MANIFOLDS ASSOCIATED TO SIMPLE GAMES

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Abstract. We describe a way of producing an $(n-3)$-dimensional manifold $\mathcal{K}(\mathcal{G})$ starting with an *Alexander self-dual simplicial complex* $\mathcal{G}$ on n vertices (or, in another terminology, by *a simple game with constant sum* with n players). The construction presents $\mathcal{K}(\mathcal{G})$ explicitly, by describing its regular cellulation.

1. Introduction

It is a usual praxis that some combinatorial data produce a geometric object. Classical examples are *permutohedron*, *associahedron* [15], other "famous" polytopes, including *graph-associahedra* and *nestohedra* [12], *small covers*, and also *Bier spheres* [5, 10], and their generalizations [1].

In the paper, we act in a similar way starting with an *Alexander self-dual simplicial complex*, or equivalently, with a *simple game* $\mathcal{G}$ as a combinatorial data. We build up a cell complex $\mathcal{K}(\mathcal{G})$, whose construction although resembles very much the combinatorics of the permutohedron, yet depends on $\mathcal{G}$. The cell complex proves to be a combinatorial manifold which we call the *manifold associated to the simple game.* Unlike Bier's construction, we obtain manifolds that are not necessarily spheres: in particular, they cover configuration spaces of all existing planar flexible polygons.

The rules of building the cell complex are borrowed from the cell decomposition of the configuration space of flexible polygons,(see [11], also Section 2). This motivates us to treat a simple game $\mathcal{G}$ as *a quasilinkage* since it provides a natural generalization of polygonal linkages. By the same reason we call the cell complex $\mathcal{K}(\mathcal{G})$ the *configuration space of the quasilinkage.*

The main result is the construction of $\mathcal{K}(\mathcal{G})$ together with Theorem 4.3, which states that $\mathcal{K}(\mathcal{G})$ is a manifold locally isomorphic to configuration space of some flexible polygon. We also establish a number of properties

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of the manifold $\mathcal{K}(\mathcal{G})$: local combinatorial analysis, PL structure, canonical smooth structure, Morse surgeries.

The paper is organized as follows. In Section 2 we give all necessary information on flexible polygons and simple games.

In Section 3 we give small examples of simple games and introduce an elementary *flip* of a simple game which amounts to a Morse surgery on the associated manifold $\mathcal{K}(\mathcal{G})$.

In Section 4 we associate to a simple game $\mathcal{G}$ a cell complex $\mathcal{K}(\mathcal{G})$ by applying the rules from [11]. We prove that $\mathcal{K}(\mathcal{G})$ is locally isomorphic to $\mathcal{K}(L)$ for some flexible polygon L (however, L depends on the location, and there may be no flexible polygon associated to the entire complex). As a corollary, we immediately see that $\mathcal{K}(\mathcal{G})$ is a $(n-3)$-manifold.

In Section 5 we show that the manifold $\mathcal{K}(\mathcal{G})$ is homeomorphic to the moduli space of stable point configurations on $\mathbb{S}^1$ for an appropriate definition of stability.

Finally, in Section 6 we compare $\mathcal{K}(\mathcal{G})$ with already existing combinatorial objects: the space $\mathcal{M}_{0,n}(R)$ of real points of the moduli space of n-punