems Δ coincide by [Kum02, Cor. 13.1.4]. Therefore the subsets $\mathring{\mathcal{X}}^f, \mathring{\mathcal{X}}_h, \mathring{\mathcal{R}}_h^f, \mathcal{C}_g$ of $\mathcal{G}/\mathcal{B}$ get sent by ψ to the corresponding subsets of $\mathcal{G}^{\min}/\mathcal{B}^{\min}$. As explained in the last paragraph of [Kum02, §13.2.8], G can be viewed as a subset of $\mathcal{G}^{\min}$ as well, and the restriction of ψ to G is the quotient map $G \to G/C$.

We justify some other statements that we used in Sections 3.7.1 and 3.8.2. For (3.7.2), see [Kum02, §13.1]. For (3.7.6), see [Kum02, §6.1.13]. For a description of $Y(\mathcal{T})$ from Section 3.8.2, see [Kum02, §13.2.2]. For a description of the pairing $\langle \cdot, \cdot \rangle : Y(\mathcal{T}) \times X(\mathcal{T}) \to \mathbb{Z}$ in the same section, see [Kum02, §13.1.1].

3.A.5 Gaussian decomposition and affine charts

By [Kum02, Thm. 7.4.14], $\mathcal{G}^{\min}$ is an *affine ind-group*. Similarly, $\mathcal{U}^{\min}, \mathcal{U}_-^{\min}, \mathcal{T}, \mathcal{B}^{\min}, \mathcal{B}_-^{\min}$ are affine ind-groups, see e.g. [Kum02, §7.4] and [Kum02, Cor. 7.3.8].

Let $\mathcal{G}_0^{\min} := \mathcal{B}_-^{\min}\mathcal{B}^{\min}$ and $g \in \tilde{W}$. Recall the subgroups $\mathcal{U}_1(g)$ and $\mathcal{U}_2(g)$ from (3.7.4). Then $\mathcal{U}_1(g)$ is a closed $\ell(g)$-dimensional subgroup of $\mathcal{U}^{\min} \cong \mathcal{U}$ and $\mathcal{U}_2(g)$ is a closed ind-subgroup of $\mathcal{U}_-^{\min} \cong \mathcal{U}_-$.

Proof of Lemma 3.8.1. For (i), see [Kum02, Prop. 7.4.11]. For (ii), we use an argument given in [Wil13, Prop. 2.5]: both maps are bijective morphisms by [Kum02, Lemma 6.1.3]. In particular, it follows that $\dot{g}\mathcal{U}_-^{\min}\dot{g}^{-1} \subset \mathcal{G}_0^{\min}$ and for $x \in \dot{g}\mathcal{U}_-^{\min}\dot{g}^{-1}$, we have $[x]_0 = 1$. The inverse maps are given by $\mu_{21}^{-1}(x) = ([x]_-, [x]_+)$, $\mu_{12}^{-1}(x) = ([x^{-1}]_+^{-1}, [x^{-1}]_-^{-1})$. They are regular morphisms by (i), which proves (ii). $\square$

Proof of (3.7.5)*.* The map $\dot{g}\mathcal{U}_-^{\min}\dot{g}^{-1} \xrightarrow{\sim} \mathcal{C}_g$ is a biregular isomorphism for $g = \mathrm{id}$ by [Kum02, Lemma 7.4.10]. Since $\tilde{W}$ acts on $\mathcal{G}^{\min}/\mathcal{B}^{\min}$ by left multiplication, the case of general $g \in \tilde{W}$ follows as well. Since $\mathcal{U}_1(g)$, $\mathcal{U}_2(g)$ are closed ind-subvarieties of $\dot{g}\mathcal{U}_-^{\min}\dot{g}^{-1}$ and $\mathring{\mathcal{X}}^g$, $\mathring{\mathcal{X}}_g$ are closed ind-subvarieties of $\mathcal{C}_g$, it suffices to show that the image of $\mathcal{U}_1(g)$ equals $\mathring{\mathcal{X}}^g$ while the image of $\mathcal{U}_2(g)$ equals $\mathring{\mathcal{X}}_g$. By [Kum02, Ex. 7.4.E(9) and Eq. 5.2.3(11)], we have

$$\mathcal{U}^{\min} = (\mathcal{U}^{\min} \cap \dot{g}\mathcal{U}_-^{\min}\dot{g}^{-1}) \cdot (\mathcal{U}^{\min} \cap \dot{g}\mathcal{U}^{\min}\dot{g}^{-1}) = \mathcal{U}_1(g) \cdot (\mathcal{U}^{\min} \cap \dot{g}\mathcal{U}^{\min}\dot{g}^{-1}),$$
$$\mathcal{U}_-^{\min} = (\mathcal{U}_-^{\min} \cap \dot{g}\mathcal{U}_-^{\min}\dot{g}^{-1}) \cdot (\mathcal{U}_-^{\min} \cap \dot{g}\mathcal{U}^{\min}\dot{g}^{-1}) = \mathcal{U}_2(g) \cdot (\mathcal{U}_-^{\min} \cap \dot{g}\mathcal{U}^{\min}\dot{g}^{-1}).$$

Thus

$$\mathcal{B}^{\min}\dot{g}\mathcal{B}^{\min} = \mathcal{U}_1(g) \cdot (\mathcal{U}^{\min} \cap \dot{g}\mathcal{U}^{\min}\dot{g}^{-1}) \cdot \dot{g}\mathcal{B}^{\min} = \mathcal{U}_1(g) \cdot \dot{g} \cdot \mathcal{B}^{\min},$$
$$\mathcal{B}_-^{\min}\dot{g}\mathcal{B}^{\min} = \mathcal{U}_2(g) \cdot (\mathcal{U}_-^{\min} \cap \dot{g}\mathcal{U}^{\min}\dot{g}^{-1}) \cdot \dot{g}\mathcal{B}^{\min} = \mathcal{U}_2(g) \cdot \dot{g} \cdot \mathcal{B}^{\min}. \qquad \square$$

4

Ising model

The goal of this chapter is to describe a cell decomposition of the space $\overline{\mathcal{X}}_n$ of planar Ising networks from Section 1.6, describe this space by inequalities, and show that it is homeomorphic to a ball. As we mentioned in Section 1.6, we also recognize Kramers–Wannier's duality [KW41] as the cyclic shift on $\mathrm{Gr}_{\geq 0}(n, 2n)$ in Theorem 4.2.4. We then explain (Remark 4.2.6) the connection between the planar Ising model at critical temperature and the unique cyclically symmetric point $X_0 \in \mathrm{Gr}_{\geq 0}(n, 2n)$ from Section 2.2.1. We also express (Theorem 4.2.13) generalized Griffiths' inequalities of [Gri67, KS68] as manifestly positive linear combinations of the Plücker coordinates of our embedding. We explain in Corollary 4.2.9 how the known formula for Plücker coordinates in terms of the dimer model gives a new expression for boundary correlation functions, which is related to Dubédat's formula for squared correlation functions [Dub11]. Finally, we solve the *inverse problem* in Section 4.2.5: given a boundary correlation matrix $M \in \mathrm{Mat}_n(\mathbb{R})$ of the Ising model on a planar graph G embedded in a disk, we show that if G is *reduced* then the edge weights of the Ising model are uniquely and explicitly determined by M.

This chapter is organized as follows. We state our main result (Theorem 4.1.3) in Section 4.1, and then list several applications of our construction in Section 4.2. We give some background on the totally nonnegative Grassmannian in Section 4.3, and study the totally nonnegative orthogonal Grassmannian in Section 4.4. After that, we prove our main results. In Section 4.5, we show that the formula for boundary correlations in terms of the dimer model indeed yields the same result as the embedding ϕ from Section 4.1. In Section 4.6, we prove that $\overline{\mathcal{X}}_n$ is homeomorphic to a ball and discuss the cyclic symmetry of this space. We explain how to express generalized Griffiths' inequalities as positive sums of Plücker coordinates in Section 4.7, and list several conjectures in Section 4.8.

4.1 Main results

We give some background necessary to formulate the main results of this chapter.

4.1.1 The Ising model

A *planar Ising network* is a pair $N = (G, J)$ where $G = (V, E)$ is a planar graph embedded in a disk and $J : E \to \mathbb{R}_{>0}$ is a function assigning positive real numbers to the edges of G. We always label the vertices of G on the boundary of the disk by $b_1, \dots, b_n \in V$ in counterclockwise order. Given a planar Ising network $N = (G, J)$, the *Ising model* on N (with *no external field* and *free boundary conditions*) is a probability measure on the space $\{-1, 1\}^V$ of *spin configurations* on the vertices of G. Given a spin configuration $\sigma : V \to \{-1, 1\}$, its probability is given by

$$\mathbf{P}(\sigma) := \frac{1}{Z} \prod_{\{u,v\} \in E} \exp\left(J_{\{u,v\}} \sigma_u \sigma_v\right), \tag{4.1.1}$$

where Z is the *partition function*:

$$Z := \sum_{\sigma \in \{-1,1\}^V} \prod_{\{u,v\} \in E} \exp\left(J_{\{u,v\}} \sigma_u \sigma_v\right). \tag{4.1.2}$$

Our main focus will be *boundary two-point correlation functions*. Let $[n] := \{1, 2, \dots, n\}$. Given $i, j \in [n]$, we define the corresponding correlation function by

$$\langle \sigma_i \sigma_j \rangle := \sum_{\sigma \in \{-1,1\}^V} \mathbf{P}(\sigma) \sigma_{b_i} \sigma_{b_j}. \tag{4.1.3}$$

Clearly, we have $\langle \sigma_i \sigma_j \rangle = \langle \sigma_j \sigma_i \rangle$, and if $i = j$ then the correlation function $\langle \sigma_i \sigma_i \rangle$ is equal to 1. We denote by $\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) \subset \mathrm{Mat}_n(\mathbb{R})$ the space of all $n \times n$ symmetric real matrices with ones on the diagonal. Thus we obtain a matrix $M = M(G, J) := (m_{i,j}) \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$ given by $m_{i,j} := \langle \sigma_i \sigma_j \rangle$. Let us denote

$$\mathcal{X}_n := \{M(G, J) \mid (G, J) \text{ is a planar Ising network with } n \text{ boundary vertices}\}.$$

Denote by $\overline{\mathcal{X}}_n$ the closure of $\mathcal{X}_n$ in the space $\mathrm{Mat}_n(\mathbb{R})$ of $n \times n$ real matrices. (In other words, $\overline{\mathcal{X}}_n$ can be defined as the space of all boundary correlation matrices $M(G, J)$ where J is allowed to take values in $[0, \infty]$, or equivalently where G is obtained from a planar graph embedded in a disk by contracting some edges that may connect boundary vertices, as we discuss in Section 4.5.) We will see later (Proposition 4.5.4) that $\overline{\mathcal{X}}_n$ admits a natural stratification into cells indexed by *matchings on* $[2n]$, that is, by perfect matchings of the complete graph K_{2n} (also called *medial pairings*). For $n = 3$, all matchings on $[2n]$ are shown in Figure 4-7.

4.1.2 The orthogonal Grassmannian

Recall the definition of $\mathrm{Gr}_{\geq 0}(k, n)$ from Section 2.2.

Definition 4.1.1. The *orthogonal Grassmannian* $\mathrm{OG}(n, 2n) \subset \mathrm{Gr}(n, 2n)$ is defined by

$$\mathrm{OG}(n, 2n) := \left\{ X \in \mathrm{Gr}(n, 2n) \mid \Delta_I(X) = \Delta_{[2n]\setminus I}(X) \text{ for all } I \in \binom{[2n]}{n} \right\}.$$

Its *totally nonnegative part* $\mathrm{OG}_{\geq 0}(n, 2n) \subset \mathrm{Gr}_{\geq 0}(n, 2n)$ is the intersection

$$\mathrm{OG}_{\geq 0}(n, 2n) := \mathrm{OG}(n, 2n) \cap \mathrm{Gr}_{\geq 0}(n, 2n).$$

The space $\mathrm{OG}_{\geq 0}(n, 2n)$ has been first considered in [HW13] in the context of the scattering amplitudes of ABJM theory. Postnikov defined a stratification of $\mathrm{Gr}_{\geq 0}(k, n)$ into *positroid cells*, which induces a stratification of $\mathrm{OG}_{\geq 0}(n, 2n)$. As it was observed in [HW13, HWX14], the strata of $\mathrm{OG}_{\geq 0}(n, 2n)$ are also naturally labeled by matchings on $[2n]$. We prove this in Section 4.4.

$$M = \begin{pmatrix} 1 & m_{12} & m_{13} & m_{14} \\ m_{12} & 1 & m_{23} & m_{24} \\ m_{13} & m_{23} & 1 & m_{34} \\ m_{14} & m_{24} & m_{34} & 1 \end{pmatrix} \quad \mapsto \quad \widetilde{M} = \begin{pmatrix} 1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\ -m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\ m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\ -m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1 \end{pmatrix}$$

Figure 4-1: An example of the map $M \mapsto \widetilde{M}$ for $n = 4$.

4.1.3 An embedding

Given a matrix $M = (m_{i,j}) \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$, one can construct an element $\phi(M) \in \mathrm{OG}(n, 2n)$ using the following rules. We will describe an $n \times 2n$ matrix $\widetilde{M} = (\widetilde{m}_{i,j})$, so that for all $i, j \in [n]$, each of $\widetilde{m}_{i,2j-1}$ and $\widetilde{m}_{i,2j}$ is equal to either $m_{i,j}$ or $-m_{i,j}$, as in Figure 4-1. Explicitly, for $i = j$ we put $\widetilde{m}_{i,2i-1} = \widetilde{m}_{i,2i} = m_{i,i} = 1$, and for $i \neq j$ we set

$$\widetilde{m}_{i,2j-1} = -\widetilde{m}_{i,2j} = (-1)^{i+j+\mathbb{1}(i<j)} m_{i,j}, \tag{4.1.4}$$

where $\mathbb{1}(i < j)$ denotes 1 if $i < j$ and 0 otherwise.

Remark 4.1.2. For each $i \in [n]$, the sum of columns $2i - 1$ and $2i$ of $\widetilde{M}$ is equal to $2e_i$, where e_i is the i-th standard basis vector in $\mathbb{R}^n$. Thus the matrix $\widetilde{M}$ has full rank, and we denote by $\phi(M) \in \mathrm{Gr}(n, 2n)$ its row span.

One can check that in fact $\phi(M)$ belongs to $\mathrm{OG}(n, 2n)$, see Corollary 4.4.6. We have thus constructed a map $\phi : \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1) \to \mathrm{OG}(n, 2n)$. Since boundary correlation matrices of planar Ising networks belong to the space $\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$, ϕ restricts to a map $\phi : \overline{\mathcal{X}}_n \to \mathrm{OG}(n, 2n)$. We are ready to state our main result.

Theorem 4.1.3. *The restriction $\phi : \overline{\mathcal{X}}_n \to \mathrm{OG}(n, 2n)$ is a stratification-preserving homeomorphism between $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$. Moreover, both spaces are homeomorphic to an $\binom{n}{2}$-dimensional closed ball.*

We prove the second part of Theorem 4.1.3 in Section 4.6, where we also deduce its first part from Theorems 4.4.17 and 4.5.5.

Remark 4.1.4. The second sentence of Theorem 4.1.3 is an application of the machinery developed in Chapter 2. The fact that the image $\phi(\overline{\mathcal{X}}_n)$ is a subset of $\mathrm{Gr}_{\geq 0}(n, 2n)$ can be deduced in a straightforward fashion from the work of Lis [Lis17], see Section 4.5.2.

Example 4.1.5. We illustrate Theorem 4.1.3 in the case $n = 2$. Let $\sigma_{12} := \langle \sigma_1\sigma_2 \rangle$, then the boundary correlation matrix M has the form

$$M = \begin{pmatrix} 1 & \sigma_{12} \\ \sigma_{12} & 1 \end{pmatrix}. \tag{4.1.5}$$

By definition, $\sigma_{12} \le 1$, and we also have $\sigma_{12} \ge 0$ by one of the *Griffiths' inequalities* [Gri67, Theorem 1]. In fact, if G has a single edge connecting the vertices b_1 and b_2 then it is easy to check that depending on $J_{\{b_1,b_2\}}$, σ_{12} can be any number strictly between 0 and 1. If we remove the edge $\{b_1, b_2\}$ from G, we get $\sigma_{12} = 0$. Thus $\mathcal{X}_n$ consists of all matrices M of the form (4.1.5) for $0 \le \sigma_{12} < 1$. If we contract the edge $\{b_1, b_2\}$, we get $\sigma_{12} = 1$. The resulting graph will no longer be embedded in a disk, because the boundary vertices b_1 and b_2 will get identified. This is an example of a *generalized planar Ising network* that we introduce in Section 4.5. We see that the closure $\overline{\mathcal{X}}_n$ of $\mathcal{X}_n$ consists of all matrices M of the form (4.1.5) for $0 \le \sigma_{12} \le 1$, and is stratified into three cells $\{\sigma_{12} = 0\}$, $\{0 < \sigma_{12} < 1\}$, and $\{\sigma_{12} = 1\}$. These three cells correspond to three possible matchings on $\{1, 2, 3, 4\}$, namely, $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$, respectively.

We have

$$\widetilde{M} = \begin{pmatrix} 1 & 1 & \sigma_{12} & -\sigma_{12} \\ -\sigma_{12} & \sigma_{12} & 1 & 1 \end{pmatrix}, \tag{4.1.6}$$

and $\phi(M) \in \mathrm{Gr}(n, 2n)$ is the row span of $\widetilde{M}$. The maximal minors of $\widetilde{M}$ are

$$\Delta_{12}(\widetilde{M}) = \Delta_{34}(\widetilde{M}) = 2\sigma_{12}, \quad \Delta_{14}(\widetilde{M}) = \Delta_{23}(\widetilde{M}) = 1 - \sigma_{12}^2,$$

$$\Delta_{13}(\widetilde{M}) = \Delta_{24}(\widetilde{M}) = 1 + \sigma_{12}^2.$$

It follows that $\phi(M)$ belongs to $\mathrm{OG}(n, 2n)$ for all $\sigma_{12} \in \mathbb{R}$, and moreover, we get $\phi(M) \in \mathrm{OG}_{\ge 0}(n, 2n)$ precisely when $0 \le \sigma_{12} \le 1$. Note that $\widetilde{M}$ is a matrix but $\phi(M)$ is an element of the Grassmannian, and thus the Plucker coordinates of $\phi(M)$ are only defined up to rescaling. Nevertheless, we can recover σ_{12} from these minors as follows:

$$\sigma_{12} = \frac{\Delta_{12}(\phi(M))}{\Delta_{13}(\phi(M)) + \Delta_{14}(\phi(M))}. \tag{4.1.7}$$

Thus we see that for $n = 2$, the map ϕ is indeed a homeomorphism, and both spaces $\overline{\mathcal{X}}_n$ and $\mathrm{OG}_{\geq 0}(n, 2n)$ are homeomorphic to $[0, 1]$, which is an $\binom{n}{2} = 1$-dimensional closed ball.

4.2 Consequences of the main construction

In this section, we give further results on the relationship between the Ising model and the orthogonal Grassmannian.

4.2.1 Reconstructing correlations from minors

Our first goal is, given an element $X \in \mathrm{OG}_{\geq 0}(n, 2n)$, to find explicitly a matrix $M = (m_{i,j}) \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$ such that X is the row span of $\widetilde{M}$. For the case $n = 2$, this was done in (4.1.7). In order to deal with the general case, we give the following important definition.

Definition 4.2.1. Given a subset $S \subset [n]$, we denote by $\mathcal{E}_n(S) \subset \binom{[2n]}{n}$ the collection of n-element subsets I of $[2n]$ such that for each $i \in [n]$, the intersection $I \cap \{2i - 1, 2i\}$ has even size if and only if $i \in S$.

The following result, proved in Section 4.5, is a simple consequence of Remark 4.1.2.

Lemma 4.2.2. *Let $M = (m_{i,j}) \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$ be a matrix. Then for each $i, j \in [n]$, we have*

$$m_{i,j} = \frac{\sum_{I \in \mathcal{E}_n(\{i,j\})} \Delta_I(\phi(M))}{\sum_{I \in \mathcal{E}_n(\emptyset)} \Delta_I(\phi(M))} = 2^{-n} \sum_{I \in \mathcal{E}_n(\{i,j\})} \Delta_I(\widetilde{M}). \tag{4.2.1}$$

We stress again that unlike $\widetilde{M}$, the maximal minors of $\phi(M)$ are defined up to a common scalar, so it only makes sense to talk about their ratios. However, for the specific matrix $\widetilde{M}$, we have

$$\sum_{I \in \mathcal{E}_n(\emptyset)} \Delta_I(\widetilde{M}) = 2^n, \tag{4.2.2}$$

by the multilinearity of the determinant, see Remark 4.1.2. Thus (4.2.2) explains why the two expressions for $m_{i,j}$ given in (4.2.1) are actually equal.

For example, for $n = 2$, (4.2.2) becomes

$$\Delta_{13}(\widetilde{M}) + \Delta_{14}(\widetilde{M}) + \Delta_{23}(\widetilde{M}) + \Delta_{24}(\widetilde{M}) = 4,$$

and for $i = 1$ and $j = 2$, Lemma 4.2.2 gives another expression for σ_{12}:

$$\sigma_{12} = \frac{\Delta_{12}(\phi(M)) + \Delta_{34}(\phi(M))}{\Delta_{13}(\phi(M)) + \Delta_{14}(\phi(M)) + \Delta_{23}(\phi(M)) + \Delta_{24}(\phi(M))},$$

which is easily seen to be equivalent to (4.1.7).

4.2.2 Cyclic symmetry and Kramers–Wannier's duality

A nice application of Theorem 4.1.3 is a cyclic symmetry of the space $\overline{\mathcal{X}}_n$, which comes from the cyclic symmetry of $\mathrm{OG}_{\geq 0}(n, 2n)$. It turns out that the cyclic shift operation on $\mathrm{OG}_{\geq 0}(n, 2n)$ corresponds to a generalization of the Kramers-Wannier duality [KW41] that switches between the *high and low temperature expansions* for the Ising model.

Let $k \leq N$, and consider a linear operator $S : \mathbb{R}^N \to \mathbb{R}^N$ mapping a row vector $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ to $v \cdot S = (v_2, v_3, \dots, v_N, (-1)^{k-1} v_1)$, see Section 2.2.1. As a matrix, S is given by $S_{i+1,i} = 1$ for $i \in [N-1]$, and $S_{1,N} = (-1)^{k-1}$. A simple observation is that multiplying a $k \times N$ matrix A with nonnegative maximal minors by S on the right yields another $k \times N$ matrix with nonnegative maximal minors. Since multiplication on the right commutes with the left $\mathrm{GL}_k(\mathbb{R})$-action, we get a cyclic shift operator on $\mathrm{Gr}_{\geq 0}(k, N)$ mapping $X \in \mathrm{Gr}_{\geq 0}(k, N)$ to $X \cdot S \in \mathrm{Gr}_{\geq 0}(k, N)$. It is clear from the definitions that for $\mathrm{Gr}_{\geq 0}(n, 2n)$, this action restricts to a cyclic shift action on $\mathrm{OG}_{\geq 0}(n, 2n)$. For example, if $X \in \mathrm{OG}_{\geq 0}(2, 4)$ is the row span of the matrix $\widetilde{M}$ given in (4.1.6) then $X \cdot S$ is represented by

$$\widetilde{M} \cdot S = \begin{pmatrix} 1 & \sigma_{12} & -\sigma_{12} & -1 \\ \sigma_{12} & 1 & 1 & \sigma_{12} \end{pmatrix}. \tag{4.2.3}$$

One can check that the row span $X \cdot S$ of this matrix again belongs to $\mathrm{OG}_{\geq 0}(2, 4)$. By

Theorem 4.1.3, there must exist a matrix $M' = \begin{pmatrix} 1 & \sigma'_{12} \\ \sigma'_{12} & 1 \end{pmatrix}$ such that $\phi(M') = X \cdot S$ in $\mathrm{OG}_{\geq 0}(2,4)$ (i.e., such that $\widetilde{M'}$ is obtained from the matrix in (4.2.3) by row operations). The value of σ'_{12} can be found from the minors of $X \cdot S$ using (4.1.7):

$$\sigma'_{12} = \frac{\Delta_{12}(X \cdot S)}{\Delta_{13}(X \cdot S) + \Delta_{14}(X \cdot S)} = \frac{1 - \sigma_{12}^2}{1 + \sigma_{12}^2 + 2\sigma_{12}} = \frac{1 - \sigma_{12}}{1 + \sigma_{12}}.$$

Thus the cyclic shift operation on $\mathrm{OG}_{\geq 0}(n, 2n)$ yields an automorphism of $\overline{\mathcal{X}}_n$ which has order $2n$ for $n > 2$ and order n for $n = 1, 2$. For $n = 2$, it sends σ_{12} to $\frac{1-\sigma_{12}}{1+\sigma_{12}}$.

Let us now formulate a generalization of the duality of [KW41].

Definition 4.2.3. Let $N = (G, J)$ be a connected[1] planar Ising network. The *dual planar Ising network* $N^* := (G^*, J^*)$ is defined as follows. The graph $G^* = (V^*, E^*)$ is the planar dual graph of G, with boundary vertices $b_1^*, \dots, b_n^*$ placed counterclockwise on the boundary of the disk so that b_i^* is between b_i and b_{i+1}. For $e \in E$, we denote by e^* the edge of G^* that crosses $e \in E$, and thus we have $E^* = \{e^* \mid e \in E\}$. The edge parameters $J^*_{e^*} \in \mathbb{R}_{>0}$ are defined uniquely by the condition that

$$\sinh(2J^*_{e^*}) = \frac{1}{\sinh(2J_e)} \tag{4.2.4}$$

for all $e \in E$.

For example, if G is the graph in Figure 4-4 (left) then its dual G^* is shown in Figure 4-4 (middle). Note that we have $\sinh(2t) = 1/\sinh(2t)$ if and only if $t = \frac{1}{2}\log(\sqrt{2}+1)$ is the *critical temperature* of the Ising model. We also remark that applying the duality twice yields the same planar Ising network except that its boundary vertex labels are cyclically shifted: $(b_i^*)^* = b_{i+1}$. We prove the following result in Section 4.6.

Theorem 4.2.4. *Let $N = (G, J)$ be a connected planar Ising network with dual planar Ising network $N^* := (G^*, J^*)$. Then the correlation matrices $M := M(G, J)$ and $M^* := M(G^*, J^*)$*

[1] This definition can be easily extended to all (not necessarily connected) generalized planar Ising networks.

are related by the cyclic shift on $\mathrm{OG}_{\geq 0}(n,2n)$*:*

$$\phi(M)\cdot S=\phi(M^*).$$

According to Theorem 1.1.1, the space $\mathrm{Gr}_{\geq 0}(k,N)$ is homeomorphic to a closed ball. The main ingredient of the proof of this result is the cyclic symmetry of $\mathrm{Gr}_{\geq 0}(k,N)$. In Section 4.6, we use the above cyclic symmetry of $\mathrm{OG}_{\geq 0}(n,2n)$ in a similar way to show that it is a closed ball, which by the first part of Theorem 4.1.3 implies that the space $\overline{\mathcal{X}}_n$ of boundary correlation matrices is homeomorphic to a closed ball as well.

Recall from Section 2.2.1 that we have a unique cyclically symmetric point $X_0\in\mathrm{Gr}_{\geq 0}(n,2n)$.

It will follow from our proof of Theorem 4.1.3 that this point X_0 actually belongs to $\mathrm{OG}_{\geq 0}(n,2n)$, and thus corresponds to some special planar Ising network with n boundary vertices. For instance, for $n=2$ this is the Ising network N with one edge e such that $J_e=\frac{1}{2}\log(\sqrt{2}+1)$. This planar Ising network is *self-dual*, i.e., satisfies $N=N^*$. However, it is easy to see that for $n=3$ there are no self-dual planar Ising networks. Nevertheless, as our next result shows, for each n, there exists a (usually not unique) planar Ising network $N=(G,J)$ with n boundary vertices such that the boundary correlation matrices of N and N^* coincide: $M(G,J)=M(G^*,J^*)$.

Proposition 4.2.5. *For each* $n\geq 1$*, there exists a unique boundary correlation matrix* $M_0\in\overline{\mathcal{X}}$ *of some planar Ising network such that the element* $\phi(M_0)\in\mathrm{OG}_{\geq 0}(n,2n)$ *is* cyclically symmetric*, i.e., satisfies* $\phi(M_0)\cdot S=\phi(M_0)$*. For any planar Ising network* $N=(G,J)$ *satisfying* $M(G,J)=M$*, we have* $M(G,J)=M(G^*,J^*)$*.*

See Section 4.6 for the proof.

Remark 4.2.6. Consider a planar Ising network $N=(G,J)$ such that G is the intersection of the square lattice of small side length δ with a disk, and let $J_e=\frac{1}{2}\log(\sqrt{2}+1)$ be critical for all $e\in E$. The dual network $N^*=(G^*,J^*)$ is "very close" to N in the sense that it is obtained by shifting N by $(\delta/2,\delta/2)$ and making some adjustments near the boundary of the disk. Thus one could argue that the boundary correlations of N are "very close" to being cyclically symmetric, in which case we can find them explicitly from (4.2.2) and (2.2.1). It

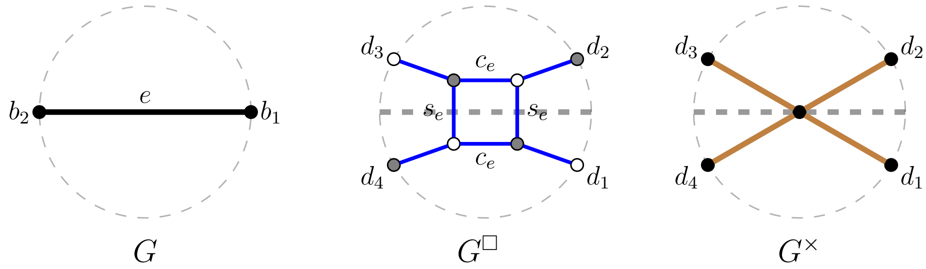


Figure 4-2: Transforming a graph G (left) with one edge e connecting two boundary vertices b_1 and b_2 into a bipartite graph $G^\square$ (middle) with eight edges. Four of those edges are incident to the boundary and have weight 1, and the rest have weights s_e, c_e, s_e, c_e, as shown in the figure. The corresponding medial graph $G^\times$ from Section 4.2.5 is shown on the right.

seems plausible to us that this approach may be applied to studying the universality of the scaling limit as $\delta \to 0$. The notion of being "very close" is asymptotic and thus is left beyond the scope of this paper.

4.2.3 Reduction to the dimer model

Lemma 4.2.2 shows that each two-point correlation function is a ratio of two sums of minors of an element of $\mathrm{OG}_{\geq 0}(n, 2n)$. In the next section, we apply a well known result that each minor of an element of $\mathrm{Gr}_{\geq 0}(k, N)$ is equal to a weighted sum of matchings in a certain planar bipartite graph.

Suppose that we are given a planar Ising network $N = (G, J)$. We introduce two functions $s, c : E \to (0, 1)$ satisfying $s_e^2 + c_e^2 = 1$ for each $e \in E$, as follows. Given $e \in E$, we set

$$s_e := \operatorname{sech}(2J_e) = \frac{2}{\exp(2J_e) + \exp(-2J_e)}; \quad c_e := \tanh(2J_e) = \frac{\exp(2J_e) - \exp(-2J_e)}{\exp(2J_e) + \exp(-2J_e)}. \tag{4.2.5}$$

Next, we transform G into a weighted planar bipartite graph (a *plabic graph* in the sense of [Pos07]) $G^\square$ embedded in a disk, as in Figures 4-2 and 4-4: we replace each edge $e \in E$ of G by a bipartite square as in Figure 4-2 (middle), and connect two such squares if the corresponding edges of G share both a vertex and a face of G. Additionally, we connect each of the $2n$ boundary vertices of $G^\square$, which we label $d_1, \ldots, d_{2n}$ in counterclockwise order, to a unique vertex of $G^\square$ in the interior of the disk in an obvious way (as in Figure 4-4). Thus d_i

is a white (resp., black) vertex if i is odd (resp., even). See Definitions 4.4.12 and 4.5.2 for a precise description of the rules for constructing $G^\square$ from G. We call $G^\square$ *the plabic graph associated with* G.

Let us now describe the *boundary measurement map* of [Pos07, Tal08], as explained in [Lam16].

Definition 4.2.7. An *almost perfect matching* of $G^\square$ is a collection $\mathcal{A}$ of edges of $G^\square$ such that every vertex of $G^\square$ is incident to at most one edge in $\mathcal{A}$, and every non-boundary vertex of $G^\square$ is incident to exactly one edge in $\mathcal{A}$. The *boundary* of $\mathcal{A}$ is a subset $\partial(\mathcal{A}) \subset [2n]$ which consists of all odd indices i such that d_i is not incident to an edge of $\mathcal{A}$ together with all even indices i such that d_i is incident to an edge of $\mathcal{A}$. We define the *weight* $\mathrm{wt}(\mathcal{A})$ of $\mathcal{A}$ to be the product of weights of all edges in $\mathcal{A}$.

It is not hard to see that $\partial(\mathcal{A})$ has size n for any almost perfect matching $\mathcal{A}$ of $G^\square$. We are prepared to give a formula for the boundary correlation functions which is very similar to Kenyon and Wilson's *grove measurement* formula [KW11]. See Section 4.6 for the proof.

Theorem 4.2.8. *Let $N = (G, J)$ be a planar Ising network and $M = M(G, J)$ be its boundary correlation matrix. Consider the element $\phi(M) \in \mathrm{OG}_{\geq 0}(n, 2n)$, and let $G^\square$ be the weighted planar bipartite graph described above. Then up to a common rescaling, for every $I \in \binom{[2n]}{n}$ we have*

$$\Delta_I(\phi(M)) = \sum_{\mathcal{A}:\partial(\mathcal{A})=I} \mathrm{wt}(\mathcal{A}), \tag{4.2.6}$$

where the sum is over almost perfect matchings $\mathcal{A}$ of $G^\square$ with boundary I.

For example, consider the graph $G^\square$ in Figure 4-2 (middle). There is a single almost perfect matching of $G^\square$ with boundary $\{1, 2\}$, shown in Figure 4-3 (left). Similarly, there are two almost perfect matchings of $G^\square$ with boundary $\{1, 3\}$ and one almost perfect matching with boundary $\{1, 4\}$, also shown in Figure 4-3. Therefore by Theorem 4.2.8 we get

$$\Delta_{12}(\phi(M)) = c_e, \quad \Delta_{13}(\phi(M)) = c_e^2 + s_e^2 = 1, \quad \Delta_{14}(\phi(M)) = s_e,$$

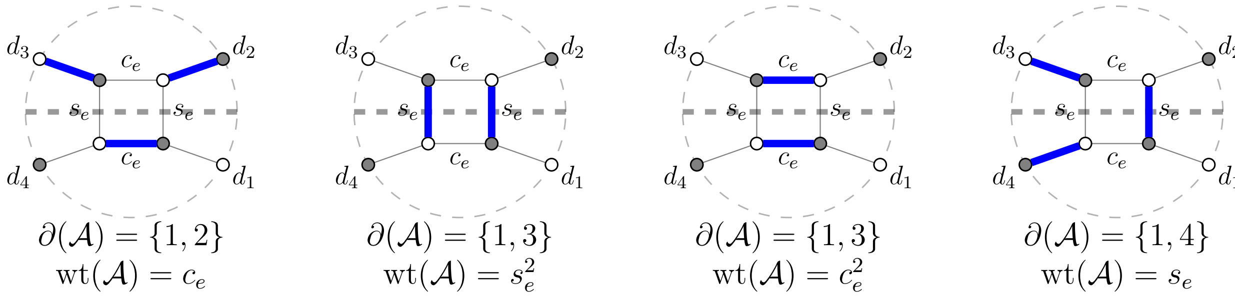


Figure 4-3: Some almost perfect matchings of $G^\square$, together with their boundaries and weights.

up to a common rescaling. By (4.1.7), we should have

$$\sigma_{12} = \frac{\Delta_{12}(\phi(M))}{\Delta_{13}(\phi(M)) + \Delta_{14}(\phi(M))} = \frac{c_e}{1+s_e}, \tag{4.2.7}$$

where $s_e = \operatorname{sech}(2J_e)$ and $c_e = \tanh(2J_e)$ are expressed in terms of J_e as in (4.2.5). Simplifying the expressions, we get

$$\sigma_{12} = \frac{\exp(J_e) - \exp(-J_e)}{\exp(J_e) + \exp(-J_e)}. \tag{4.2.8}$$

On the other hand, by the definition of the Ising model, the partition function is equal to $Z = 2(\exp(J_e) + \exp(-J_e))$ and thus the correlation $\langle \sigma_1 \sigma_2 \rangle$ is

$$\langle \sigma_1 \sigma_2 \rangle = \frac{2}{Z}\left(\exp(J_e) - \exp(-J_e)\right) = \frac{\exp(J_e) - \exp(-J_e)}{\exp(J_e) + \exp(-J_e)},$$

in agreement with Theorem 4.2.8.

Lemma 4.2.2 and Theorem 4.2.8 together give a new simple way to express boundary correlations in terms of almost perfect matchings which we summarize in the following corollary.

Corollary 4.2.9. *Let $N = (G, J)$ be a planar Ising network. Then for all $i, j \in [n]$, the corresponding boundary correlation function is given by*

$$\langle \sigma_i \sigma_j \rangle = \frac{\sum_{\mathcal{A}:\partial(\mathcal{A}) \in \mathcal{E}_n(\{i,j\})} \mathrm{wt}(\mathcal{A})}{\sum_{\mathcal{A}:\partial(\mathcal{A}) \in \mathcal{E}_n(\emptyset)} \mathrm{wt}(\mathcal{A})}, \tag{4.2.9}$$

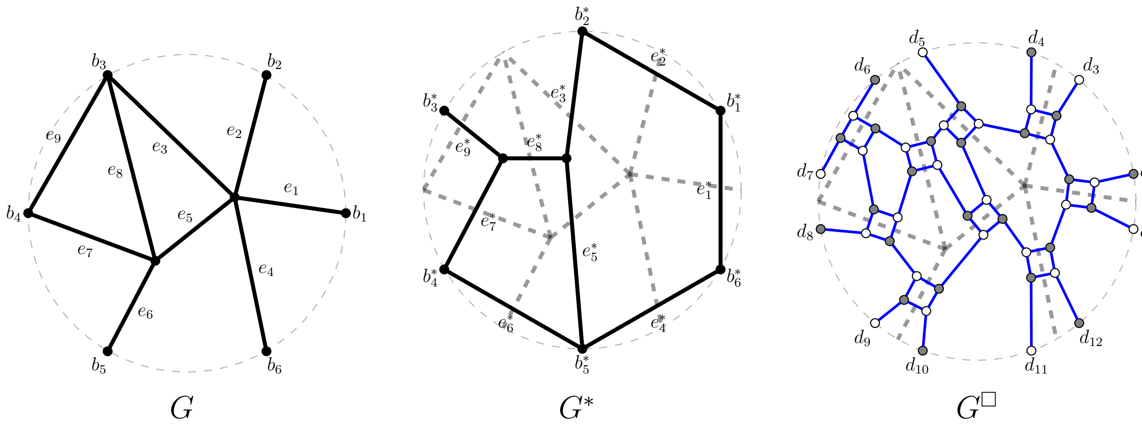


Figure 4-4: A planar graph G embedded in a disk (left), its dual G^* (middle), and the corresponding plabic graph $G^\square$ (right).

where the sums in the numerator and the denominator are over almost perfect matchings $\mathcal{A}$ of $G^\square$.

Remark 4.2.10. Corollary 4.2.9 is related to *Dubédat's bosonization identity* [Dub11]. Let us introduce a planar bipartite graph $\widehat{G}^\square$ which is obtained from $G^\square$ by simply adding an extra edge connecting d_{2i-1} to d_{2i} for all $i \in [n]$. Then there is an elegant formula (see Proposition 4.5.8) expressing the *squared* boundary correlation $\langle\sigma_i\sigma_j\rangle^2$ as a ratio of sums of perfect matchings in $\widehat{G}^\square$. We explain how to relate this formula to (4.2.9) in Section 4.5.1. We thank Marcin Lis for bringing the paper [Dub11] to our attention.

4.2.4 Generalized Griffiths' inequalities

As we have already noted, the fact that we have $J_e > 0$ for all edges e implies that all two-point correlation functions $\langle\sigma_u\sigma_v\rangle$ are nonnegative [Gri67]. Equation (4.2.1) shows that $\langle\sigma_u\sigma_v\rangle$ is a positive linear combination of the minors of $\widetilde{M}$, and thus its nonnegativity follows from Theorem 4.1.3. More generally, for every subset $A \subset [n]$, define

$$\langle\sigma_A\rangle := \langle\prod_{i\in A}\sigma_{b_i}\rangle = \sum_{\sigma\in\{-1,1\}^V}\mathbf{P}(\sigma)\prod_{i\in A}\sigma_{b_i}$$

to be the expectation of the product of the spins in A. The following *generalized Griffiths' inequalities* were proved in [KS68].

Proposition 4.2.11 ([KS68]). *For every $A \subset [n]$, we have*

$$\langle \sigma_A \rangle \geq 0.$$

For every $A, B \subset [n]$, we have

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle \geq 0.$$

Here $\langle \sigma_A \sigma_B \rangle = \langle \prod_{i \in A \oplus B} \sigma_{b_i} \rangle$, where $A \oplus B = (A \setminus B) \cup (B \setminus A)$ denotes the symmetric difference of A and B.

The goal of our next result is to explain how both inequalities in Proposition 4.2.11 also arise as positive linear combinations of the minors of the matrix $\widetilde{M}$, where $M = M(G, J)$ is the boundary correlation matrix.

Definition 4.2.12. For $A \subset [n]$, we define $\widetilde{A} := \{2i-1 \mid i \in A\} \cup \{2i \mid i \in A\}$, and for $\epsilon \in \{0, 1\}$, we let $\mathcal{D}^\epsilon(A) \subset \binom{[2n]}{n}$ be the set of all $I \in \binom{[2n]}{n}$ such that the sum of elements of $I \cap \widetilde{A}$ is equal to ϵ modulo 2.

Recall also the notation $\mathcal{E}_n(S)$ from Definition 4.2.1. We prove the following result in Section 4.7.

Theorem 4.2.13. *For every $A \subset [n]$, we have*

$$\langle \sigma_A \rangle = 2^{-n} \sum_{I \in \mathcal{E}_n(A)} \Delta_I(\widetilde{M}). \tag{4.2.10}$$

For every $A, B \subset [n]$, there exists $\epsilon \in \{0, 1\}$, given explicitly in (4.7.1)*, such that*

$$\langle \sigma_A \sigma_B \rangle - \langle \sigma_A \rangle \langle \sigma_B \rangle = 2^{-n+1} \sum_{I \in \mathcal{E}_n(A \oplus B) \cap \mathcal{D}^\epsilon(B)} \Delta_I(\widetilde{M}). \tag{4.2.11}$$

Thus the inequalities of Proposition 4.2.11 become manifestly true when expressed in terms of minors of $\widetilde{M}$, which are nonnegative by Theorem 4.1.3.

For example, when $n = 2$ and $A = B = \{1, 2\}$, we have $\epsilon = 1$ by (4.7.1), and thus (4.2.11)

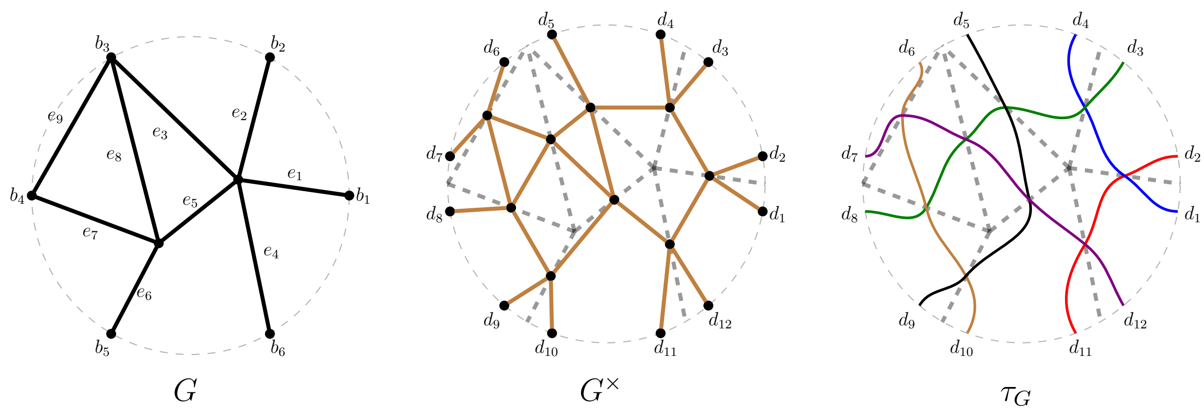


Figure 4-5: A planar graph G embedded in a disk (left), the corresponding medial graph $G^\times$ (middle) and its medial strands (right).

becomes

$$1 - \sigma_{12}^2 = \frac{\Delta_{14}(\widetilde{M}) + \Delta_{23}(\widetilde{M})}{2}.$$

4.2.5 Inverse problem

In this section, we concentrate on answering the following question.

Question 4.2.14. Given a planar Ising network $N = (G, J)$, is it possible to reconstruct J from the matrix $M(G, J)$? In other words, is it true that the function $J : E \to \mathbb{R}_{>0}$ is uniquely determined by G and $M(G, J)$?

Of course, the answer to this question is negative if, for example, G has more than $\binom{n}{2}$ edges. In order to fix this, we introduce *medial graphs*. Namely, given a planar graph G embedded in a disk, the *medial graph* $G^\times$ associated with G is a planar graph obtained from G as in Figure 4-2 (right) and Figure 4-5 (middle). It has $2n$ boundary vertices $d_1, \dots, d_{2n}$, each of degree 1, and $|E|$ interior vertices, each of degree 4. See Section 4.5 for a precise description.

Since each interior vertex of $G^\times$ has degree 4, we define a *medial strand* in $G^\times$ to be a path that starts at a boundary vertex d_i of $G^\times$, follows the only edge of $G^\times$ incident to it, and then goes "straight" at each interior vertex of degree 4, until it reaches another boundary vertex d_j of $G^\times$. (More precisely, a medial strand is determined by the condition that whenever two of its edges share a vertex, they do not share a face.) Clearly there are

n medial strands in $G^\times$, and each of them connects d_i to d_j for some $i, j \in [2n]$, giving rise to a matching τ_G on $[2n]$ called the *medial pairing associated with* G:

$$\tau_G := \left\{\{i, j\} \subset [2n] \mid \text{a medial strand in } G^\times \text{ starts at } d_i \text{ and ends at } d_j\right\}.$$

Thus τ_G is a partition of $[2n]$ into n sets, each of size 2. For example, for the graph G in Figure 4-5 (left), there is a medial strand that starts at vertex d_3, then follows the midpoints of edges e_2, e_3, e_8, e_7, and then terminates at vertex d_8. Thus the medial pairing τ_G of G contains a pair $\{3, 8\}$. Following the other five medial strands, we find

$$\tau_G = \{\{1, 4\}, \{2, 11\}, \{3, 8\}, \{5, 9\}, \{6, 10\}, \{7, 12\}\}, \tag{4.2.12}$$

see Figure 4-5 (right).

Definition 4.2.15. For $i < j, i' < j' \in [2n]$, we say that pairs $\{i, j\}$ and $\{i', j'\}$ *form a crossing* if either $i < i' < j < j'$ or $i' < i < j' < j$. For a matching τ of $[2n]$, we let $\operatorname{xing}(\tau)$ denote the number of pairs in τ that form a crossing.

For example, if τ_G is given in (4.2.12) then $\operatorname{xing}(\tau_G) = 9$. We are now ready to state an important definition, introduced in [CIM98].

Definition 4.2.16. We say that G is *reduced* if its number $|E|$ of edges equals $\operatorname{xing}(\tau_G)$.

For example, the graph G in Figure 4-5 (left) is reduced since it has $|E| = 9 = \operatorname{xing}(\tau_G)$ edges.

We will see later in Proposition 4.5.4 that if we fix G and let J vary, the space of matrices $M(G, J)$ obtained in such a way is an open ball of dimension $\operatorname{xing}(\tau_G)$. Since J varies over $\mathbb{R}^E_{>0}$, we see that if G is not reduced (in which case clearly $|E| > \operatorname{xing}(\tau_G)$) then the answer to Question 4.2.14 is negative. On the other hand, if G is reduced then the answer turns out to be always positive, as we show in Section 4.6.

Theorem 4.2.17. *Let $N = (G, J)$ be a planar Ising network such that G is reduced. Then the map $J \mapsto M(G, J)$ is injective, i.e., for each matrix $M \in \operatorname{Mat}^{\mathrm{sym}}_n(\mathbb{R}, 1)$, there is at most one function $J : E \to \mathbb{R}_{>0}$ such that $M = M(G, J)$.*

Theorem 4.1.3 combined with our results in Section 4.4 gives a simple explicit way to test whether for a given reduced graph G and a matrix $M \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$ there exists a function $J : E \to \mathbb{R}_{>0}$ such that $M = M(G, J)$.

In order to describe a recursive way of reconstructing J from $M(G, J)$, we introduce operations of *adjoining a boundary spike* and *adjoining a boundary edge* to G. An identical construction in the case of electrical networks has been considered in [CIM98].

Definition 4.2.18. Let $N' = (G', J')$ be a planar Ising network, where $G' = (V', E')$ has n boundary vertices. Given $k \in [n]$, we say that another planar Ising network $N = (G, J)$ with n boundary vertices is obtained from N' by *adjoining a boundary spike at k* if the vertex b_k in G is incident to a single edge e and contracting this edge in G yields G'. Similarly, we say that N is obtained from N' by *adjoining a boundary edge* between k and $k+1$ if G contains an edge e connecting b_k and b_{k+1}, and removing this edge from G yields G'. In both cases, we additionally require that the restriction of $J : E \to \mathbb{R}_{>0}$ to $E' = E \setminus \{e\}$ coincides with $J' : E' \to \mathbb{R}_{>0}$.

When adjoining boundary edges, we allow for $k = n$, in which case we set $k+1 := 1$. We denote $t := J_e$, and our first goal will be to reconstruct t from the matrix $M(G, J)$.

Definition 4.2.19. Let $i \in [2n]$. Consider a total order $\prec_i$ on $[2n]$ given by

$$i \prec_i i+1 \prec_i \cdots \prec_i 2n \prec_i 1 \prec_i \cdots \prec_i i-1,$$

where the indices are taken modulo $2n$. For a planar Ising network $N = (G, J)$ and $i \in [2n]$, define subsets $I_i^{\min}(G), I_i^{\max}(G) \in \binom{[2n]}{n}$ whose disjoint union is $[2n]$ as follows. For each unordered pair $\{a, b\}$ of τ_G, we may assume that $a \prec_i b$, and then we let $a \in I_i^{\min}(G)$ and $b \in I_i^{\max}(G)$. In particular, we always have $i \in I_i^{\min}(G)$ and $i-1 \in I_i^{\max}(G)$.

For example, recall that if G is the graph from Figure 4-5 (left) then τ_G is given by (4.2.12). For $i = 7$ and $i = 12$, we have

$$\begin{aligned} I_7^{\min}(G) &= \{7, 8, 9, 10, 11, 1\}, \quad I_7^{\max}(G) = \{6, 5, 4, 3, 2, 12\}, \\ I_{12}^{\min}(G) &= \{12, 1, 2, 3, 5, 6\}, \quad I_{12}^{\max}(G) = \{11, 10, 9, 8, 7, 5\}. \end{aligned} \tag{4.2.13}$$

We prove our next result in Section 4.6.

Theorem 4.2.20. *Let G be a reduced planar graph embedded in a disk.*

- *Suppose that $N = (G, J)$ is obtained from $N' = (G', J')$ by adjoining a boundary spike e at $k \in [n]$. Let $M := M(G, J)$, $\tilde{k} := 2k - 1$, and $t := J_e$. Then for $I := I^{\min}_{\tilde{k}+1}(G)$, we have*
$$s_e = \operatorname{sech}(2t) = \frac{\Delta_I(\phi(M))}{\Delta_{I \cup \{\tilde{k}\} \setminus \{\tilde{k}+1\}}(\phi(M))}.$$
- *Suppose that $N = (G, J)$ is obtained from $N' = (G', J')$ by adjoining a boundary edge e between $k \in [n]$ and $k + 1$. Let $M := M(G, J)$, $\tilde{k} := 2k$, and $t := J_e$. Then for $I := I^{\max}_{\tilde{k}+1}(G)$, we have*
$$c_e = \tanh(2t) = \frac{\Delta_I(\phi(M))}{\Delta_{I \cup \{\tilde{k}+1\} \setminus \{\tilde{k}\}}(\phi(M))}.$$

For example, the graph G in Figure 4-5 (left) can be obtained from another reduced graph by adjoining a boundary spike e_4 at $k = 6$, so we have $\tilde{k} = 11$ in the first part of Theorem 4.2.20. Since $I = I^{\min}_{12}(G) = \{1, 2, 3, 5, 6, 12\}$ by (4.2.13), we have
$$s_{e_4} = \frac{\Delta_{\{1,2,3,5,6,12\}}(\phi(M))}{\Delta_{\{1,2,3,5,6,11\}}(\phi(M))}.$$
Similarly, G can be obtained from another reduced graph by adjoining a boundary edge e_9 between $k = 3$ and $k + 1 = 4$, so we have $\tilde{k} = 6$ in the second part of Theorem 4.2.20. Since $I = I^{\max}_7(G) = \{2, 3, 4, 5, 6, 12\}$ by (4.2.13), we have
$$c_{e_9} = \frac{\Delta_{\{2,3,4,5,6,12\}}(\phi(M))}{\Delta_{\{2,3,4,5,7,12\}}(\phi(M))}.$$

Since both functions $\operatorname{sech}, \tanh : (0, \infty) \to (0, 1)$ are strictly monotone, it follows from Theorem 4.2.20 that we can reconstruct $t = J_e$ uniquely from $M(G, J)$ whenever e is either a boundary spike or a boundary edge of G. This constitutes the first step of our reconstruction algorithm, in view of the following result.

Proposition 4.2.21. *Suppose that G is a connected reduced planar graph embedded in a disk, having at least one edge. Then G is obtained from another reduced graph G' by adjoining either a boundary spike or a boundary edge.*

We note that the graph G' above need not be connected. Also, if G itself is not connected then it is clearly enough to solve the inverse problem for each connected component of G separately, and thus we may assume that G is connected. See Lemma 4.4.18 for a generalization of Proposition 4.2.21.

Proposition 4.2.21 says that given a reduced graph G and a matrix $M(G, J)$, we can reconstruct J_e for at least one edge e of G. A natural thing to do now would be to contract e if it is a boundary spike and remove e if it is a boundary edge, obtaining the reduced graph G'. Our next goal is to explain the relationship between the matrices $M(G, J)$ and $M(G', J')$ in the case when $N = (G, J)$ is obtained from $N' = (G', J')$ by adjoining either a boundary spike or a boundary edge.

We note that these two operations look like they have a very different effect on the boundary correlation matrix. For example, adjoining a boundary spike at k only changes the correlation $\langle \sigma_i \sigma_j \rangle$ when either i or j is equal to k, but adjoining a boundary edge between k and $k+1$ in general changes all entries of the boundary correlation matrix. Surprisingly, these two operations have exactly the same form when written in terms of the matrix $\widetilde{M}$, as we now explain. (In fact, it is clear that applying the duality from Section 4.2.2 switches the roles of boundary spikes and boundary edges.)

Suppose that $N = (G, J)$ is obtained from $N' = (G', J')$ by adjoining a boundary spike e at $k \in [n]$ (resp., a boundary edge e between k and $k+1$). Define $\tilde{k} := 2k-1$ (resp., $\tilde{k} := 2k$), $t := J_e$, $s_e := \operatorname{sech}(2t)$, $c_e := \tanh(2t)$, as in Theorem 4.2.20, and let $g = g_{\tilde{k}}(t)$ be a $2n \times 2n$ matrix which coincides with the identity matrix except that it contains a 2×2 block $R_{\tilde{k}}$ in rows and columns indexed by $\tilde{k}$ and $\tilde{k}+1$. When $\tilde{k}$ is odd, we set $R_{\tilde{k}} := \begin{pmatrix} 1/c_e & s_e/c_e \\ s_e/c_e & 1/c_e \end{pmatrix}$ and when $\tilde{k}$ is even, we set $R_{\tilde{k}} := \begin{pmatrix} 1/s_e & c_e/s_e \\ c_e/s_e & 1/s_e \end{pmatrix}$. In the case where we have $\tilde{k} = 2n$, i.e., when we are adding a boundary edge between $k = n$ and $k+1 = 1$, the relevant entries of g are $g_{2n,2n} = g_{1,1} = 1/s_e$ and $g_{1,2n} = g_{2n,1} = (-1)^{n-1} c_e/s_e$. This sign twist is related to the cyclic symmetry of $\mathrm{Gr}_{\geq 0}(n, 2n)$ as we explained in Section 4.2.2. For example, for $n = 2$ and $\tilde{k} = 1, 4$, we have

$$g_1(t) = \begin{pmatrix} 1/c_e & s_e/c_e & 0 & 0 \\ s_e/c_e & 1/c_e & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad g_4(t) = \begin{pmatrix} 1/s_e & 0 & 0 & -c_e/s_e \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -c_e/s_e & 0 & 0 & 1/s_e \end{pmatrix}.$$

Recall that given an element $X \in \mathrm{Gr}(n, 2n)$ and a $2n \times 2n$ invertible real matrix g, an element $X \cdot g \in \mathrm{Gr}(n, 2n)$ is well defined as the row span of $A \cdot g$ where A is any $n \times 2n$ matrix whose row span is X.

Theorem 4.2.22. *Suppose that $N = (G, J)$ is obtained from $N' = (G', J')$ by adjoining a boundary spike e at $k \in [n]$ (resp., a boundary edge e between k and $k+1$). Let $M = M(G, J)$, $M' = M(G', J')$, and $g_{\tilde{k}}(t)$ be as above. Then we have*

$$\phi(M) = \phi(M') \cdot g_{\tilde{k}}(t).$$

Theorems 4.2.20 and 4.2.22 give the following inductive algorithm for reconstructing the function $J : E \to \mathbb{R}_{>0}$ for a given reduced graph $G = (V, E)$ from the matrix $M = M(G, J)$. The problem is trivial when G has no edges. Otherwise by Proposition 4.2.21, there is either a boundary spike or a boundary edge e in G. The matrix M gives an element $\phi(M) \in \mathrm{OG}_{\geq 0}(n, 2n)$, from which we compute either s_e or c_e using Theorem 4.2.20 and thus find $t = J_e$. After that, we contract e in G if it is a boundary spike and remove it if it is a boundary edge, and also modify the matrix M accordingly: we let

$$X' := \phi(M) \cdot (g_{\tilde{k}}(t))^{-1} \in \mathrm{OG}_{\geq 0}(n, 2n),$$

where $(g_{\tilde{k}}(t))^{-1}$ can be found using

$$\begin{pmatrix} 1/c_e & s_e/c_e \\ s_e/c_e & 1/c_e \end{pmatrix}^{-1} = \begin{pmatrix} 1/c_e & -s_e/c_e \\ -s_e/c_e & 1/c_e \end{pmatrix}, \quad \begin{pmatrix} 1/s_e & c_e/s_e \\ c_e/s_e & 1/s_e \end{pmatrix}^{-1} = \begin{pmatrix} 1/s_e & -c_e/s_e \\ -c_e/s_e & 1/s_e \end{pmatrix}.$$

By Lemma 4.2.2, we have $X' = \phi(M') \in \mathrm{OG}_{\geq 0}(n, 2n)$ for a unique matrix $M' \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$. We then express the entries of M' in terms of the Plücker coordinates of X' using (4.2.1). It follows that this $n \times n$ matrix M' is equal to $M(G', J')$, so we set $G := G'$ and proceed recursively until G has no edges left, splitting G' into connected components if necessary. This finishes a constructive proof of Theorem 4.2.20. Alternatively, we deduce Theorem 4.2.20 from Theorem 4.1.3 at the end of Section 4.6.

Another question similar to Question 4.2.14 is the following.

Question 4.2.23. Given an $n \times n$ matrix $M \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$, does there exist a planar Ising network $N = (G, J)$ such that $M = M(G, J)$?

The answer to this question is provided by Theorem 4.1.3: the answer is "yes" if and only if all minors of the matrix $\widetilde{M}$ are nonnegative. There are exponentially many minors to check, that is, $\binom{2n}{n}$, and in general one needs to check all of them to ensure that $\widetilde{M}$ is totally nonnegative. However, checking whether $\phi(M) \in \mathrm{OG}(n, 2n)$ belongs to $\mathrm{OG}_{>0}(n, 2n) := \mathrm{OG}(n, 2n) \cap \mathrm{Gr}_{>0}(n, 2n)$ (defined in (4.3.1)), as opposed to $\mathrm{OG}_{\geq 0}(n, 2n)$, can be done in polynomial time. More precisely, one needs to check only $n^2 + 1$ minors of $\widetilde{M}$, as it follows from the results of [Pos07]. These minors are algebraically independent as functions on $\mathrm{Gr}(n, 2n)$, but when restricted to $\mathrm{OG}(n, 2n)$, this is no longer the case. Thus if all of them are positive then it follows that $\phi(M) \in \mathrm{OG}_{>0}(n, 2n)$, but in general one could check less minors and arrive at the same conclusion. See Section 4.8 for further discussion.

4.3 Background on the combinatorics of the totally nonnegative Grassmannian

In this section, we recall some combinatorial objects related to $\mathrm{Gr}_{\geq 0}(k, n)$ that were introduced by Postnikov [Pos07]. Most of the results in this section can be found in either [Pos07] or [Lam16].

Recall that the totally nonnegative Grassmannian $\mathrm{Gr}_{\geq 0}(k, N)$ is the subset of the real Grassmannian $\mathrm{Gr}(k, N)$ where all Plücker coordinates are nonnegative. Given a point $X \in$

$\mathrm{Gr}_{\geq 0}(k,N)$, define the *matroid* $\mathcal{M}_X \subset \binom{[N]}{k}$ of X by

$$\mathcal{M}_X := \left\{ J \in \binom{[N]}{k} \mid \Delta_J(X) > 0 \right\}.$$

Given a collection $\mathcal{M} \subset \binom{[N]}{k}$, define the *positroid cell* $\Pi^{>0}_{\mathcal{M}} \subset \mathrm{Gr}_{\geq 0}(k,N)$ by

$$\Pi^{>0}_{\mathcal{M}} := \{X \in \mathrm{Gr}_{\geq 0}(k,N) \mid \mathcal{M}_X = \mathcal{M}\}.$$

For example, one can take $\mathcal{M} = \binom{[N]}{k}$, in which case the positroid cell $\Pi^{>0}_{\mathcal{M}}$ coincides with the *totally positive Grassmannian* $\mathrm{Gr}_{>0}(k,n)$:

$$\mathrm{Gr}_{>0}(k,N) := \left\{ X \in \mathrm{Gr}(k,N) \mid \Delta_I(X) > 0 \text{ for all } I \in \binom{[N]}{k} \right\}. \tag{4.3.1}$$

A collection $\mathcal{M} \subset \binom{[N]}{k}$ is called a *positroid* if $\Pi^{>0}_{\mathcal{M}}$ is nonempty. Positroids are special kinds of matroids, and have a very nice structure which we now explain.

Recall that for $i \in [N]$, the total order $\prec_i$ on $[N]$ is given by $i \prec_i i+1 \prec_i \cdots \prec_i N \prec_i 1 \prec_i \cdots \prec_i i-1$.

Definition 4.3.1. For two sets $I, J \in \binom{[N]}{k}$, we write $I \prec_i J$ if $I = \{i_1 \prec_i \cdots \prec_i i_k\}$, $J = \{j_1 \prec_i \cdots \prec_i j_k\}$, and $i_s \prec_i j_s$ for $1 \leq s \leq k$. It turns out that if $\mathcal{M}$ is a positroid then for each i it has a unique $\prec_i$-minimal element which we denote $I^{\min}_i(\mathcal{M})$. Thus $I^{\min}_i(\mathcal{M})$ satisfies $I^{\min}_i(\mathcal{M}) \prec_i J$ for all $J \in \mathcal{M}$. Similarly, we let $I^{\max}_i(\mathcal{M})$ be the unique $\prec_i$-maximal element of $\mathcal{M}$.

Definition 4.3.2. A sequence $\mathcal{I} := (I_1, \dots, I_N)$ of k-element subsets of $[N]$ is called a *Grassmann necklace* if for each $i \in [N]$ there exists $j_i \in [N]$ such that $I_{i+1} = I_i \setminus \{i\} \cup \{j_i\}$.

Here (and everywhere in this section) the index $i+1$ is taken modulo N.

There is a simple bijection between positroids and Grassmann necklaces, which sends a positroid $\mathcal{M}$ to the sequence $\mathcal{I}(\mathcal{M}) := (I^{\min}_1(\mathcal{M}), I^{\min}_2(\mathcal{M}), \dots, I^{\min}_N(\mathcal{M}))$, which is a Grassmann necklace for each positroid $\mathcal{M}$. Each Grassmann necklace $\mathcal{I}$ is encoded by an associated *decorated permutation* $\pi_{\mathcal{I}} : [N] \to [N]$ which sends $i \in [N]$ to the index j_i from Definition 4.3.2. (When i is a fixed point of $\pi_{\mathcal{I}}$, i.e., $\pi_{\mathcal{I}}(i) = i$, there is an extra bit of data in $\pi_{\mathcal{I}}$ recording whether $i \in I_i$ or $i \notin I_i$, but this will not be important for our exposition.) The map $\mathcal{I} \mapsto \pi_{\mathcal{I}}$ is a bijection between Grassmann necklaces and decorated permutations.

Remark 4.3.3. Under the above correspondence, a positroid $\mathcal{M}$ gives rise to a decorated permutation $\pi_{\mathcal{M}}$ such that for $i \in [N]$, $\pi_{\mathcal{M}}(i)$ is equal to the unique element of the set $I_{i+1}^{\min}(\mathcal{M}) \setminus I_i^{\min}(\mathcal{M})$, if it is nonempty, and is equal to i otherwise. It is not hard to see that $\pi_{\mathcal{M}}^{-1}(i)$ is the unique element of the set $I_i^{\max}(\mathcal{M}) \setminus I_{i+1}^{\max}(\mathcal{M})$.

See [Pos07, Section 16] for a detailed description of these objects and bijections between them.

A *plabic graph* is a planar bipartite graph $G^{\square} = (V^{\square}, E^{\square})$ embedded in a disk such that it has N boundary vertices $d_1, \ldots, d_N$, each of degree 1. (Postnikov considers more general plabic graphs where vertices of the same color are allowed to be connected by an edge, but for our purposes it is sufficient to work with bipartite graphs.) Recall that the notion of an almost perfect matching is given in Definition 4.2.7. Given an almost perfect matching $\mathcal{A}$ of G, we define its *boundary* $\partial(\mathcal{A}) \subset [N]$ to be the set

$$\begin{aligned} \partial(\mathcal{A}) = &\{i \in [N] \mid d_i \text{ is black and is not incident to an edge of } \mathcal{A}\} \cup \\ &\{i \in [N] \mid d_i \text{ is white and is incident to an edge of } \mathcal{A}\}. \end{aligned}$$

It turns out that for every $G^{\square}$ there exists an integer $0 \leq k \leq N$ such that every almost perfect matching $\mathcal{A}$ of $G^{\square}$ satisfies $|\partial(\mathcal{A})| = k$. The number k is given explicitly in terms of the number of black and white vertices of $G^{\square}$, see [Lam16, Eq. (9)].

Definition 4.3.4. Each plabic graph $G^{\square}$ gives rise to a decorated permutation $\pi_{G^{\square}}$, as follows. A *strand* in $G^{\square}$ is a path that turns maximally right (resp., maximally left) at each black (resp., white) vertex. If a strand that starts at b_i ends at b_j for some $i, j \in [N]$ then we put $\pi_{G^{\square}}(i) := j$, which defines a decorated permutation $\pi_{G^{\square}} : [n] \to [n]$. (For each i such that $\pi_{G^{\square}}(i) = i$, $\pi_{G^{\square}}$ also contains the information whether i was black or white in $G^{\square}$.)

Since decorated permutations are in bijection with Grassmann necklaces and positroids, each plabic graph $G^{\square}$ gives rise to a Grassmann necklace $\mathcal{I}_{G^{\square}}$ and a positroid $\mathcal{M}_{G^{\square}}$.

A *weighted plabic graph* is a pair $(G^{\square}, \mathrm{wt})$ where $G^{\square}$ is a plabic graph and $\mathrm{wt} : E^{\square} \to \mathbb{R}_{>0}$ is a *weight function* assigning positive real numbers to the edges of $G^{\square}$. For an almost perfect matching $\mathcal{A}$ of $G^\square$, recall that $\mathrm{wt}(\mathcal{A})$ is the product of weights of edges of $\mathcal{A}$. We can consider a collection $\mathrm{Meas}(G^\square, \mathrm{wt}) := (\Delta_I(G^\square, \mathrm{wt}))_{I \in \binom{[N]}{k}} \in \mathbb{RP}^{\binom{N}{k}-1}$ of polynomials given for $I \in \binom{[N]}{k}$ by

$$\Delta_I(G^\square, \mathrm{wt}) := \sum_{\mathcal{A}:\partial(\mathcal{A})=I} \mathrm{wt}(\mathcal{A}), \tag{4.3.2}$$

where the sum is over all almost perfect matchings $\mathcal{A}$ of $G^\square$ with boundary I. It turns out that $(\Delta_I(G^\square, \mathrm{wt}))_{I \in \binom{[N]}{k}}$ is the collection of Plücker coordinates of some point $X \in \mathrm{Gr}_{\geq 0}(k, N)$. The following result is implicit in [PSW09].

Theorem 4.3.5 ([Lam16, Corollary 7.14])**.** *Given a weighted plabic graph* $(G^\square, \mathrm{wt})$*, there exists a unique point* $X \in \mathrm{Gr}(k, N)$ *such that*

$$\mathrm{Meas}(G^\square, \mathrm{wt}) = (\Delta_I(X))_{I \in \binom{[N]}{k}}$$

as elements of the projective space $\mathbb{RP}^{\binom{N}{k}-1}$*. The point* X *belongs to* $\mathrm{Gr}_{\geq 0}(k, N)$ *and in fact to the positroid cell* $\Pi^{>0}_{\mathcal{M}_{G^\square}}$*, where* $\mathcal{M}_{G^\square}$ *is the positroid whose decorated permutation is* $\pi_{G^\square}$*. Every point* $X \in \Pi^{>0}_{\mathcal{M}_{G^\square}}$ *arises in this way from some weight function* $\mathrm{wt} : E^\square \to \mathbb{R}_{>0}$*.*

The map $\mathrm{Meas}(G^\square, \cdot) : \mathbb{R}^{E^\square}_{>0} \to \mathrm{Gr}_{\geq 0}(k, N)$ sending $\mathrm{wt} \mapsto X$ is not usually injective. To see this, observe that each interior vertex of $G^\square$ is incident to precisely one edge of each almost perfect matching $\mathcal{A}$. Thus rescaling the weights of all edges incident to a single interior vertex (i.e. applying a *gauge transformation*) does not change the value of Meas. We denote by $\mathbb{R}^{E^\square}_{>0} / \mathrm{Gauge}$ the space of gauge-equivalence classes of functions $\mathrm{wt} : E^\square \to \mathbb{R}_{>0}$, so that wt and wt$'$ are the same in $\mathbb{R}^{E^\square}_{>0} / \mathrm{Gauge}$ if and only if wt$'$ can be obtained from wt by a sequence of gauge transformations. It is not hard to see that $\mathbb{R}^{E^\square}_{>0} / \mathrm{Gauge}$ is homeomorphic to an open ball of dimension $F(G^\square) - 1$, where $F(G^\square)$ denotes the number of faces of $G^\square$.

Thus by Theorem 4.3.5, Meas gives rise to a map

$$\overline{\mathrm{Meas}} : \mathbb{R}^{E^\square}_{>0} / \mathrm{Gauge} \to \Pi^{>0}_{\mathcal{M}_{G^\square}} \subset \mathrm{Gr}_{\geq 0}(k, N)$$

which turns out to be injective for some plabic graphs $G^\square$. More precisely, let us say that $G^\square$ is *reduced* if all of the following conditions are satisfied:

- no strand in $G^\square$ intersects itself;
- there are no closed strands in $G^\square$;
- no two strands in $G^\square$ have a *bad double crossing*.

Here two strands are said to form a *bad double crossing* if there are two vertices $u, v \in V^\square$ such that both strands first pass through u and then through v. The following result can be found in [Pos07, Lam16].

Theorem 4.3.6. *For each positroid $\mathcal{M}$, there exists a reduced plabic graph $G^\square$ such that $\mathcal{M} = \mathcal{M}_{G^\square}$. Given a reduced plabic graph $G^\square$, the map $\overline{\mathrm{Meas}} : \mathbb{R}_{>0}^{E^\square} / \mathrm{Gauge} \to \Pi^{>0}_{\mathcal{M}_{G^\square}}$ is a homeomorphism. Thus the positroid cell $\Pi^{>0}_{\mathcal{M}_{G^\square}}$ is homeomorphic to $\mathbb{R}^{F(G^\square)-1}$. In addition, we have*

$$\mathrm{Gr}_{\geq 0}(k, N) = \bigsqcup_{\mathcal{M}} \Pi^{>0}_{\mathcal{M}}, \tag{4.3.3}$$

where the union is over all positroids $\mathcal{M} \subset \binom{[N]}{k}$.

The last ingredient from the theory of plabic graphs that we will need is *BCFW bridges*, introduced in [AHBC+16, BCFW05]. Our exposition will follow [Lam16, Section 7].

Recall that each boundary vertex of a plabic graph is incident to a unique edge.

Definition 4.3.7. Given $i \in [N]$, we say that a plabic graph $G^\square$ has a *removable bridge between i and $i+1$* if there exists a path of length 3 between d_i and d_{i+1} in $G^\square$. (In particular, these vertices have to be of different color).

Here we again allow $i = N$ and $i + 1 = 1$. We refer to the middle edge of this path of length 3 as a *bridge between i and $i + 1$*. There are two types of bridges, since i can be incident either to a white or to a black interior vertex. It turns out that the weight of the bridge can always be recovered from the minors of the corresponding element of the Grassmannian. The following result can be found in [Lam16, Proposition 7.10] and [Lam18, Proposition 3.10], and is the main ingredient of the proof of Theorem 4.2.20.

Theorem 4.3.8. *Let $(G^\square, \mathrm{wt})$ be a weighted reduced plabic graph, and suppose that it has a removable bridge between i and $i + 1$. Assume that the weights of the edges incident to*

d_i and d_{i+1} are both equal to 1 (which can always be achieved using gauge transformations). Let $e \in E^{\square}$ be the bridge between i and $i+1$, and denote $X := \overline{\mathrm{Meas}}(G^{\square}, \mathrm{wt}) \in \mathrm{Gr}_{\geq 0}(k, N)$.

- *If i is white then for $I := I_{i+1}^{\min}(\mathcal{M}_{G^{\square}})$, we have*

$$\mathrm{wt}(e) = \frac{\Delta_I(X)}{\Delta_{I \cup \{i\} \setminus \{i+1\}}(X)}.$$

- *If i is black then for $I := I_{i+1}^{\max}(\mathcal{M}_{G^{\square}})$, we have*

$$\mathrm{wt}(e) = \frac{\Delta_I(X)}{\Delta_{I \cup \{i+1\} \setminus \{i\}}(X)}.$$

We will also need to explain how removing a bridge changes the corresponding element of the Grassmannian. For $i \in [N-1]$ and $t \in \mathbb{R}$, define $x_i(t) \in \mathrm{Mat}_N(\mathbb{R})$ to be a $N \times N$ matrix with ones on the diagonal and a single nonzero off-diagonal entry in row i and column $i+1$ equal to t. We also define $x_N(t)$ to be the matrix with ones on the diagonal and the entry in row N, column 1 equal to $(-1)^{k-1}t$. We define $y_i(t)$ to be the matrix transpose of $x_i(t)$ for $i \in [N]$.

Lemma 4.3.9 ([Lam16, Lemma 7.6])**.** *Let $(G^{\square}, \mathrm{wt})$ be a weighted plabic graph, and suppose that it has a removable bridge between i and $i+1$. Assume that the weights of the edges incident to d_i and d_{i+1} are both equal to 1. Let $e \in E^{\square}$ be the bridge between i and $i+1$ with weight $\mathrm{wt}(e) = t$, and denote $X := \overline{\mathrm{Meas}}(G^{\square}, \mathrm{wt}) \in \mathrm{Gr}_{\geq 0}(k, N)$. Let $(G^{\square\prime}, \mathrm{wt}')$ be obtained from $(G^{\square}, \mathrm{wt})$ by removing e, and define $X' := \overline{\mathrm{Meas}}(G^{\square\prime}, \mathrm{wt})$. Then for all $I \in \binom{[N]}{k}$ we have the following.*

- *If i is white then $X' = X \cdot x_i(-t)$, and*

$$\Delta_I(X') = \begin{cases} \Delta_I(X) - t\Delta_{I \setminus \{i+1\} \cup \{i\}}(X), & \text{if } i+1 \in I \text{ but } i \notin I; \\ \Delta_I(X), & \text{otherwise.} \end{cases}$$

- *If i is black then $X' = X \cdot y_i(-t)$, and*

$$\Delta_I(X') = \begin{cases} \Delta_I(X) - t\Delta_{I\setminus\{i\}\cup\{i+1\}}(X), & \text{if } i \in I \text{ but } i+1 \notin I; \\ \Delta_I(X), & \text{otherwise.} \end{cases}$$

4.4 The totally nonnegative orthogonal Grassmannian

In this section, we discuss how the stratification of $\mathrm{Gr}(n, 2n)$ induces a stratification of the totally nonnegative orthogonal Grassmannian $\mathrm{OG}_{\geq 0}(n, 2n)$. We remark that some of the statements below have appeared in [HW13, HWX14], but mostly without proofs.

Recall from Definition 4.1.1 that the orthogonal Grassmannian $\mathrm{OG}(n, 2n) \subset \mathrm{Gr}(n, 2n)$ is the set of $X \in \mathrm{Gr}(n, 2n)$ such that $\Delta_I(X) = \Delta_{[2n]\setminus I}(X)$ for all n-element sets $I \subset [2n]$. In the literature, the term "orthogonal Grassmannian" usually refers to the set of subspaces where a certain non-degenerate symmetric bilinear form vanishes. Over the complex numbers, there is only one such bilinear form up to isomorphism, but over the real numbers, one needs to choose a signature. Following [HW13], define a non-degenerate symmetric bilinear form $\eta : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}$ by $\eta(u, v) := u_1v_1 - u_2v_2 + \cdots + u_{2n-1}v_{2n-1} - u_{2n}v_{2n}$. Let us also introduce another subset $\mathrm{OG}_-(n, 2n) \subset \mathrm{Gr}(n, 2n)$ consisting of all $X \in \mathrm{Gr}(n, 2n)$ such that $\Delta_I(X) = -\Delta_{[2n]\setminus I}(X)$ for all n-element sets $I \subset [2n]$.[2] We justify our terminology as follows.

Proposition 4.4.1. *For a subspace $X \in \mathrm{Gr}(n, 2n)$, the following are equivalent:*

- *$X \in \mathrm{OG}(n, 2n) \sqcup \mathrm{OG}_-(n, 2n)$;*
- *for any two vectors $u, v \in X \subset \mathbb{R}^{2n}$, we have $\eta(u, v) = 0$.*

Proof. Given an $k \times N$ matrix $A = (a_{i,j})$, define another $k \times N$ matrix $\mathrm{alt}(A) := ((-1)^j a_{i,j})$. Taking row spans and setting $k := n$, $N := 2n$, we get a map $\mathrm{alt} : \mathrm{Gr}(n, 2n) \to \mathrm{Gr}(n, 2n)$. It is a classical result (see e.g. [Hoc75, Section 7] or [Kar17, Lemma 1.11]) that for $X \in \mathrm{Gr}(n, 2n)$ and $I \in \binom{[2n]}{n}$, we have $\Delta_{[2n]\setminus I}(\mathrm{alt}(X^\perp)) = c\Delta_I(X)$, where $\perp$ denotes the orthogonal complement of $X \subset \mathbb{R}^{2n}$ with respect to the standard scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^{2n}$ and $c \in \mathbb{R}$

[2] We thank David Speyer for suggesting to consider both $\mathrm{OG}(n, 2n)$ and $\mathrm{OG}_-(n, 2n)$.

is some nonzero constant. Note that $\eta(u,v) = \langle \operatorname{alt}(u), v\rangle$ for $u,v \in \mathbb{R}^{2n}$, which shows that η vanishes on X if and only if $\Delta_I(X) = c\Delta_{[2n]\setminus I}(X)$ for all $I \in \binom{[2n]}{n}$. Applying this equality twice, we get $\Delta_I(X) = c\Delta_{[2n]\setminus I}(X) = c^2\Delta_I(X)$, and thus $c = \pm 1$. We are done with the proof. $\square$

Remark 4.4.2. Lusztig [Lus94] has defined the totally nonnegative part $(G/P)_{\geq 0}$ of any partial flag variety G/P inside a split reductive algebraic group G over $\mathbb{R}$. Rietsch showed that the space $\mathrm{Gr}_{\geq 0}(k,n)$ is a special case of $(G/P)_{\geq 0}$, see [Lam16, Remark 3.8]. For a specific choice of $G = O(n,n)$ (i.e. the *split orthogonal group*, which corresponds to the Dynkin diagram of type D_n) and a maximal parabolic subgroup $P \cong \mathrm{SL}_n(\mathbb{R})$ (corresponding to the Dynkin diagram of type A_{n-1}, obtained from D_n by removing a leaf adjacent to a degree 3 vertex), G/P becomes equal to $\mathrm{OG}(n,2n)$. If we had $(G/P)_{\geq 0} = \mathrm{OG}_{\geq 0}(n,2n)$ then the fact that $\mathrm{OG}_{\geq 0}(n,2n)$ is a closed ball would follow from the results of [GKL18]. However, the relationship between Lusztig's $(G/P)_{\geq 0}$ and $\mathrm{OG}_{\geq 0}(n,2n)$ remains unclear to us. For instance, the cell decomposition of $(G/P)_{\geq 0}$ conjectured by Lusztig and proved by Rietsch [Rie98, Rie99] appears to have a different number of cells than the cell decomposition of $\mathrm{OG}_{\geq 0}(n,2n)$ indexed by matchings on $[2n]$ that we consider in this paper. David Speyer [Spe18] has also informed us that the space E_n of electrical networks can be realized as a subset of the *Lagrangian Grassmannian* $\mathrm{LG}(n-1,2n-2)$, which is also equal to G/P when G is the *symplectic group* $\mathrm{Sp}_{2n-2}(\mathbb{C})$ (corresponding to the Dynkin diagram of type C_{n-1}). The relationship between this subset and $(G/P)_{\geq 0}$ is again unclear in this case.

Remark 4.4.3. A different relation between the Ising model and spin representations of the orthogonal group can be found in [Kau49, SMJ78, Pal07].

Remark 4.4.4. The generators $g_i(t)$ from Section 4.2.5 belong to $O(n,n)$, and moreover, they are *hyperbolic rotation matrices*, since for $t \in \mathbb{R}_{>0}$ and $c := \tanh(2t)$, $s := \operatorname{sech}(2t)$, there exists a unique $r(t) \in \mathbb{R}$ such that $\begin{pmatrix} 1/c & s/c \\ s/c & 1/c \end{pmatrix} = \begin{pmatrix} \cosh(r(t)) & \sinh(r(t)) \\ \sinh(r(t)) & \cosh(r(t)) \end{pmatrix}$. It would thus be interesting to find an analog of the theory of [LP15] for the orthogonal group rather than the symplectic group.

Remark 4.4.5. A more standard choice of coordinates for $\mathrm{OG}(n,2n)$ is to consider the set $\mathrm{OG}'(n,2n)$ of all $X \in \mathrm{Gr}(n,2n)$ where another symmetric bilinear form, $\eta'(u,v) := u_1v_{n+1} + u_2v_{n+2} + \cdots + u_nv_{2n}$, vanishes. Consider a $2n \times 2n$ matrix J with the only nonzero entries given by $J_{2j-1,j} = J_{2j,j} = J_{2j-1,j+n} = 1/2$, $J_{2j,j+n} = -1/2$, for all $j \in [n]$. Then the map $X \mapsto X \cdot J$ gives a bijection between $\mathrm{OG}(n,2n) \sqcup \mathrm{OG}_-(n,2n)$ and $\mathrm{OG}'(n,2n)$. Moreover, for $M \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1)$, the matrix $M \cdot J$ has the form $[I_n|M']$ for a skew-symmetric matrix M' given by $m'_{i,j} = (-1)^{i+j+1}m_{i,j}$ for $i \neq j$ and $m'_{i,j} = 0$ for $i = j$. A standard way to work with $\mathrm{OG}'(n,2n)$ is to consider *spinor coordinates*, which are essentially *Pfaffians* of the matrix M' above, see e.g. [HS10, Section 5]. It was shown in [GBK78] that these Pfaffians are multi-point boundary correlation functions for the Ising model, as we explain in Proposition 4.7.7. We thank David Speyer for this remark.

Proposition 4.4.1 allows one to deduce that the image of the map ϕ is contained inside the orthogonal Grassmannian.

Corollary 4.4.6. *We have* $\phi(\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1)) \subset \mathrm{OG}(n,2n)$*.*

Proof. Let $M \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1)$. It is obvious from the definition of $\widetilde{M}$ that if $u,v \in \mathbb{R}^{2n}$ are any two rows of $\widetilde{M}$ then we have $\eta(u,v) = 0$, and thus by Proposition 4.4.1 we get $\phi(\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1)) \subset \mathrm{OG}(n,2n) \sqcup \mathrm{OG}_-(n,2n)$. But note that $\mathrm{OG}(n,2n)$ and $\mathrm{OG}_-(n,2n)$ are not connected to each other inside $\mathrm{OG}(n,2n) \sqcup \mathrm{OG}_-(n,2n)$, however, $\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1) \cong \mathbb{R}^{\binom{n}{2}}$ is connected. Thus $\phi(\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1))$ is connected, and clearly the image of the identity matrix $I_n \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R},1)$ belongs to $\mathrm{OG}(n,2n)$ and not to $\mathrm{OG}_-(n,2n)$. The result follows. $\square$

Proposition 4.4.7. *Let $X \in \mathrm{OG}_{\geq 0}(n,2n)$, and let $\mathcal{M} := \mathcal{M}_X$ be the positroid of X with decorated permutation $\pi_{\mathcal{M}}$. Then $\pi_{\mathcal{M}}$ is a* fixed-point free involution*: if $\pi_{\mathcal{M}}(i) = j$ then $i \neq j$ and $\pi_{\mathcal{M}}(j) = i$.*

Proof. It is clear from Definition 4.3.1 that we have $I_i^{\min}(\mathcal{M}) = [2n] \setminus I_i^{\max}(\mathcal{M})$, because $X \in \mathrm{OG}(n,2n)$. Suppose now that $\pi_{\mathcal{M}}(i) = j$ and that $i \neq j$. By Remark 4.3.3, $\pi_{\mathcal{M}}^{-1}(i)$ is the unique element of the set

$$I_i^{\max}(\mathcal{M}) \setminus I_{i+1}^{\max}(\mathcal{M}) = ([2n] \setminus I_i^{\min}(\mathcal{M})) \setminus ([2n] \setminus I_{i+1}^{\min}(\mathcal{M})) = I_{i+1}^{\min}(\mathcal{M}) \setminus I_i^{\min}(\mathcal{M}),$$

which is equal to $\{\pi_{\mathcal{M}}(i)\} = \{j\}$. Thus $\pi_{\mathcal{M}}^{-1}(i) = j$, equivalently, $\pi_{\mathcal{M}}(j) = i$, so $\pi_{\mathcal{M}}$ is an involution. It remains to show that it is *fixed-point free*, i.e., that $\pi_{\mathcal{M}}(i) \neq i$ for all $i \in [2n]$. We can have $\pi_{\mathcal{M}}(i) = i$ if either i is a *loop* or a *coloop* of $\mathcal{M}$, that is, if either $i \in I$ for all $I \in \mathcal{M}$ or $i \notin I$ for all $I \in \mathcal{M}$, respectively. Choose some $I \in \mathcal{M}$. Then $[2n] \setminus I$ also belongs to $\mathcal{M}$, which shows that i is neither a loop nor a coloop of $\mathcal{M}$. We are done with the proof. □

Remark 4.4.8. Recall that given a matching τ on $[2n]$, Definition 4.2.19 gives two disjoint sets $I_i^{\min}(\tau)$ and $I_i^{\max}(\tau)$ for each $i \in [2n]$. It is easy to check that if π is the fixed-point free involution corresponding to τ then $I_i^{\min}(\mathcal{M}_\pi) = I_i^{\min}(\tau)$ and $I_i^{\max}(\mathcal{M}_\pi) = I_i^{\max}(\tau)$.

In Section 4.2.5, we described how to transform a planar graph G embedded in a disk into a *medial graph* $G^\times$, and then how to obtain a medial pairing τ_G from $G^\times$. Not all matchings can be obtained in this way, for example, when $n = 2$, the matching $\{\{1,4\},\{2,3\}\}$ is not a medial pairing of any graph G. It will thus be more convenient for us to work with medial graphs rather than matchings. In Section 4.5, we introduce *generalized planar Ising networks* which correspond to *all* matchings on $[2n]$.

Definition 4.4.9. A *medial graph* is a planar graph $G^\times = (V^\times, E^\times)$ embedded in a disk, such that it has $2n$ boundary vertices $d_1, d_2, \dots, d_{2n} \in V^\times$ in counterclockwise order, each of degree 1, and such that every other vertex of $G^\times$ has degree 4.

The non-boundary vertices (the ones that have degree 4) are called *interior vertices* of $G^\times$, and we let $V_{\text{int}}^\times := V^\times \setminus \{d_1, \dots, d_{2n}\}$ denote the set of such vertices. Each medial graph $G^\times$ gives rise to a *medial pairing* $\tau_{G^\times}$, as in Section 4.2.5. We say that a medial graph $G^\times$ is *reduced* if the number of its interior vertices equals $\operatorname{xing}(\tau_{G^\times})$. Equivalently, $G^\times$ is reduced if every edge of $G^\times$ belongs to some medial strand connecting two boundary vertices, no medial strand intersects itself, and no two medial strands intersect more than once.

Lemma 4.4.10. *For every matching τ on $[2n]$, there exists a reduced medial graph $G^\times$ satisfying $\tau_{G^\times} = \tau$.*

Proof. For each pair $\{i, j\} \in \tau$, connect d_i with d_j by a straight line segment. Then perturb each line segment slightly so that every point inside the disk would belong to at most two segments, obtaining a *pseudoline arrangement*. Let $G^\times$ be obtained from this pseudoline arrangement by putting an interior vertex at each intersection point. It is clear that $G^\times$ is a reduced medial graph whose medial pairing is τ. Alternatively, $G^\times$ can be constructed by induction on $\operatorname{xing}(\tau)$ in an obvious way using the poset P_n from Definition 4.4.15. $\square$

Let us say that a *medial network* $N^\times = (G^\times, J^\times)$ is a medial graph $G^\times$ together with a function $J^\times : V^\times_{\text{int}} \to \mathbb{R}_{>0}$. Thus if $N = (G, J)$ is a planar Ising network then the edges of G correspond to the interior vertices of the corresponding medial graph $G^\times$ and thus the Ising network N gives rise to a medial network $N^\times = (G^\times, J^\times)$, as described in Sections 4.2.5 and 4.5. In the remainder of this section, we will work with medial networks rather than with planar Ising networks.

Every medial graph gives rise to a plabic graph. In order to describe this correspondence, we first introduce a canonical way to orient each medial graph, as described in [HWX14].

Proposition 4.4.11. *Let $G^\times$ be a medial graph. Then there exists a unique orientation of the edges of $G^\times$ such that:*

1. *for $i \in [2n]$, d_i is a source if and only if i is odd;*
2. *each interior vertex $v \in V^\times_{\text{int}}$ of $G^\times$ is incident to two incoming and two outgoing arrows so that their directions alternate around v.*

Proof. If $G^\times$ is connected then it is easy to see that there are just two orientations satisfying the second condition, since we can color the faces of $G^\times$ in a bipartite way and then orient all black faces clockwise and all white faces counterclockwise, or vice versa. One of these two orientations will satisfy the first condition. If $G^\times$ has C connected components then there are 2^C orientations of G satisfying the second condition, but there will still be one of them that satisfies the first condition, because the number of vertices of each connected component is even. $\square$

Definition 4.4.12. Given a medial network $N^\times = (G^\times, J^\times)$, the associated weighted plabic graph $(G^\square, \mathrm{wt})$ is constructed as follows. First, orient the edges of $G^\times$ as in Proposition 4.4.11, and then for each oriented edge e of $G^\times$, put a white vertex e° of $G^\square$ close to the source of e and a black vertex $e^\bullet$ of $G^\square$ close to the target of e. If the source (resp.,

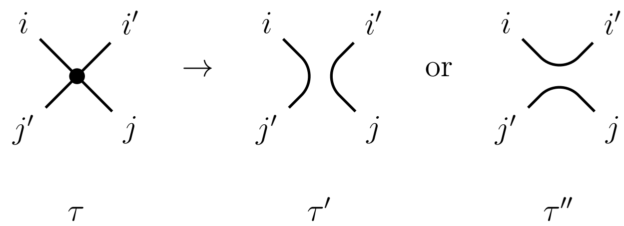


Figure 4-6: Uncrossing the pairs $\{i, j\}$ and $\{i', j'\}$.

the target) of e is a boundary vertex d_i then we set $e^\circ := d_i$ (resp., $e^\bullet := d_i$). Now, for each edge $e \in E^\square$ of $G^\square$, connect $e^\bullet$ and e° by an edge of $G^\square$, and set its weight to 1: $\mathrm{wt}(\{e^\bullet, e^\circ\}) := 1$. Additionally for every interior vertex $v \in V_{\mathrm{int}}^\times$ of $G^\square$ incident to edges $e_1, e_2, e_3, e_4 \in E^\square$ in counterclockwise order so that v is the target of e_1 and e_3 and the source of e_2 and e_4, add four edges $\{e_1^\bullet, e_2^\circ\}$, $\{e_2^\circ, e_3^\bullet\}$, $\{e_3^\bullet, e_4^\circ\}$, $\{e_4^\circ, e_1^\bullet\}$ to $G^\square$. The weights of these edges are given by

$$\mathrm{wt}(\{e_1^\bullet, e_2^\circ\}) = \mathrm{wt}(\{e_3^\bullet, e_4^\circ\}) := s_v, \quad \mathrm{wt}(\{e_2^\circ, e_3^\bullet\}) = \mathrm{wt}(\{e_4^\circ, e_1^\bullet\}) := c_v, \tag{4.4.1}$$

where s_v and c_v are given by (4.2.5), that is,

$$s_v := \mathrm{sech}(2J_v^\times) = \frac{2}{\exp(2J_v^\times) + \exp(-2J_v^\times)}; \quad c_v := \tanh(2J_v^\times) = \frac{\exp(2J_v^\times) - \exp(-2J_v^\times)}{\exp(2J_v^\times) + \exp(-2J_v^\times)}.$$

This defines a weighted plabic graph $(G^\square, \mathrm{wt})$ associated to the medial network $N^\times = (G^\times, J^\times)$.

Recall that for a medial graph $G^\times$, the corresponding medial pairing $\tau_{G^\times} = \{\{i_1, j_1\}, \dots, \{i_n, j_n\}\}$ is a matching on $[2n]$. We define a permutation $\pi_{G^\times} : [2n] \to [2n]$ by setting $\pi_{G^\times}(i_k) := j_k$ and $\pi_{G^\times}(j_k) := i_k$ for all $k \in [2n]$. Thus $\pi_{G^\times}$ is a fixed-point free involution.

Lemma 4.4.13. *A medial graph $G^\times$ is reduced if and only if the corresponding plabic graph $G^\square$ is reduced. We have $\pi_{G^\times} = \pi_{G^\square}$.*

Proof. This is straightforward to check from the definitions, since the medial strands correspond to the strands in $G^\square$ from Definition 4.3.4. $\square$

Proposition 4.4.14. *Given a medial network $N^\times = (G^\times, J^\times)$, let $(G^\square, \mathrm{wt})$ be the corresponding weighted plabic graph. Then $\overline{\mathrm{Meas}}(G^\square, \mathrm{wt})$ yields an element of $\Pi_{\mathcal{M}}^{>0} \cap \mathrm{OG}_{\geq 0}(n, 2n)$, where $\mathcal{M} = \mathcal{M}_{\pi_{G^\times}}$ is the positroid corresponding to the fixed-point free involution $\pi_{G^\times}$.*

Proof. By Lemma 4.4.13, we have $\pi_{G^\times} = \pi_{G^\square}$. Let $X \in \mathrm{Gr}_{\geq 0}(n, 2n)$ be the point given by $\overline{\mathrm{Meas}}(G^\square, \mathrm{wt})$. By Theorem 4.3.5, X belongs to $\Pi_{\mathcal{M}}^{>0}$, and it remains to show that X belongs to $\mathrm{OG}_{\geq 0}(n, 2n)$. Given an almost perfect matching $\mathcal{A}$ of $G^\square$, let us define $S^\times(\mathcal{A}) \subset E^\times$ to be the set of edges e of $G^\times$ such that the edge $\{e^\bullet, e^\circ\}$ of $G^\square$ belongs to $\mathcal{A}$. We claim that for all sets $R \subset E^\times$ of edges of $G^\times$, we have

$$\sum_{S^\times(\mathcal{A}) = R} \mathrm{wt}(\mathcal{A}) = \sum_{S^\times(\mathcal{A}) = E^\times \setminus R} \mathrm{wt}(\mathcal{A}), \tag{4.4.2}$$

where the sums are over almost perfect matchings of $G^\square$. The left hand side of (4.4.2) is equal to the product over all interior vertices $v \in V_{\mathrm{int}}^\times$ of $G^\times$ of $q(v)$, where $q(v)$ is equal to either c_e, s_e, $c_e^2 + s_e^2$, 1, or 0, depending on which of the four edges of $G^\times$ adjacent to v belong to R. It is clear from Figure 4-3 that replacing R with its complement does not affect this product. (The only non-trivial change is replacing $c_e^2 + s_e^2$ with 1, but recall that we have $c_e^2 + s_e^2 = 1$ by construction.) This proves (4.4.2), and clearly if two almost perfect matchings $\mathcal{A}, \mathcal{A}'$ of $G^\square$ satisfy $S^\times(\mathcal{A}') = E^\times \setminus S^\times(\mathcal{A})$ then they also satisfy $\partial(\mathcal{A}') = [2n] \setminus \partial(\mathcal{A})$, finishing the proof. $\square$

For a matching τ on $[2n]$, let $\mathcal{M}_\tau$ be the positroid corresponding to the fixed-point free involution $\pi : [2n] \to [2n]$ associated with τ, and denote by $\Pi_\tau^{>0} := \Pi_{\mathcal{M}_\tau}^{>0}$. Following [Lam18], denote by P_n the partially ordered set (*poset*) of all matchings τ on $[2n]$. (It is easy to see that P_n has $\frac{(2n)!}{n!2^n}$ elements.) The covering relations of P_n are described as follows. Given a matching τ on $[2n]$, suppose that the pairs $\{i, j\}, \{i', j'\} \in \tau$ form a crossing, as in Definition 4.2.15. Introduce two matchings

$$\begin{aligned} \tau' &:= \tau \setminus \{\{i, j\}, \{i', j'\}\} \cup \{\{i, j'\}, \{i', j\}\}; \\ \tau'' &:= \tau \setminus \{\{i, j\}, \{i', j'\}\} \cup \{\{i, i'\}, \{j, j'\}\}. \end{aligned} \tag{4.4.3}$$

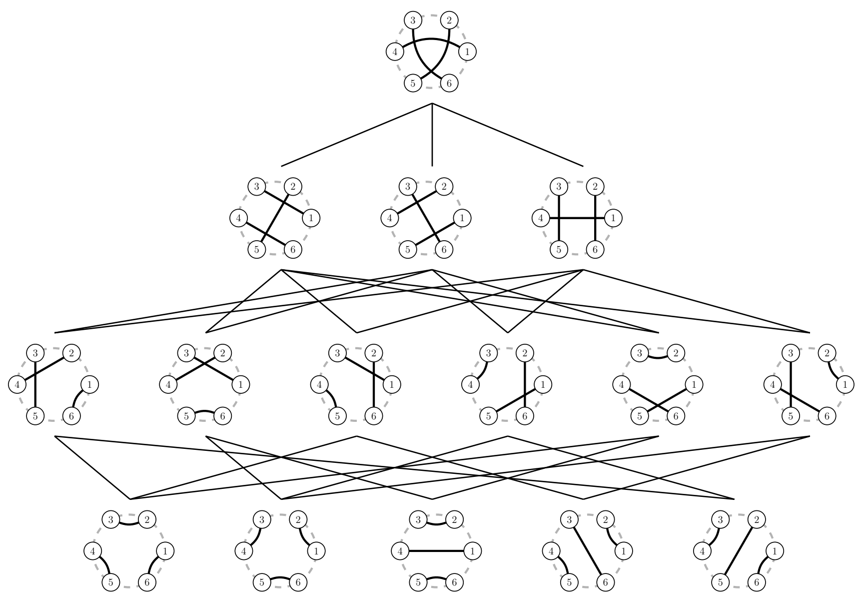


Figure 4-7: The Hasse diagram of the poset P_3.

Definition 4.4.15. We say that τ' and τ'' are obtained from τ by *uncrossing the pairs* $\{i, j\}$ *and* $\{i', j'\}$ (see Figure 4-6). In addition, if $\operatorname{xing}(\tau') + 1 = \operatorname{xing}(\tau)$ (resp., $\operatorname{xing}(\tau'') + 1 = \operatorname{xing}(\tau)$), we write $\tau' \lessdot \tau$ (resp., $\tau'' \lessdot \tau$), and let P_n be the poset whose order relation $\leq$ is the transitive closure of $\lessdot$.

Remark 4.4.16. Equivalently, as explained in [Lam18, Section 4.5], given a medial graph $G^\times$ with medial pairing τ, we have $\tau' \lessdot \tau$ if and only if "uncrossing" the unique vertex $v \in V^\times_{\mathrm{int}}$ of $G^\times$ that belongs to the intersection of medial strands connecting d_i to d_j and $d_{i'}$ to $d_{j'}$ yields a *reduced* medial graph with medial pairing τ'. Here *uncrossing* an interior vertex of a medial graph means replacing its neighborhood in one of the two ways shown in Figure 4-6.

By [Lam18, Lemma 4.13], the poset P_n is graded with grading given by $\operatorname{xing}(\tau)$, and by [HK18, Lam15], P_n is a shellable Eulerian poset. See Figure 4-7 for the case $n = 3$.

We are now ready to state the main result of this section.

Theorem 4.4.17.

(i) Given a reduced medial graph $G^\times$, let $G^\square$ be the corresponding plabic graph with positroid $\mathcal{M} := \mathcal{M}_{G^\square}$. Then the map $J^\times \mapsto \overline{\operatorname{Meas}}(G^\square, \mathrm{wt})$ is a homeomorphism between $\mathbb{R}^{E^\times}_{>0}$ and $\Pi^{>0}_{\mathcal{M}} \cap \operatorname{OG}_{\geq 0}(n, 2n)$.

(ii) The set $\mathrm{OG}_{\geq 0}(n,2n)$ *is a disjoint union of cells*

$$\mathrm{OG}_{\geq 0}(n,2n) = \bigsqcup_{\tau \in P_n} \left(\Pi_\tau^{>0} \cap \mathrm{OG}_{\geq 0}(n,2n)\right), \tag{4.4.4}$$

and each cell $\Pi_\tau^{>0} \cap \mathrm{OG}_{\geq 0}(n,2n)$ *is homeomorphic to* $\mathbb{R}^{\mathrm{xing}(\tau)}$.

(iii) For $\tau \in P_n$*, the closure of the cell* $\Pi_\tau^{>0} \cap \mathrm{OG}_{\geq 0}(n,2n)$ *in* $\mathrm{Gr}(n,2n)$ *equals*

$$\overline{(\Pi_\tau^{>0} \cap \mathrm{OG}_{\geq 0}(n,2n))} = \bigsqcup_{\sigma \in P_n : \sigma \leq \tau} \left(\Pi_\sigma^{>0} \cap \mathrm{OG}_{\geq 0}(n,2n)\right). \tag{4.4.5}$$

Proof. As we have shown in Proposition 4.4.7, for every $X \in \mathrm{OG}_{\geq 0}(n,2n)$, the decorated permutation $\pi = \pi_{\mathcal{M}_X}$ associated with the positroid $\mathcal{M}_X$ of X is a fixed-point free involution, and thus (4.4.4) follows from (4.3.3). The remainder of part (ii) (that each cell is an open ball) follows from part (i), which we prove now. Thus we fix a reduced medial graph $G^\times$ and the corresponding plabic graph $G^\square$, reduced by Lemma 4.4.13. Let $\psi : \mathbb{R}_{>0}^{E^\times} \to \Pi_{\mathcal{M}}^{>0} \cap \mathrm{OG}_{\geq 0}(n,2n)$ be the map that sends $J^\times \mapsto \overline{\mathrm{Meas}}(G^\square, \mathrm{wt})$. We first show that ψ is injective. By Theorem 4.3.6, it suffices to show that the map $J^\times : V_{\mathrm{int}}^\times \to \mathbb{R}_{>0}$ can be reconstructed from the corresponding weight function $\mathrm{wt} \in \mathbb{R}_{>0}^{E^\square} / \mathrm{Gauge}$. Fix an interior vertex $v \in V_{\mathrm{int}}^\times$ of $G^\times$ and consider the corresponding four vertices $v_1, v_2, v_3, v_4 \in V^\square$ of $G^\square$ on the four edges of $G^\times$ incident to v. Let e_{12}, e_{23}, e_{34}, e_{14} be the four edges of $G^\square$ forming a square around v. We have $\mathrm{wt}(e_{12}) = \mathrm{wt}(e_{34}) = s_v$ and $\mathrm{wt}(e_{23}) = \mathrm{wt}(e_{14}) = c_v$, as in (4.4.1). Suppose that we have applied a gauge transformation to wt obtaining another weight function wt'. Thus we have rescaled all edges adjacent to the vertex v_k by some number $t_k \in \mathbb{R}_{>0}$ for $1 \leq k \leq 4$. Therefore

$$\mathrm{wt}'(e_{12}) = t_1 t_2 s_v, \quad \mathrm{wt}'(e_{34}) = t_3 t_4 s_v, \quad \mathrm{wt}'(e_{23}) = t_2 t_3 c_v, \quad \mathrm{wt}'(e_{14}) = t_1 t_4 c_v.$$

In order for wt' to come from some other map $(J^\times)' : V_{\mathrm{int}}^\times \to \mathbb{R}_{>0}$, we must have

$$t_1 t_2 s_v = t_3 t_4 s_e = s'_v, \quad t_2 t_3 c_v = t_1 t_4 c_e = c'_v, \quad (s'_v)^2 + (c'_v)^2 = 1,$$

where $s'_v = \operatorname{sech}(2(J^\times)'_v)$ and $c'_v = \tanh(2(J^\times)'_v)$. But the above equations imply that $t_1 = t_2 = t_3 = t_4 = 1$, and it follows that ψ is injective.

Clearly ψ is continuous, and we now prove that it is surjective, and that its inverse is also continuous. We need the following simple observation, whose proof we leave as an exercise to the reader.

Lemma 4.4.18. *Suppose that $G^\times$ is a connected medial graph having at least one interior vertex. Then there exists an interior vertex $v \in V^\times_{\mathrm{int}}$ and an index $i \in [2n]$ such that v is connected in $G^\times$ to both d_i and d_{i+1} (modulo $2n$).*

Note that Proposition 4.2.21 follows from Lemma 4.4.18 as an immediate corollary.

We now return to the proof of Theorem 4.4.17, part (i). Let τ be a matching on $[2n]$, π be the corresponding fixed-point free involution, $\mathcal{M} = \mathcal{M}_\pi$ be the corresponding positroid. Choose a reduced medial graph $G^\times$ with medial pairing $\tau_{G^\times} = \tau$ (which exists by Lemma 4.4.10), and let $G^\square$ be the associated plabic graph. If $G^\times$ is not connected then each of its connected components contains an even number of vertices. Moreover, in this case $G^\square$ induces the same partition of boundary vertices into connected components, and it is clear from the definition of the map $\overline{\mathrm{Meas}}$ and Theorem 4.3.5 that each minor of $\overline{\mathrm{Meas}}(G^\square, \mathrm{wt})$ is a product of the individual minors for each of the connected components. Thus the problem naturally separates into several independent problems, one for each connected component of $G^\times$, and in what follows, we assume that $G^\times$ is connected.

Let $X \in \Pi^{>0}_{\mathcal{M}} \cap \mathrm{OG}_{\geq 0}(n, 2n)$. By Theorem 4.3.6, there exists a weight function $\mathrm{wt} : E^\square \to \mathbb{R}_{>0}$ such that $\overline{\mathrm{Meas}}(G^\square, \mathrm{wt}) = X$, and our goal is to show that there exists a unique function $J^\times : V^\times_{\mathrm{int}} \to \mathbb{R}_{>0}$ such that $\psi(J^\times) : E^\square \to \mathbb{R}_{>0}$ is obtained from wt using gauge transformations. Choose $v \in V^\times_{\mathrm{int}}$ and $i \in [2n]$ as in Lemma 4.4.18. Thus $G^\square$ contains a removable bridge between i and $i+1$ (modulo $2n$). Denote by v_1 and v_2 the vertices of $G^\square$ adjacent to d_i and d_{i+1} respectively, and denote by v_3 and v_4 the other two vertices of $G^\square$ so that v_1, v_2, v_3, v_4 surround v in counterclockwise order. Applying gauge transformations to v_1 and v_2, we may assume that $\mathrm{wt}(\{v_1, d_i\}) = \mathrm{wt}(\{v_2, d_{i+1}\}) = 1$. Let $s := \mathrm{wt}(\{v_1, v_2\}) > 0$. Applying gauge transformations to v_3 and v_4, we may assume that $\mathrm{wt}(\{v_3, v_4\}) = s$, and $\mathrm{wt}(\{v_1, v_4\}) = \mathrm{wt}(\{v_2, v_3\}) = c$ for some $c \in \mathbb{R}_{>0}$. Now, let $I := I^{\min}_{i+1}(\mathcal{M})$ and $J := [2n] \setminus I =$

$I_{i+1}^{\max}(\mathcal{M})$. Thus $i+1 \in I$ and $i \in J$. Choose some $n \times 2n$ matrix representing X and denote by $X_k \in \mathbb{R}^n$ its k-th column. For $u \in \mathbb{R}^n$ and $k \in [2n]$, let $X(k \to u)$ denote the matrix obtained from X by replacing its k-th column with u. We introduce linear functions $h_I, h_J : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$h_I(u) := \Delta_I(X(i+1 \to u)), \quad h_J(u) := \Delta_J(X(i \to u)).$$

Denote $u := X_i$ and $w := X_{i+1}$. Since $X \in \mathrm{OG}(n, 2n)$, we get $h_I(w) = h_J(u)$ and $h_I(u) = h_J(w)$. Let $(G^{\square\prime}, \mathrm{wt}')$ be obtained from $(G^{\square}, \mathrm{wt})$ by removing the bridge $\{v_1, v_2\}$, and let $X' \in \mathrm{Gr}_{\geq 0}(n, 2n) := \overline{\mathrm{Meas}}(G^{\square\prime}, \mathrm{wt}')$. By the first part of Theorem 4.3.8, we have $s = h_I(w)/h_I(u)$. By Lemma 4.3.9, we have $(X')_i = u$ while $(X')_{i+1} = w - su$. Now, after removing degree 2 vertices[3] v_1 and v_2 from $G^{\square\prime}$ and denoting the resulting graph $G^{\square\prime\prime}$, each of the vertices d_i and d_{i+1} changes color and becomes adjacent to an edge of weight $\mathrm{wt}'(\{d_i, v_4\}) = \mathrm{wt}'(\{d_{i+1}, v_3\}) = c$. Let us define wt'' to be the same as wt' except that $\mathrm{wt}''(\{d_i, v_4\}) = \mathrm{wt}''(\{d_{i+1}, v_3\}) := 1$, and let $X'' := \overline{\mathrm{Meas}}(G^{\square\prime\prime}, \mathrm{wt}'')$. It is clear from the definition of $\overline{\mathrm{Meas}}$ that $(X'')_i = cu$ and $(X'')_{i+1} = \frac{1}{c}(w - su)$. Finally, $G^{\square\prime\prime}$ has a removable bridge between i and $i+1$ so by the second part of Theorem 4.3.8, the weight $\mathrm{wt}''(\{v_3, v_4\})$ of this bridge must be equal to

$$\mathrm{wt}''(\{v_3, v_4\}) = \frac{\Delta_J(X'')}{\Delta_{J \cup \{i+1\} \setminus \{i\}}(X'')} = \frac{h_J(cu)}{h_J\left(\frac{1}{c}(w - su)\right)},$$

which after substituting $s := h_I(w)/h_I(u)$, $h_J(w) := h_I(u)$, $h_J(u) := h_I(w)$, and using the linearity of h_I, transforms into

$$\mathrm{wt}''(\{v_3, v_4\}) = \frac{c^2 h_I(u) h_I(w)}{h_I(u)^2 - h_I(w)^2}.$$

By construction, $\mathrm{wt}''(\{v_3, v_4\})$ is equal to s, and thus after substituting $h_I(w) := s h_I(u)$ the above equation becomes

$$s = \frac{c^2 s h_I(u)^2}{(1 - s^2) h_I(u)^2} = \frac{c^2 s}{1 - s^2},$$

[3] It is well known that adding/removing vertices of degree 2 does not affect the result of $\overline{\mathrm{Meas}}$, see [Lam16, Section 4.5 (M2)].

which is equivalent to $c^2+s^2=1$. Since we have $s,c>0$, it follows that $0<s,c<1$ and thus there exists a unique $t\in\mathbb{R}_{>0}$ satisfying $s=\operatorname{sech}(2t)$ and $c=\tanh(2t)$. Moreover, it is clear that t depends continuously on the minors of X, since the denominators $\Delta_I(X)=\Delta_J(X)$ must be positive. Setting $J_v^\times := t$, we uncross (in the sense of Remark 4.4.16) the interior vertex v in $G^\times$ so that the corresponding graph $G^\square$ would be obtained by removing the bridges $\{v_1,v_2\}$ and $\{v_3,v_4\}$, and proceed by induction, finishing the proof of part (i) (and therefore of part (ii) as well).

It remains to prove (4.4.5). There is a certain partial order (called the *affine Bruhat order*) on the set of decorated permutations such that two decorated permutations π,σ satisfy $\sigma\le\pi$ if and only if the closure of the positroid cell $\Pi^{>0}_{\mathcal{M}_\pi}$ contains $\Pi^{>0}_{\mathcal{M}_\sigma}$. We have

$$\Pi_{\mathcal{M}_\pi}=\bigsqcup_{\sigma\le\pi}\Pi^{>0}_{\mathcal{M}_\sigma},$$

see e.g. [Lam16, Theorem 8.1]. Moreover, the restriction of the affine Bruhat order to the set of fixed-point free involutions coincides with the poset P_n from Definition 4.4.15. Thus we have

$$\overline{(\Pi^{>0}_\tau\cap \mathrm{OG}_{\ge 0}(n,2n))}\subset\bigsqcup_{\sigma\in P_n:\sigma\le\tau}\left(\Pi^{>0}_\sigma\cap\mathrm{OG}_{\ge0}(n,2n)\right),$$

and it remains to prove that the left hand side of (4.4.5) contains the right hand side, i.e., that for all pairs $\sigma\le\tau$ in P_n, the cell $\Pi^{>0}_\sigma\cap\mathrm{OG}_{\ge0}(n,2n)$ is contained inside the closure of $\Pi^{>0}_\tau\cap\mathrm{OG}_{\ge0}(n,2n)$. Clearly it is enough to consider the case $\sigma\lessdot\tau$. Then σ is obtained from τ by uncrossing some pairs $\{i,j\}$ and $\{i',j'\}$. Moreover, since $G^\times$ is reduced, it contains a unique vertex $v\in V^\times_{\mathrm{int}}$ which belongs to the medial strands connecting i to j and i' to j', and one of the two ways of uncrossing v yields a reduced medial graph with medial pairing σ, see Remark 4.4.16. But the two ways of uncrossing v correspond to sending $J_v^\times$ to either 0 or ∞, or equivalently, sending either $s_v\to1,c_v\to0$ or $s_v\to0,c_v\to1$. By part (i), we indeed see that $\Pi^{>0}_\sigma\cap\mathrm{OG}_{\ge0}(n,2n)$ is a subset of the closure of $\Pi^{>0}_\tau\cap\mathrm{OG}_{\ge0}(n,2n)$, finishing the proof of Theorem 4.4.17. □

4.5 From the Ising model to the orthogonal Grassmannian

In this section, we study the relationship between the space $\overline{\mathcal{X}}_n$ and the space $\mathrm{OG}_{\geq 0}(n, 2n)$.

We start by slightly extending the notion of a planar Ising network so that contracting an edge in such a network would yield another such network. Throughout, we assume that a planar graph embedded in a disk has no *loops* (i.e. edges connecting a vertex to itself) or interior vertices of degree 1.

Definition 4.5.1. A *generalized planar Ising network* is a pair $N = (G, J)$ where $G = (V, E)$ is a planar graph embedded in a disk and $J : E \to \mathbb{R}_{>0} \cup \{\infty\}$. We denote $E = E_{\text{fin}} \sqcup E_\infty$, where $E_\infty := \{e \in E \mid J_e = \infty\}$. The *Ising model* associated to N is a probability measure on the space

$$\{-1, 1\}^{V/E_\infty} := \{\sigma : V \to \{-1, 1\} \mid \sigma_u = \sigma_v \text{ for all } \{u, v\} \in E_\infty\}.$$

The definitions of the probability $\mathbf{P}(\sigma)$ of a spin configuration $\sigma \in \{-1, 1\}^{V/E_\infty}$, the partition function Z, and a two-point boundary correlation $\langle \sigma_i \sigma_j \rangle$ are obtained from the corresponding definitions (4.1.1), (4.1.2), and (4.1.3) by replacing $\{-1, 1\}^V$ with $\{-1, 1\}^{V/E_\infty}$ and E with E_{fin}. As before, we let $M(G, J) = (\langle \sigma_i \sigma_j \rangle) \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$ denote the boundary correlation matrix. Thus we have $m_{i,j} = 1$ whenever there exists a path connecting b_i to b_j by edges in E_∞.

Definition 4.5.2. To each generalized planar Ising network $N = (G, J)$ we associate a medial network $N^\times = (G^\times, J^\times)$. First suppose that $E = E_{\text{fin}}$, i.e., that J only takes values in $\mathbb{R}_{>0}$. Then the medial graph $G^\times = (V^\times, E^\times)$ is obtained from G as in Figures 4-2 and 4-4. More precisely, the vertex set $V^\times$ is given by

$$V^\times = \{d_1, \dots, d_{2n}\} \sqcup \{v_e \mid e \in E\},$$

where the $d_1, \dots, d_{2n}$ are boundary vertices placed counterclockwise on the boundary of the disk so that b_i is between d_{2i-1} and d_{2i}, while v_e is the midpoint of the edge $e \in E$ of G.

The edges of $G^\times$ are described as follows. If $e, e' \in E$ share both a vertex and a face then we connect v_e to $v_{e'}$ in $G^\times$. In addition, for each $i \in [n]$, we connect d_{2i-1} (resp., d_{2i}) with v_e where $e \in E$ is the first in the clockwise (resp., counterclockwise) order edge of G incident to b_i. If b_i is isolated in G then we connect d_{2i-1} to d_{2i} in $G^\times$. Thus each vertex $v_e \in V^\times$ has degree 4, and each boundary vertex d_i, $i \in [2n]$, has degree 1 in $G^\times$. Finally, we set $J^\times_{v_e} := J_e$.

Suppose now that $E \neq E_{\text{fin}}$, and thus J takes the value of ∞ on some edges of G. Let $N_{\text{fin}} = (G, J_{\text{fin}})$ be obtained from N by setting $(J_{\text{fin}})_e := 1$ for all $e \in E_\infty$ and $(J_{\text{fin}})_e := J_e$ for $e \in E_{\text{fin}}$. Let $N^\times_{\text{fin}} := (G^\times_{\text{fin}}, J^\times_{\text{fin}})$ be the medial network associated to N_{fin}. Then the medial network $N^\times = (G^\times, J^\times)$ associated to N is obtained from $N^\times_{\text{fin}}$ by "uncrossing" (see Remark 4.4.16) the vertices v_e of $G^\times_{\text{fin}}$ for all $e \in E_\infty$. There are two ways to uncross the vertex v_e as in Figure 4-6, and we choose the one where no edge of the resulting graph $G^\times$ intersects the corresponding edge e of G. This uniquely defines the medial graph $G^\times$, and we set $J^\times_{v_e} := J_e$ for all $e \in E_{\text{fin}}$.

The notion of a generalized planar Ising network is equivalent to the notion of a *cactus network* introduced in [Lam18, Section 4.1], where he also assigns a medial graph to it in the same way as in Definition 4.5.2.

Remark 4.5.3. Given a planar Ising network $N = (G, J)$, the above procedure assigns a medial network $N^\times = (G^\times, J^\times)$ to it. In Section 4.4, we assign a weighted plabic graph $(G^\square, \text{wt})$ to $N^\times$. It is trivial to check that the same weighted plabic graph $(G^\square, \text{wt})$ gets assigned to $N = (G, J)$ in the construction described in Section 4.2.3.

To each medial graph $G^\times$ (and thus to each generalized planar Ising network) we have associated a medial pairing τ in Section 4.2.5. Let us denote

$$\mathcal{X}_\tau := \{M(G, J) \mid N = (G, J) \text{ is a generalized planar Ising network with medial pairing } \tau\}.$$

The following stratification of $\overline{\mathcal{X}}_n$ will be deduced from Theorem 4.1.3 at the end of Section 4.6.

Proposition 4.5.4. *The space* $\overline{\mathcal{X}}_n$ *decomposes as*

$$\overline{\mathcal{X}}_n = \bigsqcup_{\tau \in P_n} \mathcal{X}_\tau, \tag{4.5.1}$$

and for each $\tau \in P_n$, $\mathcal{X}_\tau$ *is homeomorphic to* $\mathbb{R}^{\mathrm{xing}(\tau)}$, *with closure relations given by the poset* P_n.

Given a (generalized) planar Ising network $N = (G, J)$, we have described two ways to assign an element of $X \in \mathrm{OG}_{\geq 0}(n, 2n)$ to N. First, one can take the boundary correlation matrix $M = M(G, J)$, and let $X := \phi(M)$, as we did in Section 4.1.3. Second, one can construct a medial network $N^\times = (G^\times, J^\times)$ as above, transform it into a weighted plabic graph $(G^\square, \mathrm{wt})$, and then put $X' := \overline{\mathrm{Meas}}(G^\square, \mathrm{wt})$, as we did in Section 4.2.3. Theorem 4.4.17, part (i) shows that the second map $J \mapsto \overline{\mathrm{Meas}}(G^\square, \mathrm{wt})$ gives a homeomorphism between $\mathbb{R}^E_{>0}$ and $\Pi^{>0}_\tau \cap \mathrm{OG}_{\geq 0}(n, 2n)$, where τ is the medial pairing of $G^\times$. The goal of the rest of this section is to show that the outputs $X = \phi(M)$ and $X' = \overline{\mathrm{Meas}}(G^\square, \mathrm{wt})$ of these two maps coincide.

Theorem 4.5.5. *Let* $N = (G, J)$ *be a (generalized) planar Ising network with boundary correlation matrix* $M = M(G, J)$. *Define* $X := \phi(M)$. *Let* $N^\times = (G^\times, J^\times)$ *and* $(G^\square, \mathrm{wt})$ *be the medial network and the weighted plabic graph corresponding to* N, *and put* $X' := \overline{\mathrm{Meas}}(G^\square, \mathrm{wt})$. *Then* $X = X'$ *in* $\mathrm{Gr}(n, 2n)$.

We give two proofs of Theorem 4.5.5, one using Dubédat's results [Dub11], and one using a formula of Lis [Lis17] for boundary correlations in terms of the *random alternating flow model*. Note that it suffices to prove Theorem 4.5.5 only for planar Ising networks, since the corresponding statement for generalized planar Ising networks is obtained by taking the limit $J_e \to \infty$ for all $e \in E_\infty$.

Before we proceed with the proofs, we need several preliminary results.

Proof of Lemma 4.2.2. By (4.2.2), it is enough to show that $m_{i,j} = 2^{-n} \sum_{I \in \mathcal{E}_n(\{i,j\})} \Delta_I(\widetilde{M})$. For $k \in [n]$, denote by $e_k \in \mathbb{R}^n$ the k-th standard basis vector, and for $k \in [2n]$, denote by

$(\widetilde{M})_k$ the k-th column of $\widetilde{M}$. Consider an $n \times n$ matrix A with columns

$$e_1, e_2, \ldots, e_{i-1}, (\widetilde{M})_{2i-1}, (\widetilde{M})_{2i}, e_{i+1}, \ldots, e_{j-1}, e_{j+1}, \ldots, e_n.$$

Since by Remark 4.1.2, $2e_k = (\widetilde{M})_{2k-1} + (\widetilde{M})_{2k}$ for all $k \in [n]$, we can expand $\det A$ in terms of minors of $\widetilde{M}$:

$$\det A = 2^{-n+2} \sum_{I \in \mathcal{E}_n(\{i,j\}) : 2i-1, 2i \in I} \Delta_I(\widetilde{M}).$$

On the other hand, since most of the columns of A are basis vectors, we can compute its determinant directly: $\det A = 2m_{i,j}$, where the sign in (4.1.4) is chosen so that we would have $\det A = 2m_{i,j}$ and not $\det A = -2m_{i,j}$. Similarly, we can define an $n \times n$ matrix B with columns

$$e_1, e_2, \ldots, e_{i-1}, e_{i+1}, \ldots, e_{j-1}, (\widetilde{M})_{2j-1}, (\widetilde{M})_{2j}, e_{j+1}, \ldots, e_n.$$

We have $\det B = 2m_{i,j}$ as well, and

$$\det B = 2^{-n+2} \sum_{I \in \mathcal{E}_n(\{i,j\}) : 2j-1, 2j \in I} \Delta_I(\widetilde{M}).$$

It remains to note that for all $I \in \mathcal{E}_n(\{i,j\})$, we have either $2i-1, 2i \in I$ or $2j-1, 2j \in I$, but not both. Thus

$$4m_{i,j} = \det A + \det B = 2^{-n+2} \sum_{I \in \mathcal{E}_n(\{i,j\})} \Delta_I(\widetilde{M}),$$

finishing the proof. $\square$

Lemma 4.5.6. *Let $J := \{1, 3, \ldots, 2n-1\}$ and $X' \in \mathrm{OG}_{\geq 0}(n, 2n)$. Then for all $I \in \binom{[2n]}{n}$, we have*

$$\Delta_I(X') \leq \Delta_J(X').$$

Proof. This follows from Skandera's inequalities [Ska04] for $\mathrm{Gr}_{\geq 0}(n, 2n)$. Namely, by [FP16, Theorem 6.1], we have $\Delta_I(X')\Delta_{[2n]\setminus I}(X') \leq \Delta_J(X')\Delta_{[2n]\setminus J}(X')$ for all $X' \in \mathrm{Gr}_{\geq 0}(n, 2n)$. In particular, if $X' \in \mathrm{OG}_{\geq 0}(n, 2n)$, this becomes $(\Delta_I(X'))^2 \leq (\Delta_J(X'))^2$, which finishes the proof. $\square$

An important consequence of the above lemma is that for $J := \{1, 3, \dots, 2n-1\}$ and all $X' \in \mathrm{OG}_{\geq 0}(n, 2n)$, we have $\Delta_J(X') > 0$, since we must have $\Delta_I(X') > 0$ for some $I \in \binom{[2n]}{n}$.

Lemma 4.5.7. *The image* $\phi(\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1))$ *contains* $\mathrm{OG}_{\geq 0}(n, 2n)$. *Equivalently, for any* $X' \in \mathrm{OG}_{\geq 0}(n, 2n)$, *there exists a matrix* $M' \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$ *such that* $X' = \phi(M')$ *as elements of* $\mathrm{Gr}(n, 2n)$.

Proof. We are going to use Lemma 4.2.2. Choose some $n \times 2n$ matrix A representing X' in $\mathrm{Gr}(n, 2n)$, and let $\widetilde{I}_n$ be the $n \times 2n$ matrix given by $(\widetilde{I}_n)_{i,2i-1} = (\widetilde{I}_n)_{i,2i} = 1$ and the remaining entries being zero. Remark 4.1.2 says that for a matrix $M' \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$, we have $\widetilde{M'} \cdot (\widetilde{I}_n)^T = 2I_n$, where I_n is the $n \times n$ identity matrix, and ^{T} denotes matrix transpose. Let $B := A \cdot (\widetilde{I}_n)^T$. We claim that if $X' \in \mathrm{OG}_{\geq 0}(n, 2n)$ then B is an invertible matrix. Indeed, by the multilinearity of the determinant, we have $\det B = \sum_{I \in \mathcal{E}_n(\emptyset)} \Delta_I(A)$. This sum contains only nonnegative terms, and the term $\Delta_{\{1,3,\dots,2n-1\}}(A)$ is positive by Lemma 4.5.6. Thus $B \in \mathrm{GL}_n(\mathbb{R})$ is invertible, and we can consider the matrix $C := 2 \cdot B^{-1} \cdot A$, which represents the same element X' in $\mathrm{Gr}(n, 2n)$. The matrix C satisfies $C \cdot (\widetilde{I}_n)^T = 2I_n$, in particular, $C_{i,2j-1} = -C_{i,2j}$ for $i \neq j \in [n]$. We define the $n \times n$ matrix $M' = (m'_{i,j})$ by $m'_{i,i} := 1$ and $m'_{i,j} := (-1)^{i+j+\mathbb{1}(i<j)} C_{i,2j-1}$ for $i \neq j \in [n]$, in agreement with (4.1.4). It turns out that M' is a symmetric matrix, since its entries can be recovered from the minors of C as follows. As we have mentioned in the proof of Lemma 4.2.2, for each $I \in \mathcal{E}_n(\{i, j\})$, we have either $2i-1, 2i \in I$ or $2j-1, 2j \in I$, but not both. Thus we can write $m'_{i,j} = 2^{-n+2} \sum_I \Delta_I(C)$, where the sum is over all $I \in \mathcal{E}_n(\{i, j\})$ such that $2i-1, 2i \in I$. Similarly, we have $m'_{j,i} = 2^{-n+2} \sum_I \Delta_I(C)$, where the sum is over all $I \in \mathcal{E}_n(\{i, j\})$ such that $2j-1, 2j \in I$. Since $\Delta_I(C) = \Delta_{[2n]\setminus I}(C)$ (because C represents $X' \in \mathrm{OG}_{\geq 0}(n, 2n)$), we see that $m'_{i,j} = m'_{j,i}$, and thus M' belongs to $\mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$. Similarly, using

$$2^{n-1} C_{i,2i-1} = \sum_{I \in \mathcal{E}_n(\emptyset) : 2i-1 \in I} \Delta_I(C) = \sum_{I \in \mathcal{E}_n(\emptyset) : 2i \in I} \Delta_I(C) = 2^{n-1} C_{i,2i},$$

and $C_{i,2i-1} + C_{2i} = 2$, we get $C_{i,2i-1} = C_{i,2i} = 1$, and thus $\phi(M') = X'$. We are done with the proof of Lemma 4.5.7. □

In order to prove Theorem 4.5.5, we need to show that $X := \phi(M)$ equals to $X' :=$

$\overline{\text{Meas}}(G^\square, \text{wt})$ as an element of $\text{Gr}(n, 2n)$. By Theorem 4.4.17, we know that $X' \in \text{OG}_{\geq 0}(n, 2n)$.

By Lemma 4.5.7, we get a matrix $M' \in \text{Mat}_n^{\text{sym}}(\mathbb{R}, 1)$ such that $\phi(M') = X'$. Since $X = \phi(M)$, we have $X = X'$ if and only if $M = M'$. By Lemma 4.2.2, the entries $m'_{i,j}$ of M' can be written as ratios of sums of minors of X'. By Theorem 4.3.5, each such minor is a sum over almost perfect matchings of $G^\square$ with prescribed boundary. Putting it all together, we get the following: for $i \neq j \in [n]$,

$$m'_{i,j} = \frac{\sum_{\mathcal{A}:\partial(\mathcal{A}) \in \mathcal{E}_n(\{i,j\})} \text{wt}(\mathcal{A})}{\sum_{\mathcal{A}:\partial(\mathcal{A}) \in \mathcal{E}_n(\emptyset)} \text{wt}(\mathcal{A})}, \tag{4.5.2}$$

where the sums are over almost perfect matchings in $G^\square$. Our goal is to show that $m'_{i,j}$ equals to $m_{i,j} := \langle \sigma_i \sigma_j \rangle$.

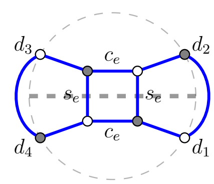
4.5.1 Dubédat's bosonization identity

Recall from Remark 4.2.10 that the planar bipartite graph $\widehat{G}^\square = (\widehat{V}^\square, \widehat{E}^\square)$ is obtained from $G^\square$ by adding an edge of weight 1 connecting d_{2i-1} to d_{2i} for each $i \in [n]$. We view $\widehat{G}^\square$ as a weighted graph embedded in the sphere, and for each $i \in [n]$, we view the vertex b_i of G as a point inside of the square face of $\widehat{G}^\square$ which contains d_{2i-1} and d_{2i}. We let $\text{Match}(\widehat{G}^\square)$ denote the set of *perfect matchings* of $\widehat{G}^\square$. Given such a perfect matching $\mathcal{A} \in \text{Match}(\widehat{G}^\square)$, its *weight* is the product of weights of its edges. We say that a *generic path* is a continuous path P in the sphere whose endpoints belong to the interiors of faces of $\widehat{G}^\square$, and which intersects every edge of $\widehat{G}^\square$ at most once. Clearly every path in G is a generic path. We denote by $\widehat{E}^\square_-(P) \subset \widehat{E}^\square$ the set of edges of $\widehat{G}^\square$ that intersect P. Given a perfect matching $\mathcal{A} \in \text{Match}(\widehat{G}^\square)$ and a generic path P, we define $\text{wt}_-(\mathcal{A}, P) := (-1)^{|\widehat{E}^\square_-(P) \cap \mathcal{A}|} \text{wt}(\mathcal{A})$. In other words, $\text{wt}_-(\mathcal{A}, P)$ is the product of weights of edges in $\mathcal{A}$, where the weight of every edge that intersects P is negated. We are ready to state Dubédat's formula, as explained in [dT14, Corollary 1]. See also [BdT12] and [DCL17, Remark 4] for related results.

Proposition 4.5.8 ([Dub11])**.** *Let $N = (G, J)$ be a planar Ising network with n boundary vertices. Let $i, j \in [n]$, and choose some path P connecting b_i to b_j in G. Then the* squared *boundary correlation function is given by*

$$\langle \sigma_i \sigma_j \rangle^2 = \frac{\sum_{\mathcal{A} \in \mathrm{Match}(\widehat{G}^\square)} \mathrm{wt}_-(\mathcal{A}, P)}{\sum_{\mathcal{A} \in \mathrm{Match}(\widehat{G}^\square)} \mathrm{wt}(\mathcal{A})}. \tag{4.5.3}$$

Example 4.5.9. If G has two vertices and one edge as in Figure 4-2, then $\widehat{G}^\square$ has the following form:



There is exactly one path P in G that connects b_1 to b_2, and $\widehat{E}^\square_-(P)$ consists of two edges of weight s_e. There are five perfect matchings of $\widehat{G}^\square$ with weights s_e^2, s_e, s_e, 1, and c_e^2, respectively. Thus (4.5.3) in this case becomes

$$\langle \sigma_1 \sigma_2 \rangle^2 = \frac{1 - 2s_e + s_e^2 + c_e^2}{1 + 2s_e + s_e^2 + c_e^2} = \frac{1 - s_e}{1 + s_e}.$$

Recall from (4.2.7) that (4.2.9) in this case yields $\langle \sigma_1 \sigma_2 \rangle = \frac{c_e}{1+s_e}$. These two formulas agree:

$$\left(\frac{c_e}{1 + s_e} \right)^2 = \frac{1 - s_e}{1 + s_e} \quad \iff \quad c_e^2 = (1 + s_e)(1 - s_e) \quad \iff \quad c_e^2 + s_e^2 = 1.$$

First proof of Theorem 4.5.5. Recall that our goal is to show that $m'_{i,j} = \langle \sigma_i \sigma_j \rangle$, where $m'_{i,j}$ is given by (4.5.2). Since the right hand side of (4.5.2) is manifestly positive, it is enough to show $(m'_{i,j})^2 = \langle \sigma_i \sigma_j \rangle^2$, which is given by (4.5.3). Observe that every perfect matching $\mathcal{A}'$ of $\widehat{G}^\square$ restricts to an almost perfect matching $\mathcal{A}$ of $G^\square$ such that $\partial(\mathcal{A}) \in \mathcal{E}_n(\emptyset)$. Moreover, this gives a weight-preserving bijection between $\mathrm{Match}(\widehat{G}^\square)$ and such almost perfect matchings of $G^\square$. Thus the denominators of (4.5.2) and (4.5.3) are equal, and therefore it suffices to show

$$\left(\sum_{\mathcal{A} : \partial(\mathcal{A}) \in \mathcal{E}_n(\{i,j\})} \mathrm{wt}(\mathcal{A}) \right)^2 = \left(\sum_{\mathcal{A} \in \mathrm{Match}(\widehat{G}^\square)} \mathrm{wt}_-(\mathcal{A}, P) \right) \left(\sum_{\mathcal{A} \in \mathrm{Match}(\widehat{G}^\square)} \mathrm{wt}(\mathcal{A}) \right), \tag{4.5.4}$$

where the sum on the left is over almost perfect matchings of $G^{\square}$.

Denote $\partial(P)$ to be the set of $i \in [n]$ such that $\{d_{2i-1}, d_{2i}\}$ intersects P. The following simple observation shows that the right hand side of (4.5.3) does not really depend on the choice of a generic path P.

Lemma 4.5.10. *Let P, P' be two generic paths with the same endpoints. Then we have*

$$(-1)^{|\widehat{E}^{\square}_{-}(P)|+|\partial(P)|}\left(\sum_{\mathcal{A}\in \mathrm{Match}(\widehat{G}^{\square})} \mathrm{wt}_{-}(\mathcal{A}, P)\right) = (-1)^{|\widehat{E}^{\square}_{-}(P')|+|\partial(P')|}\left(\sum_{\mathcal{A}\in \mathrm{Match}(\widehat{G}^{\square})} \mathrm{wt}_{-}(\mathcal{A}, P')\right).$$

Proof. Since we can transform P continuously into P' in a generic way, it is enough to show the above equality when there is a single vertex v of $\widehat{G}^{\square}$ that lies inside the region bounded by P and P'. If v belongs to the interior of the disk then it has degree 3 in $\widehat{G}^{\square}$, so $|\partial(P)| = |\partial(P')|$ but $|\widehat{E}^{\square}_{-}(P)|$ and $|\widehat{E}^{\square}_{-}(P')|$ have different parity. Similarly, if $v = d_j$ for some $j \in [2n]$ then it has degree 2 in $\widehat{G}^{\square}$ and thus $|\widehat{E}^{\square}_{-}(P)| + |\partial(P)|$ and $|\widehat{E}^{\square}_{-}(P')| + |\partial(P')|$ again have different parity. On the other hand, each perfect matching $\mathcal{A}$ of $\widehat{G}^{\square}$ contains exactly one edge incident to v, and thus we also have $\mathrm{wt}_{-}(\mathcal{A}, P) = -\mathrm{wt}_{-}(\mathcal{A}, P')$. The result follows. ☐

Observe that when P is a path in G, it necessarily intersects an even number of edges of $\widehat{G}^{\square}$, and $\partial(P) = \emptyset$. Suppose now that P connects b_i to b_j. Consider another generic path P' which connects b_i to b_j but only intersects two edges of $\widehat{G}^{\square}$, namely, $\{d_{2i-1}, d_{2i}\}$ and $\{d_{2j-1}, d_{2j}\}$. By Lemma 4.5.10, we have $\sum_{\mathcal{A}\in \mathrm{Match}(\widehat{G}^{\square})} \mathrm{wt}_{-}(\mathcal{A}, P) = \sum_{\mathcal{A}\in \mathrm{Match}(\widehat{G}^{\square})} \mathrm{wt}_{-}(\mathcal{A}, P')$, and thus it suffices to show (4.5.4) for P replaced with P'.

For simplicity, we denote $a := d_{2i-1}$, $b := d_{2i}$, $c := d_{2j-1}$, $d := d_{2j}$. Define $\widehat{G}^{\square}_{i,j}$ to be the subgraph obtained from $\widehat{G}^{\square}$ by removing the edges $\{a, b\}$ and $\{c, d\}$. Given a set $S \subset \{a, b, c, d\}$, denote by $\widehat{G}^{\square}_{i,j}(S)$ the graph obtained from $\widehat{G}^{\square}_{i,j}$ by removing the vertices of $\{a, b, c, d\} \setminus S$ (each such vertex is removed together with the unique edge incident to it). Finally, define

$$\kappa_S := \sum_{\mathcal{A}\in \mathrm{Match}(\widehat{G}^{\square}_{i,j}(S))} \mathrm{wt}(\mathcal{A})$$

for each $S \subset \{a, b, c, d\}$. Thus we have $\kappa_S = \kappa_{\{a,b,c,d\}\setminus S}$ for all S by Proposition 4.4.14. After replacing P with P', (4.5.4) becomes

$$\left(\kappa_{a,d}+\kappa_{b,c}\right)^2 = \left(\kappa_{a,b,c,d}-\kappa_{a,b}-\kappa_{c,d}+\kappa_{\emptyset}\right)\left(\kappa_{a,b,c,d}+\kappa_{a,b}+\kappa_{c,d}+\kappa_{\emptyset}\right), \tag{4.5.5}$$

where we denote $\kappa_{\{a,d\}} = \kappa_{a,d}$, etc. Since $\kappa_{a,d} = \kappa_{b,c}$, the left hand side equals $4\kappa_{a,d}^2 = 4\kappa_{a,d}\kappa_{b,c}$. Similarly, the right hand side equals $4(\kappa_{\emptyset}-\kappa_{a,b})(\kappa_{\emptyset}+\kappa_{a,b}) = 4(\kappa_{\emptyset}^2-\kappa_{a,b}^2) = 4(\kappa_{a,b,c,d}\kappa_{\emptyset}-\kappa_{a,b}\kappa_{c,d})$. Thus we need to show $\kappa_{a,d}\kappa_{b,c} = \kappa_{a,b,c,d}\kappa_{\emptyset}-\kappa_{a,b}\kappa_{c,d}$, or equivalently,

$$\kappa_{a,d}\kappa_{b,c}+\kappa_{a,b}\kappa_{c,d} = \kappa_{a,b,c,d}\kappa_{\emptyset}. \tag{4.5.6}$$

This is easy to prove bijectively using standard double-dimer arguments. For instance, superimposing a pair of matchings $(\mathcal{A},\mathcal{A}') \in \operatorname{Match}(\widehat{G}^{\square}_{i,j}(\{a,d\})) \times \operatorname{Match}(\widehat{G}^{\square}_{i,j}(\{b,c\}))$ gives a union of cycles in $\widehat{G}^{\square}_{i,j}$ together with a path Q connecting a to d and a path R connecting b to c. Thus $\mathcal{A}\oplus R$ (the symmetric difference of sets of edges) is a perfect matching of $\widehat{G}^{\square}_{i,j}(\{a,b,c,d\})$, while $\mathcal{A}'\oplus R$ is a perfect matching of $\widehat{G}^{\square}_{i,j}(\emptyset)$. The case $(\mathcal{A},\mathcal{A}') \in \operatorname{Match}(\widehat{G}^{\square}_{i,j}(\{a,b\})) \times \operatorname{Match}(\widehat{G}^{\square}_{i,j}(\{c,d\}))$ is completely similar, and together they give a bijection between the left and the right hand sides of (4.5.6). We are done with the first proof of Theorem 4.5.5. $\square$

4.5.2 Random alternating flows of Lis

For our second proof, we use a formula due to Lis [Lis17], which he proved using the *random currents model* of [GHS70], see also [DC16, LW16]. Let us say that a *clockwise bidirected edge* (resp., a *counterclockwise bidirected edge*) is a directed cycle of length two in the plane which is oriented clockwise (resp., counterclockwise).

Suppose we are given a planar Ising network $N = (G, J)$ with n boundary vertices and two disjoint subsets $A, B \subset [n]$ of the same size. We define $G^{A\sqcup B} = (V^{A\sqcup B}, E^{A\sqcup B})$ to be the graph obtained from G by adding a boundary spike at b_i for all $i \in A \sqcup B$.

An (A,B)-*alternating flow* F on G is a graph obtained from $G^{A\sqcup B}$ by replacing each edge $\{u,v\} \in E^{A\sqcup B}$ of $G^{A\sqcup B}$ by either

(a) an undirected edge, or

(b) a directed edge $u \to v$ or $v \to u$, or

(c) a clockwise or a counterclockwise bidirected edge,

so that the vertex b_i is incident to an outgoing (resp., incoming) edge if $i \in A$ (resp., if $i \in B$), and so that every other vertex $v \in V$ of G is incident to an even number of directed edges of F, and their directions alternate around v. The set of all (A, B)-alternating flows on G is denoted $\mathcal{F}_{A,B}(G)$.

For $e \in E$, we put

$$x_e := \tanh(J_e) = \frac{\exp(J_e) - \exp(-J_e)}{\exp(J_e) + \exp(-J_e)}, \quad y_e := \operatorname{sech}(J_e) = \frac{2}{\exp(J_e) + \exp(-J_e)}.$$

(Recall that x_e and y_e are not the same as $c_e = \tanh(2J_e)$ and $s_e = \operatorname{sech}(2J_e)$.) Given an edge $e \in E^{A \sqcup B}$ and an (A, B)-alternating flow $F \in \mathcal{F}_{A,B}(G)$, we set

$$\mathrm{w}(F, e) = \begin{cases} 2x_e/y_e^2, & \text{if } e \text{ is a directed edge in } F; \\ 2x_e^2/y_e^2, & \text{if } e \text{ is a bidirected edge in } F; \\ 1, & \text{otherwise.} \end{cases}$$

Following [Lis17, Eq. (4.2)], the *weight* of an (A, B)-alternating flow F is given by

$$\mathrm{w}(F) := 2^{|A|-|V(F)|} \prod_{e \in E^{A \sqcup B}} \mathrm{w}(F, e), \tag{4.5.7}$$

where $V(F)$ denotes the set of vertices $v \in V^{A \sqcup B} \setminus \{b_i \mid i \in A \sqcup B\}$ incident to a directed or a bidirected edge in F (note that b_i is always incident to a directed edge in F when $i \in A \sqcup B$).

Remark 4.5.11. The equivalence of (4.5.7) and [Lis17, Eq. (4.2)] is explained in the proof of [Lis17, Lemma 4.2].

We will be interested in the two special cases $A = B = \emptyset$ and $A = \{a\}$, $B = \{b\}$ for $a \neq b \in [n]$. We denote the corresponding graphs by $G^{\emptyset}$ and $G^{a,b}$, respectively. Denote also $\mathcal{F}_{\emptyset}(G) := \mathcal{F}_{\emptyset,\emptyset}(G)$ and $\mathcal{F}_{a,b}(G) := \mathcal{F}_{\{a\},\{b\}}(G)$.

Lemma 4.5.12 ([Lis17, Lemma 5.2]). *Let $N = (G, J)$ be a planar Ising network with n boundary vertices, and let $i \neq j \in [n]$. Then the boundary correlation $\langle \sigma_i \sigma_j \rangle$ equals*

$$\langle \sigma_i \sigma_j \rangle = \frac{\sum_{F \in \mathcal{F}_{a,b}(G)} \mathrm{w}(F)}{\sum_{F \in \mathcal{F}_{\emptyset}(G)} \mathrm{w}(F)}. \tag{4.5.8}$$

Second proof of Theorem 4.5.5. For a flow $F \in \mathcal{F}_{A,B}(G)$, let $U(F)$ denote the set of vertices $v \in V^{A \sqcup B}$ that are *not* incident to a directed or a bidirected edge of F. Thus $|U(F)| = |V| - |V(F)|$, and we set

$$\tilde{\mathrm{w}}(F) := 2^{|V|} \mathrm{w}(F) = 2^{|A|+|U(F)|} \prod_{e \in E^{A \sqcup B}} \mathrm{w}(F, e).$$

Suppose that we are given a flow $F \in \mathcal{F}_{A,B}(G)$ together with a map $\alpha : U(F) \to \{-1, 1\}$. We say that the pair (F, α) is a *spinned flow*. The weight of a spinned flow is defined to be $\tilde{\mathrm{w}}(F, \alpha) = 2^{|A|} \prod_{e \in E^{A \sqcup B}} \mathrm{w}(F, e)$, so that $\tilde{\mathrm{w}}(F) = \sum_{\alpha \in \{-1,1\}^{U(F)}} \tilde{\mathrm{w}}(F, \alpha)$. We then define an order relation $\leq$ on spinned flows by writing $(F, \alpha) \leq (F', \alpha')$ if all of the following conditions are satisfied:

- F' is obtained from F by making some undirected edges bidirected (thus $U(F') \subset U(F)$),
- the restriction of α to $U(F')$ equals α', and
- for every vertex $v \in U(F) \setminus U(F')$ such that $\alpha(v) = 1$ (resp., $\alpha(v) = -1$), all bidirected edges of F' incident to v are clockwise (resp., counterclockwise) bidirected edges.

Even though α' can be obtained from α by restricting it to $U(F') \subset U(F)$, we can also reconstruct α from (F', α'), since every vertex $v \in U(F) \setminus U(F')$ is incident to at least one bidirected edge of F', and either all such edges are clockwise (in which case we must have $\alpha(v) = 1$) or counterclockwise (in which case we must have $\alpha(v) = -1$).

Given a spinned flow (F, α), we say that an undirected edge e of F is *active* if there exists a spinned flow $(F', \alpha') > (F, \alpha)$ such that e is bidirected in F'. Thus any $(F', \alpha') > (F, \alpha)$ is obtained from (F, α) by making some active edges bidirected. (An active edge can become either a clockwise or a counterclockwise bidirected edge but not both.) Equivalently, for

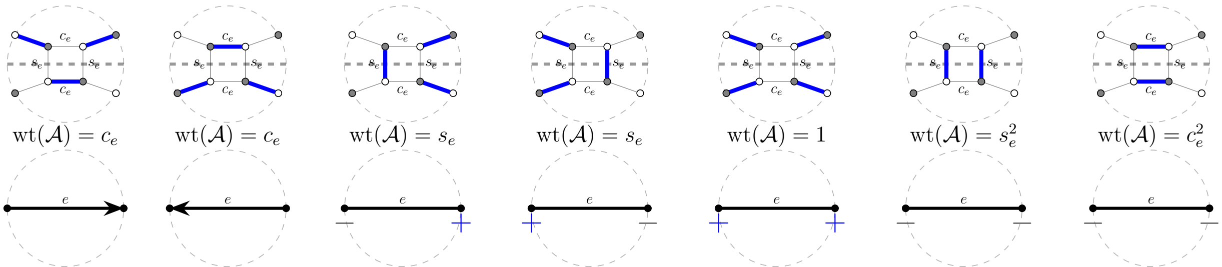


Figure 4-8: The correspondence θ between almost perfect matchings of $G^\square$ (top) and minimal spinned flows (bottom), where $+$ (resp., $-$) next to a vertex v denotes $\alpha(v,e) = 1$ (resp., $\alpha(v,e) = -1$).

every undirected edge e of F and a vertex v incident to e, we set $\alpha(v,e) := \alpha(v)$ if $v \in U(F)$, and $\alpha(v,e) := 1$ (resp., $\alpha(v,e) := -1$) if $v \in V \setminus U(F)$ and after replacing e by a clockwise (resp., counterclockwise) bidirected edge, the directions of edges still alternate around v. Then an undirected edge $e = \{u, v\}$ of F is active if and only if we have $\alpha(v,e) = \alpha(u,e)$.

We say that a spinned flow (F, α) is *minimal* if it is minimal with respect to our order relation $\le$. Equivalently, (F, α) is minimal if F has no bidirected edges. We denote $\mathcal{F}^{\min}_{A,B}(G)$ the set of all minimal spinned flows (F, α) where $F \in \mathcal{F}_{A,B}(G)$. For $(F, \alpha) \in \mathcal{F}^{\min}_{A,B}(G)$, we define its weight

$$\bar{\mathrm{w}}(F,\alpha) := \sum_{(F',\alpha') \ge (F,\alpha)} \tilde{\mathrm{w}}(F',\alpha') = 2^{|A|} \prod_{e \in E^{A \sqcup B}} \bar{\mathrm{w}}(F,\alpha,e),$$

where

$$\bar{\mathrm{w}}(F,\alpha,e) = \begin{cases} 2x_e/y_e^2, & \text{if } e \text{ is a directed edge in } F; \\ 1 + 2x_e^2/y_e^2 = (1+x_e^2)/y_e^2, & \text{if } e \text{ is an active edge of } (F,\alpha); \\ 1, & \text{otherwise.} \end{cases} \tag{4.5.9}$$

Here we used the fact that $x_e^2 + y_e^2 = 1$.

It thus follows that

$$\sum_{F \in \mathcal{F}_{A,B}(G)} \tilde{\mathrm{w}}(F) = \sum_{(F,\alpha) \in \mathcal{F}^{\min}_{A,B}(G)} \bar{\mathrm{w}}(F,\alpha).$$

Our goal is to give a map θ from almost perfect matchings of $G^\square$ to minimal spinned flows, which locally is defined in Figure 4-8. Namely, each edge $e = \{u, v\}$ of G corresponds to four interior vertices of $G^\square$, as in Figure 4-2. Every almost perfect matching $\mathcal{A}$ of $G^\square$ assigns a single edge to each of those four vertices, and there are seven ways to do so, as in Figure 4-8 (top). The product of the weights of edges of $\mathcal{A}$ incident to one of the four vertices of $G^\square$ equals, respectively, to $c_e, c_e, s_e, s_e, 1, s_e^2, c_e^2$, see Figure 4-8 (top).

Similarly, for every minimal spinned flow (F, α), e may be directed from u to v, or directed from v to u, or undirected, in which case the functions $\alpha(u, e), \alpha(v, e) \in \{-1, 1\}$ are well defined. As shown in Figure 4-8, the two matchings of weight c_e correspond to the case where e is directed in F, and the remaining five matchings correspond to e being undirected in F. Specifically, the two matchings of weight s_e correspond to the two cases where $\alpha(u, e) \neq \alpha(v, e)$, the matching of weight 1 corresponds to the case $\alpha(u, e) = \alpha(v, e) = 1$, and the two matchings of weights s_e^2, c_e^2 correspond to a single case $\alpha(u, e) = \alpha(v, e) = -1$.

It is straightforward to check that these rules give a well defined map θ from the set of almost perfect matchings of $G^\square$ to the set of minimal spinned flows on G. Moreover, it is easy to check that the set $J := \partial(\mathcal{A}) \subset [2n]$ determines uniquely two disjoint sets $A, B \subset [n]$ such that $\theta(\mathcal{A}) \in \mathcal{F}_{A,B}^{\min}(G)$. Namely, we have $A = \{i \in [n] \mid 2i - 1, 2i \notin J\}$ and $B = \{i \in [n] \mid 2i - 1, 2i \in J\}$. Finally, let $(F, \alpha) \in \mathcal{F}_{A,B}^{\min}(G)$ be a minimal spinned flow, then we claim that

$$\bar{\mathrm{w}}(F, \alpha) = \frac{1}{\prod_{e \in E} s_e} \sum_{\mathcal{A}:\theta(\mathcal{A})=(F,\alpha)} \mathrm{wt}(\mathcal{A}), \tag{4.5.10}$$

where the sum is over almost perfect matchings $\mathcal{A}$ of $G^\square$. To see why this is the case, note that the multiplicative contribution of an edge $e \in E$ to $\bar{\mathrm{w}}(F, \alpha)$ is given by (4.5.9). On the other hand, it is clear from Figure 4-8 that for any two almost perfect matchings $\mathcal{A}, \mathcal{A}'$ such that $\theta(\mathcal{A}) = \theta(\mathcal{A}')$, we have $S^\times(\mathcal{A}) = S^\times(\mathcal{A}')$, where $S^\times(\mathcal{A})$ is defined in the proof of Proposition 4.4.14. Thus the total weight of almost perfect matchings in the preimage of (F, α) under θ equals to the product over all edges $e \in E$ of $q(e)$, defined in the proof of

Proposition 4.4.14 as

$$q(e) = \begin{cases} c_e, & \text{if } e \text{ is a directed edge in } F; \\ 1 = s_e^2 + c_e^2, & \text{if } e \text{ is an active edge of } (F,\alpha); \\ s_e, & \text{otherwise.} \end{cases} \tag{4.5.11}$$

Indeed, if $e = \{u, v\}$ is an active edge of (F, α) then we either have $\alpha(u, e) = \alpha(v, e) = 1$ in which case e corresponds locally to a single matching of weight 1, or we have $\alpha(u, e) = \alpha(v, e) = -1$ in which case e corresponds locally to two matchings of weights s_e^2 and c_e^2, which can be interchanged in every almost perfect matching in the preimage of (F, α) under θ. It remains to note that the right hand side of (4.5.11) can be obtained from the right hand side of (4.5.9) by multiplying by s_e:

$$s_e\bar{\mathrm{w}}(F,\alpha,e) = \begin{cases} 2x_e s_e/y_e^2 = c_e, & \text{if } e \text{ is a directed edge in } F; \\ (1+x_e^2)s_e/y_e^2 = 1, & \text{if } e \text{ is an active edge of } (F,\alpha); \\ s_e, & \text{otherwise.} \end{cases}$$

Thus $s_e\bar{\mathrm{w}}(F, \alpha, e) = q(e)$, which proves (4.5.10). This implies that the right hand sides of (4.5.8) and (4.5.2) are equal, finishing the second proof of Theorem 4.5.5. □

4.6 Cyclic symmetry and a homeomorphism with a ball

By Theorems 4.5.5 and 4.4.17, the map ϕ is a stratification-preserving homeomorphism from $\overline{\mathcal{X}}_n$ to $\mathrm{OG}_{\geq 0}(n, 2n)$, which is the first part of Theorem 4.1.3. In this section, we follow the strategy of Chapter 2 to prove the second part of Theorem 4.1.3, which states that $\overline{\mathcal{X}}_n$ is homeomorphic to a closed ball of dimension $\binom{n}{2}$.

Recall that the cyclic shift $2n \times 2n$ matrix S was defined in Section 4.2.2. We let S^T denote the matrix transpose of S. Also, recall from Corollary 2.2.4 that $\exp(t(S+S^T))$ sends $\mathrm{Gr}_{\geq 0}(k, N)$ to $\mathrm{Gr}_{>0}(k, N)$ for all $t > 0$. Here the totally positive Grassmannian $\mathrm{Gr}_{>0}(k, N)$ is defined in (4.3.1). Let us define the *totally positive orthogonal Grassmannian* to be the intersection

$$\mathrm{OG}_{>0}(n,2n) := \mathrm{Gr}_{>0}(n,2n) \cap \mathrm{OG}(n,2n).$$

Lemma 4.6.1. *For all $X \in \mathrm{OG}_{\geq 0}(n,2n)$ and all $t > 0$, we have $X \cdot \exp(t(S+S^T)) \in \mathrm{OG}_{>0}(n,2n)$.*

Proof. In view of Corollary 2.2.4, it suffices to show that $X \cdot \exp(t(S+S^T)) \in \mathrm{OG}(n,2n)$. By Proposition 4.4.1, it is enough to prove that $\exp(t(S+S^T))$ belongs to the Lie group $O(n,n)$ consisting of all $2n \times 2n$ matrices g preserving the bilinear form η, i.e., satisfying $\eta(gu,gv) = \eta(u,v)$ for all $u,v \in \mathbb{R}^{2n}$. It is a standard fact from Lie theory that $\exp(t(S+S^T))$ is such a matrix if and only if $S+S^T$ belongs to the Lie algebra of $O(n,n)$. Let $D := \mathrm{diag}(1,-1,1,-1,\dots,1,-1)$ be a $2n \times 2n$ diagonal matrix with $D_{i,i} = (-1)^{i-1}$ for $1 \leq i \leq 2n$. Then the Lie algebra of $O(n,n)$ consists of all $2n \times 2n$ matrices B such that $B \cdot D = -D \cdot B^T$. It is easy to check that $S+S^T$ belongs to this Lie algebra. We are done with the proof. $\square$

Example 4.6.2. For $n=2$, the computation we need to check that $S+S^T$ belongs to the Lie algebra of $O(n,n)$ goes as follows.

$$(S+S^T) \cdot D = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{pmatrix};$$

$$D \cdot (S+S^T)^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{pmatrix}.$$

This shows $(S+S^T) \cdot D = -D \cdot (S+S^T)^T$ for $n=2$.

Remark 4.6.3. For all $X \in \mathrm{Gr}_{\geq 0}(n,2n)$, it was shown in Chapter 2 that the limit of $X \cdot \exp(t(S+S^T))$ as $t \to \infty$ is the unique cyclically symmetric element $X_0 \in \mathrm{Gr}_{\geq 0}(n,2n)$ from Section 2.2.1. It follows from Lemma 4.6.1 that this point X_0 belongs to $\mathrm{OG}_{\geq 0}(n,2n)$.

Proof of Theorem 4.1.3. As we have already discussed, the first part is a direct consequence of Theorems 4.5.5 and 4.4.17. The second part follows from Lemma 4.6.1 together with an argument completely identical to the one in Chapter 2, which we briefly outline here.

It was shown in Section 2.2 that the space $\mathrm{Gr}_{\geq 0}(n,2n)$ can be explicitly realized as a subset of $\mathbb{R}^N$ so that the image of $\mathrm{Gr}_{>0}(n,2n)$ would be an embedded submanifold of $\mathbb{R}^N$, and that the action of $\exp(t(S+S^T))$ on $\mathrm{Gr}_{\geq 0}(n,2n)$ extends to a contractive flow on $\mathbb{R}^N$. Since $\mathrm{OG}(n,2n)$ is an embedded submanifold of $\mathrm{Gr}(n,2n)$, we see that $Q := \mathrm{OG}_{>0}(n,2n)$ becomes an embedded submanifold of $\mathbb{R}^N$ whose closure is $\overline{Q} := \mathrm{OG}_{\geq 0}(n,2n)$ in $\mathbb{R}^N$. By Lemma 4.6.1, the contractive flow $\exp(t(S+S^T))$ restricts to $\mathrm{OG}_{\geq 0}(n,2n)$ and satisfies (2.1.1). The result follows. $\square$

Theorem 4.1.3 establishes the correspondence between the planar Ising model and the totally nonnegative orthogonal Grassmannian. Having finished its proof, we are in a position to deduce several other results stated in Section 4.2.

Proof of Theorem 4.2.4. This follows easily from studying the relationship of the map

$$(G,J) \mapsto (G^{\square}, \mathrm{wt}) \mapsto \overline{\mathrm{Meas}}(G^{\square}, \mathrm{wt}) \in \mathrm{OG}_{\geq 0}(n,2n)$$

with the duality map $(G,J) \mapsto (G^*, J^*)$. Namely, a planar Ising network $N = (G,J)$ corresponds to a weighted plabic graph $(G^{\square}, \mathrm{wt})$ then the dual planar Ising network $N^* = (G^*, J^*)$ corresponds to a weighted plabic graph $((G^*)^{\square}, \mathrm{wt}^*)$ so that $(G^*)^{\square}$ is obtained from $G^{\square}$ by switching the colors of all vertices and cyclically relabeling boundary vertices (i.e., $d_i^* := d_{i+1}$), and wt^* is obtained from wt by swapping s_e and c_e for all $e \in E$. More precisely, for each $e \in E$ we have $\sinh(2J_e)\sinh(2J^*_{e^*}) = 1$ by (4.2.4). On the other hand, by (4.2.5), we have $\sinh(2J_e) = \frac{c_e}{s_e}$ and $\sinh(2J^*_{e^*}) = \frac{c_{e^*}}{s_{e^*}}$, so $s_{e^*} = c_e$ and $c_{e^*} = s_e$. It thus follows from the definition of $\overline{\mathrm{Meas}}$ given in (4.3.2) that the minor Δ_I of $\overline{\mathrm{Meas}}(G^{\square}, \mathrm{wt})$ equals the minor $\Delta_{I'}$ of $\overline{\mathrm{Meas}}((G^*)^{\square}, \mathrm{wt}^*)$ where $I' = \{i+1 \mid i \in I\}$ (modulo $2n$) for all $I \in \binom{[2n]}{n}$. This is equivalent to having $\overline{\mathrm{Meas}}(G^{\square}, \mathrm{wt}) \cdot S = \overline{\mathrm{Meas}}((G^*)^{\square}, \mathrm{wt}^*)$, which finishes the proof. $\square$

Proof of Proposition 4.2.5. We know from Section 2.2.1 that there exists a unique cyclically symmetric element $X_0 \in \mathrm{Gr}_{\geq 0}(n,2n)$, and by Remark 4.6.3, we have $X_0 \in \mathrm{OG}_{\geq 0}(n,2n)$. By

Theorem 4.1.3, X_0 corresponds to a unique boundary correlation matrix $M_0 \in \overline{\mathcal{X}}$ of a planar Ising network (i.e., $\phi(M_0) = X_0$). Since the operation $N = (G, J) \mapsto N^* = (G^*, J^*)$ amounts to applying the cyclic shift on $\mathrm{OG}_{\geq 0}(n, 2n)$ by Theorem 4.2.4, we see that $M_0 = M(G, J)$ if and only if $M(G^*, J^*) = M(G, J)$. □

Proof of Theorem 4.2.8. This also follows easily from Theorem 4.1.3 combined with (4.3.2). □

Proof of Theorem 4.2.17. Follows from Theorem 4.1.3 and part (i) of Theorem 4.4.17. □

Proof of Theorem 4.2.20. Follows from Theorems 4.3.8 and 4.1.3. □

Proof of Theorem 4.2.22. Indeed, adjoining a boundary spike e to G' amounts to adding a pair of bridges to $(G')^\square$. Adding bridges to $(G')^\square$ translates into acting by $x_{\tilde{k}}(s_e)$ and $y_{\tilde{k}+1}(s_e)$ on $\overline{\mathrm{Meas}}((G')^\square, \mathrm{wt}')$ by Lemma 4.3.9. However, we also rescale the edges incident to $\tilde{k}$ and $\tilde{k}+1$ by c_e between adding the two bridges, which amounts to multiplying $\overline{\mathrm{Meas}}((G')^\square, \mathrm{wt})$ by a diagonal matrix $D_{\tilde{k}}(c_e)$ whose $(\tilde{k}, \tilde{k})$-th and $(\tilde{k}+1, \tilde{k}+1)$-th entries are equal to c_e and $1/c_e$, respectively. Thus if $N = (G, J)$ is obtained from $N' = (G', J')$ by adjoining a boundary spike, then the matrices $M = M(G, J)$ and $M' = M(G', J')$ are related by

$$\phi(M) = \overline{\mathrm{Meas}}(G^\square, \mathrm{wt}) = \overline{\mathrm{Meas}}((G')^\square, \mathrm{wt}') \cdot x_{\tilde{k}}(s_e) \cdot D_{\tilde{k}}(c_e) \cdot y_{\tilde{k}+1}(s_e) = \overline{\mathrm{Meas}}((G')^\square, \mathrm{wt}') \cdot g_{\tilde{k}}(t),$$

which is equal to $\phi(M') \cdot g_{\tilde{k}}(t)$. Here $x_{\tilde{k}}(s_e) \cdot D_{\tilde{k}}(c_e) \cdot y_{\tilde{k}+1}(s_e) = g_{\tilde{k}}(t)$ reduces to the following 2×2 matrix computation, which relies on $s_e^2 + c_e^2 = 1$:

$$\begin{pmatrix} 1 & s_e \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} c_e & 0 \\ 0 & 1/c_e \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ s_e & 1 \end{pmatrix} = \begin{pmatrix} 1/c_e & s_e/c_e \\ s_e/c_e & 1/c_e \end{pmatrix}.$$

We are done with the case of adjoining a boundary spike. The case of adjoining a boundary edge is completely similar, and also follows by applying the duality from Section 4.2.2, which switches between s_e and c_e due to (4.2.4). We are done with the proof. □

Proof of Proposition 4.5.4. It follows from Theorem 4.5.5 that ϕ sends $\mathcal{X}_\tau$ homeomorphically onto the cell $\Pi_\tau^{>0} \cap \mathrm{OG}_{\geq 0}(n, 2n)$, and thus the result follows from Theorem 4.1.3 combined with Theorem 4.4.17 (ii). $\square$

4.7 Generalized Griffiths' inequalities

In this section, our goal is to prove Theorem 4.2.13. Note that (4.2.10) follows from (4.2.11) by taking disjoint A and B such that $|B| = 1$. We thus focus on proving (4.2.11). Let us fix two subsets $A, B \subset [n]$, and let $C := A \oplus B$ be their symmetric difference. If C has odd size then both sides of (4.2.11) become zero. Thus we assume that the size of C is even. Recall that $\mathcal{E}_n(C) \subset \binom{[2n]}{n}$ consists of all n-element subsets I of $[2n]$ such that $I \cap \{2i-1, 2i\}$ has even size if and only if $i \in C$. (In particular, $\mathcal{E}_n(C)$ is empty when C has odd size.)

Throughout, we also fix a matrix $M = (m_{i,j}) \in \mathrm{Mat}_n^{\mathrm{sym}}(\mathbb{R}, 1)$, and we treat the entries $m_{i,j} = m_{j,i}$ as indeterminates for $i \neq j$.

Our first goal is to give a formula for the minors $\Delta_I(\widetilde{M})$ for $I \in \mathcal{E}_n(C)$.

Definition 4.7.1. Denote $n' := n - |C|/2$. Let $\alpha : [2n] \to [2n']$ be the unique order-preserving map such that $\alpha(2i-1) = \alpha(2i)$ if and only if $i \in C$. Let $\beta : [2n'] \to [n]$ be the unique order-preserving map such that the composition $\beta \circ \alpha : [2n] \to [n]$ sends both $2i-1$ and $2i$ to i for all $i \in [n]$.

Example 4.7.2. Suppose that $n = 4$ and $C = \{1, 3\}$. Then $n' = 3$, and the map $\alpha : [8] \to [6]$ sends the top row entries of the 2-line array $\left|\begin{array}{cc|c|c|cc|c|c} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 2 & 3 & 4 & 4 & 5 & 6 \end{array}\right|$ to the corresponding bottom row entries (i.e., $\alpha(1) = \alpha(2) = 1, \alpha(3) = 2$, etc.). Similarly, $\beta : [6] \to [4]$ sends the top row entries of $\left|\begin{array}{c|cc|c|cc} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 3 & 4 & 4 \end{array}\right|$ to its bottom row entries, giving rise to a composite map $\beta \circ \alpha$ represented by a 3-line array $\left|\begin{array}{cc|cc|cc|cc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 1 & 2 & 3 & 4 & 4 & 5 & 6 \\ 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{array}\right|$.

For disjoint subsets $I, J \subset [2N]$ of the same size, we say that π is a matching *between I and J* if π contains $|I| = |J|$ pairs, and for each pair $\{i, j\} \in \pi$, we have either $i \in I, j \in J$ or $i \in J, j \in I$. The set of matchings between I and J is denoted by $\mathrm{Match}(I, J)$. For a subset $K \subset [2N]$ of even size, a *matching on K* is a partition of K into $|K|/2$ disjoint subsets of size 2, and we let $\mathrm{Match}(K)$ denote the set of matchings on K. Thus $\mathrm{Match}(I, J) \subset \mathrm{Match}(I \sqcup J)$, and Match($[2n]$) is as a set equal to P_n. The function xing naturally extends to Match(I, J) and Match(K).

For each $I \in \mathcal{E}_n(C)$, we denote $I' := \alpha(I)$, and it is easy to check that we have $I' \in \binom{[2n']}{n'}$ for $I \in \mathcal{E}_n(C)$. Given a matching π on $[2n']$, we define a monomial $m_{\beta,\pi} := \prod_{\{i,j\}\in\pi} m_{\beta(i),\beta(j)}$. Similarly, given a subset $K \subset [n]$ of even size and a matching $\pi \in$ Match(K), we set $m_\pi := \prod_{\{i,j\}\in\pi} m_{i,j}$.

Proposition 4.7.3. *For $I \in \mathcal{E}_n(C)$, we have*

$$\Delta_I(\widetilde{M}) = 2^{|C|/2} \sum_{\pi\in \mathrm{Match}(I',[2n']\setminus I')} (-1)^{\mathrm{xing}(\pi)} m_{\beta,\pi}.$$

Proof. This is essentially [Pos07, Proposition 5.2], see also [Lis17, Eq. (2.2)]. □

Remark 4.7.4. For any $I \in \binom{[2n]}{n}$, there exists a unique $C \subset [n]$ such that $I \in \mathcal{E}_n(C)$. Thus Proposition 4.7.3 actually gives a formula for all maximal minors of $\widetilde{M}$ in terms of the entries of M.

Example 4.7.5. Let $n = 4$ and $C = \{1, 3\}$ as in Example 4.7.2, so $n' = 3$. The matrices M and $\widetilde{M}$ are given in Figure 4-1. Let $I := \{1, 2, 4, 7\}$. We have $I \in \mathcal{E}_n(C)$ since $|I \cap \{1, 2\}| = 2$ and $|I \cap \{5, 6\}| = 0$ are both even, while $|I \cap \{3, 4\}| = |I \cap \{7, 8\}| = 1$ are both odd. Next, $I' = \alpha(I) = \{1, 3, 5\} \in \binom{[2n']}{n'}$. Computing the maximal minor $\Delta_I(\widetilde{M})$, we find

$$\Delta_I(\widetilde{M}) = 2(m_{14}m_{23}m_{24} - m_{13}m_{24}^2 + m_{12}m_{24}m_{34} + m_{12}m_{23} + m_{14}m_{34} + m_{13}).$$

These six terms correspond to the six elements of Match($I', [2n']\setminus I'$) = Match($\{1, 3, 5\}, \{2, 4, 6\}$). For instance, the term $-m_{13}m_{24}^2$ comes from the matching $\pi = \{\{1, 4\}, \{3, 6\}, \{5, 2\}\}$ with $\mathrm{xing}(\pi) = 3$, while the term m_{13} comes from the matching $\pi = \{\{1, 4\}, \{3, 2\}, \{5, 6\}\}$ with $\mathrm{xing}(\pi) = 0$.

Definition 4.7.6. We introduce two disjoint subsets $A', B' \subset [2n']$ by:

$$A' := \{i \in [2n'] \mid \beta(i) \in A \setminus B\} \cup \{i \in [2n'] \mid \beta(i) = \beta(i + 1) \in A \cap B\};$$
$$B' := \{i \in [2n'] \mid \beta(i) \in B \setminus A\} \cup \{i + 1 \in [2n'] \mid \beta(i) = \beta(i + 1) \in A \cap B\}.$$

Define the number $\epsilon \in \{0,1\}$ mentioned in Theorem 4.2.13 by

$$\epsilon \equiv 1 + \sum_{i \in B'} i \pmod 2. \tag{4.7.1}$$

Next, we state a classical result expressing correlations of the Ising model in terms of *Pfaffians*. Given a set $K \subset [n]$ of even size, we define

$$\mathrm{Pf}_K(M) := \sum_{\pi \in \mathrm{Match}(K)} (-1)^{\mathrm{xing}(\pi)} m_\pi.$$

If the size of K is odd, we set $\mathrm{Pf}_K(M) := 0$. The following classical result expresses multi-point correlations in terms of two-point correlations.

Proposition 4.7.7 ([GBK78, Theorem A]). *Given a planar Ising network $N = (G, J)$, let $M = M(G, J)$ be its boundary correlation matrix. Then for every set $K \subset [n]$, we have*

$$\langle \sigma_K \rangle = \mathrm{Pf}_K(M) = \sum_{\pi \in \mathrm{Match}(K)} (-1)^{\mathrm{xing}(\pi)} \prod_{\{i,j\} \in \pi} \langle \sigma_i \sigma_j \rangle.$$

Thus Theorem 4.2.13 becomes a consequence of the following result.

Theorem 4.7.8. *We have*

$$\mathrm{Pf}_C(M) - \mathrm{Pf}_A(M)\,\mathrm{Pf}_B(M) = \frac{1}{2^{n-1}} \sum_{I \in \mathcal{E}_n(A \oplus B) \cap \mathcal{D}^\epsilon(B)} \Delta_I(\widetilde{M}). \tag{4.7.2}$$

Both sides of (4.7.2) are polynomials in the entries of M by Propositions 4.7.3 and 4.7.7.

Remark 4.7.9. It may look like the right hand side of (4.7.2) is not symmetric with respect to A and B, but in fact it is easy to see that

$$\mathcal{E}_n(A \oplus B) \cap \mathcal{D}^\epsilon(B) = \mathcal{E}_n(A \oplus B) \cap \mathcal{D}^{\epsilon'}(A),$$

where $\epsilon' \equiv 1 + n + \sum_{i \in A'} i \pmod 2$.

Before we prove Theorem 4.7.8, we state a lemma which will be used repeatedly later.

Lemma 4.7.10. *Let* $[2N] = K_1 \sqcup K_2$ *for two sets* K_1, K_2 *of even size. Let* $\pi_1 \in \mathrm{Match}(K_1)$*,* $\pi_2 \in \mathrm{Match}(K_2)$*, and let* $\pi_1 \sqcup \pi_2 \in \mathrm{Match}([2N])$ *be obtained by superimposing* π_1 *and* π_2*. Then*

$$\mathrm{xing}(\pi_1 \sqcup \pi_2) - \mathrm{xing}(\pi_1) - \mathrm{xing}(\pi_2) \equiv |K_1|/2 + \sum_{i \in K_1} i \equiv |K_2|/2 + \sum_{i \in K_2} i \pmod 2. \quad (4.7.3)$$

Proof. Suppose that there is $i \in K_1$ such that $i > 1$ and $i - 1 \notin K_1$. Then replacing K_1 with $K_1 \setminus \{i\} \cup \{i-1\}$ and modifying π_1, π_2 accordingly changes the parity of each side of (4.7.3). Applying this operation repeatedly until $K_1 = [|K_1|]$, the result follows. $\square$

Proof of Theorem 4.7.8. First, it is straightforward to check that if $i \in [n] \setminus (A \cup B)$ then removing i from $[n]$ does not affect the left and right hand sides of (4.7.2). Thus *from now on we assume that* $A \cup B = [n]$*.*

Assume first that $A \cap B = \emptyset$. This implies $C = A \sqcup B = [n]$, n is even, $n' = n/2$, $A' = A$, and $B' = B$. For a matching $\pi \in \mathrm{Match}([n])$, we are going to compare the coefficients of m_π on both sides of (4.7.2), and show that in all cases they are equal.

Here for two disjoint subsets I and J, we say that a matching $\pi \in \mathrm{Match}(I \sqcup J)$ *restricts to* I *and* J and write $\pi \in \mathrm{Match}|^{\mathrm{res}}_{I,J}$ if for all $\{i,j\} \in \pi$ we have either $\{i,j\} \subset I$ or $\{i,j\} \subset J$. We denote by $\pi|_I \in \mathrm{Match}(I)$ and $\pi|_J \in \mathrm{Match}(J)$ the corresponding restricted matchings. Thus the set $\mathrm{Match}|^{\mathrm{res}}_{I,J} \subset \mathrm{Match}(I \sqcup J)$ is in bijection with $\mathrm{Match}(I) \times \mathrm{Match}(J)$.

For $\pi \in \mathrm{Match}([n])$, the coefficient of m_π in $\mathrm{Pf}_C(M) - \mathrm{Pf}_A(M)\,\mathrm{Pf}_B(M)$ is equal to

$$c_{\mathrm{left}}(\pi) = \begin{cases} (-1)^{\mathrm{xing}(\pi)} - (-1)^{\mathrm{xing}(\pi|_A)}(-1)^{\mathrm{xing}(\pi|_B)} & \text{if } \pi \in \mathrm{Match}|^{\mathrm{res}}_{A,B}; \\ (-1)^{\mathrm{xing}(\pi)} & \text{otherwise.} \end{cases}$$

For the right hand side of (4.7.2), observe that by Definition 4.2.12, a given set $I \in \mathcal{E}_n(C)$ belongs to $\mathcal{D}^\epsilon(B)$ if and only if

$$\sum_{i \in I \cap \widetilde{B}} i \equiv 1 + \sum_{i \in B} i \pmod 2,$$

because $B = B'$. Since $C = [n]$, we have $I \in \mathcal{E}_n(C)$ if and only if $I \cap \{2i-1, 2i\}$ has even size for all $i \in [n]$. Let us say that a set $I \in \mathcal{E}_n(C)$ is *compatible with* π if $\pi \in \mathrm{Match}(I', [2n'] \setminus I')$. It is clear that the coefficient of m_π in the right hand side of (4.7.2) is equal to

$$c_{\mathrm{right}}(\pi) := \frac{2^{n/2}}{2^{n-1}}(-1)^{\mathrm{xing}(\pi)} N(\pi),$$

where $N(\pi)$ is the number of $I \in \mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$ compatible with π. We claim that $N(\pi)$ is given by

$$N(\pi) = \begin{cases} 2^{n/2} & \text{if } \pi \in \mathrm{Match}\,|^{\mathrm{res}}_{A,B} \text{ and } |B|/2 \equiv 1 + \sum_{i \in B} i \pmod 2; \\ 0 & \text{if } \pi \in \mathrm{Match}\,|^{\mathrm{res}}_{A,B} \text{ and } |B|/2 \not\equiv 1 + \sum_{i \in B} i \pmod 2; \\ 2^{n/2-1} & \text{if } \pi \notin \mathrm{Match}\,|^{\mathrm{res}}_{A,B}. \end{cases} \tag{4.7.4}$$

Indeed, assume first $\pi \notin \mathrm{Match}\,|^{\mathrm{res}}_{A,B}$. Then there exists a pair $\{i,j\} \in \pi$ such that $i \in A$ and $j \in B$. Note that there are a total of $2^{n/2}$ sets $I \in \mathcal{E}_n(C)$ compatible with π. Each such set satisfies either $2i-1, 2i \in I, 2j-1, 2j \notin I$ or $2j-1, 2j \in I, 2i-1, 2i \notin I$, so they naturally split into pairs $\{I, I \oplus \{2i-1, 2i, 2j-1, 2j\}\}$. Exactly one set I in each pair satisfies $\sum_{i \in I \cap \widetilde{B}} i \equiv \epsilon \pmod 2$. Thus the total number $N(\pi)$ of sets $I \in \mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$ compatible with π equals $2^{n/2-1}$ in this case.

Assume now that $\pi \in \mathrm{Match}\,|^{\mathrm{res}}_{A,B}$. Then for any $I \in \mathcal{E}_n(C)$ compatible with π, we have $\sum_{i \in I \cap \widetilde{B}} i \equiv |B|/2 \pmod 2$. Thus, either all I compatible with π belong to $\mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$, in which case we get $2^{n/2}$ of them, or they all belong to $\mathcal{E}_n(C) \cap \mathcal{D}^{1-\epsilon}(B)$, in which case we get $N(\pi) = 0$. It is easy to check that the former case happens exactly when $|B|/2 \equiv 1 + \sum_{i \in B} i \pmod 2$. This shows (4.7.4), which, combined with (4.7.3), clearly implies $c_{\mathrm{left}}(\pi) = c_{\mathrm{right}}(\pi)$. We are done with the case $A \cap B = \emptyset$.

Assume now that $A \cap B \neq \emptyset$. Since we are assuming $A \cup B = [n]$, we have $2n' = n + |A \cap B|$, and $[2n'] = A' \sqcup B'$.

For $k, k+1 \in [2n']$ such that $\beta(k) = \beta(k+1) = j$, let the *flipping* of a matching $\pi \in \mathrm{Match}([2n'])$ *at* j be a matching π' obtained from π' by "swapping" the elements $k, k+1$, i.e., $\pi' = \pi \setminus \{\{a,k\},\{b,k+1\}\} \cup \{\{a,k+1\},\{b,k\}\}$ for some $a, b \in [2n']$. (If $\{k,k+1\} \in \pi$ then we set $\pi' := \pi$.)

Note that two different matchings $\pi, \pi' \in \mathrm{Match}([2n'])$ can yield the same monomial $m_{\beta,\pi}$ if they differ by a flipping at some $j \in A \cap B$. We write in this case $\pi \sim \pi'$, and denote $\Pi = [\pi]$ the equivalence class of matchings $\pi \in \mathrm{Match}([2n'])$ with respect to this equivalence relation. Thus we have $m_{\beta,\pi} = m_{\beta,\pi'}$ if and only if $\pi \sim \pi'$, and we denote $m_{\beta,[\pi]} = m_{\beta,\pi}$.

We say that π is *trivial on* $j \in A \cap B$, denoted $\pi \perp j$, if the pair $\{k, k+1\} = \beta^{-1}(j)$ belongs to π. We say that π is *trivial on* $A \cap B$, denoted $\pi \perp A \cap B$, if π is trivial on all elements of $A \cap B$. It is easy to see that triviality depends only on the equivalence class of π, justifying the notation $\Pi \perp j$ and $\Pi \perp A \cap B$. In the case $\Pi \perp A \cap B$, Π consists of just a single element π, so we define $\mathrm{xing}(\Pi) := \mathrm{xing}(\pi)$ in this case.

Let $\pi \in \mathrm{Match}([2n'])$ be a matching. Consider a graph $\Gamma_\pi = ([2n'], E(\pi))$ with vertex set $[2n']$ and edge set

$$E(\pi) = \pi \cup \{\{k, k+1\} \mid k \in [2n'] \text{ is such that } \beta(k) = \beta(k+1)\} = \pi \cup \{\beta^{-1}(j) \mid j \in A \cap B\}.$$

(Here if π is trivial on j then the corresponding pair $\{k, k+1\} = \beta^{-1}(j)$ belongs to both π and $\{\beta^{-1}(j) \mid j \in A \cap B\}$, so Γ_π contains *two* edges connecting k to $k+1$.)

Each connected component of Γ_π contains an even number of vertices and is either a cycle or a path. We denote by $\mathrm{Conn}(\Gamma_\pi)$ the set of connected components of Γ_π and by $\mathrm{Cyc}(\Gamma_\pi) \subset \mathrm{Conn}(\Gamma_\pi)$ the set of cycles of Γ_π. Clearly, flipping π at $j \in A \cap B$ preserves the set of vertices of each connected component of Γ_π. In particular, we have $\mathrm{cyc}(\pi) := |\,\mathrm{Cyc}(\pi)| = |\,\mathrm{Cyc}(\pi')|$ for all $\pi \sim \pi'$, and thus we set $\mathrm{cyc}([\pi]) := \mathrm{cyc}(\pi)$.

For each equivalence class Π of matchings we are going to compare the coefficients of $m_{\beta,\Pi}$ on both sides of (4.7.2), and show that they are equal. (Recall that we have $m_{\beta,\pi} = m_{\beta,\pi'}$ if and only if $\pi \sim \pi'$, and in particular we have $m_{\beta,\Pi} \neq m_{\beta,\Pi'}$ for $\Pi \neq \Pi'$.)

The coefficient of $m_{\beta,\Pi}$ in the left hand side of (4.7.2) equals

$$c_{\mathrm{left}}(\Pi) = \begin{cases} (-1)^{\mathrm{xing}(\Pi)} & \text{if } \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} = \emptyset \text{ and } \Pi \perp A \cap B; \\ -(-1)^{\mathrm{xing}(\pi|_{A'}) + \mathrm{xing}(\pi|_{B'})} 2^{\mathrm{cyc}(\Pi)} & \text{if } \pi \in \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} \text{ and } \Pi \not\perp A \cap B; \\ 0 & \text{if } \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} = \emptyset \text{ and } \Pi \not\perp A \cap B. \end{cases}$$

Note that the case $\Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} \neq \emptyset$, $\Pi \perp A \cap B$ is impossible because $A \cap B \neq \emptyset$. For the second case $\pi \in \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'}$, $\Pi \not\perp A \cap B$, the parity of $\mathrm{xing}(\pi|_{A'}) + \mathrm{xing}(\pi|_{B'})$ is uniquely determined, even if π itself may not be uniquely determined. Indeed, any two $\pi, \pi' \in \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'}$ can be obtained from each other by flipping all $j \in S$ for some $S \subset A \cap B$ such that $\beta^{-1}(S)$ is a union of cycles of Γ_π (and thus a union of cycles of $\Gamma_{\pi'}$). Clearly in this case we have $\mathrm{xing}(\pi|_{A'}) + \mathrm{xing}(\pi|_{B'}) = \mathrm{xing}(\pi'|_{A'}) + \mathrm{xing}(\pi'|_{B'})$.

Recall that $I \in \mathcal{E}_n(C)$ is *compatible* with π if $\pi \in \mathrm{Match}(I', [2n'] \setminus I')$. In this case we also say that I' is *compatible* with π. Note that the map $I \mapsto I' = \alpha(I)$ is injective on $\mathcal{E}_n(C)$, and we denote by $\mathcal{E}'_n(C) := \{I' \mid I \in \mathcal{E}_n(C)\} \subset \binom{[2n']}{n'}$ the image of this map. Thus $I' \in \mathcal{E}'_n(C)$ if and only if $|I'| = n'$ and $|I' \cap \{k, k+1\}| = 1$ for all $k \in [2n']$ such that $\beta(k) = \beta(k+1)$.

It is clear that the coefficient of $m_{\beta,\Pi}$ in the right hand side of (4.7.3) is equal to

$$c_{\mathrm{right}}(\Pi) = \frac{2^{n-n'}}{2^{n-1}} \sum_{(\pi,J)} (-1)^{\mathrm{xing}(\pi)},$$

where the sum is over all pairs (π, J) such that $\pi \in \Pi$ and $J \in \mathcal{E}_n(C)$ is compatible with π. We claim that this sum equals

$$\sum_{(\pi,J)} (-1)^{\mathrm{xing}(\pi)} = \begin{cases} (-1)^{\mathrm{xing}(\Pi)} 2^{n'-1} & \text{if } \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} = \emptyset \text{ and } \Pi \perp A \cap B; \\ -(-1)^{\mathrm{xing}(\pi|_{A'}) + \mathrm{xing}(\pi|_{B'})} 2^{n'-1+\mathrm{cyc}(\Pi)} & \text{if } \pi \in \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} \text{ and } \Pi \not\perp A \cap B; \\ 0 & \text{if } \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} = \emptyset \text{ and } \Pi \not\perp A \cap B. \end{cases}$$

Consider the first case $\Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} = \emptyset$, $\Pi \perp A \cap B$. Let π be the unique element of Π. Pick some $j \in A \cap B$, and let $\{k, k+1\} := \beta^{-1}(j)$. For each pair $\{i, i'\} \in \pi$ except for $\{k, k+1\}$, choose arbitrarily which of i and i' belongs to J' and which does not. There are total $2^{n'-1}$ ways to do this. For each of the $2^{n'-1}$ ways, the condition $\sum_{i \in J \cap \widetilde{B}} i \equiv \epsilon \pmod 2$ uniquely determines whether k or $k+1$ must belong to J' in order for J to belong to $\mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$. We are done with the first case.

Consider now the third case $\Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} = \emptyset$, $\Pi \not\perp A \cap B$. It follows that there is a pair $\{i, i'\}$ common to all $\pi \in \Pi$ such that $\beta(i) \in A \setminus B$ and $\beta(i') \in B \setminus A$. There is also a pair $\{k, k+1\} = \beta^{-1}(j)$ for some $j \in A \cap B$ such that k and $k+1$ are not connected to each other in any $\pi \in \Pi$. Consider a map $\gamma : \mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B) \to \mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$ defined as follows. We put $\gamma(I) = J$ for $I, J \in \mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$ if $J' = I' \oplus \{i, i', k, k+1\}$. Let π' be obtained from π by flipping at j. We claim that $I \in \mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$ is compatible with π if and only if $\gamma(I) \in \mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$ is compatible with π'. Moreover, $\mathrm{xing}(\pi')$ differs from $\mathrm{xing}(\pi)$ by 1. Thus, we have a sign-reversing involution that cancels all the terms in $\sum_{(\pi,J)}(-1)^{\mathrm{xing}(\pi)}$, proving that it is equal to 0 in the third case.

Finally, consider the second case $\pi \in \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'}$, $\Pi \not\perp A \cap B$. We are going to show that

$$\sum_{(\pi,J)}(-1)^{\mathrm{xing}(\pi)} = -(-1)^{\mathrm{xing}(\pi|_{A'})+\mathrm{xing}(\pi|_{B'})}2^{n'-1+\mathrm{cyc}(\Pi)}.$$

Fix a matching $\pi \in \Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'}$. We claim that for any $\pi' \in \Pi$, there exists $\epsilon_{\pi'} \in \{0,1\}$ such that for all $I \in \mathcal{E}_n(C)$ compatible with π', we have $I \in \mathcal{D}^{\epsilon_{\pi'}}(B)$, that is,

$$\sum_{i \in I \cap \widetilde{B}} i \equiv \epsilon_{\pi'} \pmod 2.$$

Indeed, each component of $\Gamma_{\pi'}$ is a bipartite graph (a path or a cycle with an even number of vertices) so let us color its vertices black and white in a bipartite way. It is easy to check that $I \in \mathcal{E}_n(C)$ is compatible with π' if and only if for each connected component of $\Gamma_{\pi'}$, I' contains either all white vertices or all black vertices of this component. Let $S \subset [2n']$ be the set of vertices of a connected component of $\Gamma_{\pi'}$, and let $J \in \mathcal{E}_n(C)$ be such that $J' = I' \oplus S$ (thus J' is obtained from I' by switching from white to black inside the component S). It is straightforward to check that because π' is equivalent to $\pi \in \mathrm{Match}\,|^{\mathrm{res}}_{A',B'}$, we have

$$\sum_{i \in I \cap \widetilde{B}} i \equiv \sum_{i \in J \cap \widetilde{B}} i \pmod 2.$$

We thus define $\epsilon_{\pi'} := \sum_{i \in I \cap \widetilde{B}} i$ for some $I \in \mathcal{E}_n(C)$ compatible with π', and we have shown that $\epsilon_{\pi'}$ does not depend on the choice of I.

Next, flipping π' at some $j \in A \cap B$ changes $\epsilon_{\pi'}$ into $1 - \epsilon_{\pi'}$. Thus we have $\epsilon_{\pi'} = \epsilon$ for precisely half of the matchings $\pi' \in \Pi$, and for each such matching π', there are $2^{|\,\mathrm{Conn}(\Gamma_{\pi'})|} = 2^{n'-|A\cap B|}$ sets $J \in \mathcal{E}_n(C)$ compatible with π'. Since $\Pi \cap \mathrm{Match}\,|^{\mathrm{res}}_{A',B'} \neq \emptyset$, we have $\Pi \not\perp j$ for each $j \in A \cap B$, and thus $|\Pi| = 2^{|A\cap B|}$. Therefore the total number of pairs (π', J) such that $\pi' \in \Pi$ and $J \in \mathcal{E}_n(C) \cap \mathcal{D}^\epsilon(B)$ equals $2^{n'-1}$, and for each of them, the parity of $\mathrm{xing}(\pi')$ is the same, because it satisfies

$$\epsilon_\pi - \epsilon \equiv \mathrm{xing}(\pi) - \mathrm{xing}(\pi').$$

Thus in order to finish the proof, it suffices to show that

$$\mathrm{xing}(\pi) - \mathrm{xing}(\pi|_{A'}) - \mathrm{xing}(\pi|_{B'}) \not\equiv \epsilon_\pi - \epsilon \pmod 2. \tag{4.7.5}$$

Let $J \in \mathcal{E}_n(C)$ be compatible with π. Then by the definition of ϵ_π and ϵ, we have

$$\epsilon_\pi - \epsilon \equiv \sum_{i \in J \cap \widetilde{B}} i + \sum_{i \in B'} i + 1 \pmod 2.$$

Combining this with Lemma 4.7.10, Equation (4.7.5) transforms into

$$|B'|/2 + \sum_{i \in B'} i \equiv \sum_{i \in J \cap \widetilde{B}} i + \sum_{i \in B'} i \pmod 2,$$

equivalently, $|B'|/2 \equiv \sum_{i \in J \cap \widetilde{B}} i \pmod 2$, which follows in a straightforward way since $\pi \in \mathrm{Match}\,|^{\mathrm{res}}_{A',B'}$, J contains either all white or all black vertices in each connected component of Γ_π, and hence the contribution of each connected component to the left and right hand side is the same. We are done with the proof of Theorem 4.7.8, which implies Theorem 4.2.13 as discussed previously. □

4.8 Open problems and future directions

In this section, we briefly list several questions that in our opinion would be worth exploring further.

According to (4.4.4), $\mathrm{OG}_{\geq 0}(n, 2n)$ is a union of cells labeled by matchings τ on $[2n]$, and each such cell $\Pi_\tau^{>0} \cap \mathrm{OG}_{\geq 0}(n, 2n)$ is homeomorphic to $\mathbb{R}^{\mathrm{xing}(\tau)}$. It would be nice to understand the topology closures of these cells. In fact, we have a conjecture, analogous to [Pos07, Conjecture 3.6].

Conjecture 4.8.1. The cell decomposition (4.4.4) gives a regular CW complex structure on $\mathrm{OG}_{\geq 0}(n, 2n)$. In other words, the closure of each cell $\Pi_\tau^{>0} \cap \mathrm{OG}_{\geq 0}(n, 2n)$, given by (4.4.5), is homeomorphic to a closed $\mathrm{xing}(\tau)$-dimensional ball.

As we have already mentioned, the poset P_n of cells in $\mathrm{OG}_{\geq 0}(n, 2n)$ has been studied in the context of electrical networks. In particular, it has been shown to be shellable and Eulerian by [Lam15, HK18], which shows that P_n is the face poset of *some* regular CW complex by a result of [Bjö84]. This leads to our next question.

Question 4.8.2. Does there exist a natural stratification-preserving homeomorphism between the compactification E_n of the space of response matrices of planar electrical networks (as studied in [Lam18]) and the space $\overline{\mathcal{X}}_n$ of boundary correlation matrices of planar Ising networks?

Recall that both spaces have cell decompositions into cells indexed by matchings on $[2n]$, and both spaces are homeomorphic to a closed $\binom{n}{2}$-dimensional ball by Theorems 1.5.1 and 4.1.3. Similarly to Conjecture 4.8.1, the space E_n is believed to be a regular CW complex with face poset P_n. There are many more surprising analogies between the two spaces:

- In both cases, a planar graph yields a point in the cell corresponding to its medial pairing.
- Two reduced planar graphs yield the same point if and only if they are connected by the corresponding $Y - \Delta$ (or *star-triangle*) moves.[4]
- In both cases, there is an embedding of the space of boundary measurements into the totally nonnegative Grassmannian, as in Theorem 4.1.3 and [Lam18, Theorem 5.8].[5]

[4]In fact, under our map $G \mapsto G^{\square}$, applying a $Y - \Delta$ move to G corresponds to applying the *superurban renewal* of [KP16] to $G^{\square}$.

[5]The corresponding decorated permutations differ by a "shift by 1", i.e., if $\pi : [n] \to [n]$ is a fixed-point free involution then Lam embeds the electrical response matrix into the cell $\Pi_{\pi'}^{>0}$ of $\mathrm{Gr}_{\geq 0}(n-1, 2n)$, where $\pi'(i) := \pi(i) - 1$ modulo n for all $i \in [n]$. An analogous construction in the context of the *amplituhedron* of [AHT14] is related to going from the *momentum space* to the *momentum-twistor space*, where one performs a "shift by 2". It remains an open problem to define the amplituhedron and related objects in the context of ABJM amplitudes. We thank Thomas Lam for pointing this out to us.

- Both spaces can be realized as subsets of the partial flag variety G/P for a suitable choice of G and P, see Remark 4.4.2.

- The cyclic shift inside the corresponding Grassmannian amounts to the duality operation for Ising networks as in Section 4.2.2, and for electrical networks it corresponds to taking the dual graph and replacing each conductance by its reciprocal, as easily follows from the results of [Lam18, Section 5].

- Adding boundary spikes and boundary edges translates into adding pairs of bridges to the corresponding plabic graph, see Theorem 4.2.20 and [Lam18, Proposition 5.12].

Our next question is related to Remark 4.2.6.

Problem 4.8.3. Explain rigorously the relationship between the scaling limit of planar Ising networks at critical temperature and the unique cyclically symmetric point $X_0 \in \mathrm{OG}_{\geq 0}(n, 2n)$ from Section 2.2.1.

Our main result establishes a correspondence between total positivity and planar Ising networks, and thus potentially allows to apply results and intuition from one area to another. For example, asymptotic properties of plabic graphs have not yet been studied, while asymptotic properties of planar Ising networks have rich and important well-studied structure. Similarly, the space $\mathrm{Gr}_{\geq 0}(k, n)$ is usually studied in the context of *cluster algebras* and *canonical bases* of Lusztig, see e.g. [FZ02, Lus97]. For instance, Theorem 4.2.13 expresses Griffiths' inequalities as positive linear sums of minors of $\phi(M)$. But the theory of cluster algebras gives a much larger family of rational functions of the minors that all take positive values on $\mathrm{Gr}_{\geq 0}(k, n)$.

Problem 4.8.4. Give an interpretation of the values of other cluster variables in the cluster algebra of the Grassmannian in terms of the planar Ising model.

Another direction is related to Question 4.2.23 and the discussion after it: what is the minimal number of minors one needs to check in order to test whether a given element $X \in \mathrm{OG}(n, 2n)$ belongs to $\mathrm{OG}_{\geq 0}(n, 2n)$? A similar question for electrical networks has been discussed in [Ken12, Section 4.5.3]. This question also makes sense when X belongs to a lower-dimensional cell inside $\mathrm{OG}_{\geq 0}(n, 2n)$. Note also that in the case of the Grassmannian, collections of such minors have a very nice structure [OPS15] as they form *clusters* in the associated cluster algebra. It is not clear to us whether there exists a similar "cluster structure" on $\mathrm{OG}_{\geq 0}(n, 2n)$.

Finally, there has been a rich interplay between the areas of scattering amplitudes and total positivity, giving rise to *canonical differential forms* on positroid cells inside $\mathrm{Gr}_{\geq 0}(k, n)$, see [AHT14, AHBC^{+}16, AHBL17, GL18]. A similar result for electrical networks can be found in [Ken12, Theorem 4.13], which gives an explicit expression for the Jacobian of a certain natural map. In [HWX14, Section 2.4.2], an expression for another Jacobian was given for $\mathrm{OG}_{\geq 0}(n, 2n)$ in the context of ABJM scattering amplitudes. It would thus be interesting to understand their Jacobian in the language of planar Ising networks, as well as develop an analog of the amplituhedron for which $\mathrm{OG}_{\geq 0}(n, 2n)$ plays the role of $\mathrm{Gr}_{\geq 0}(k, n)$.

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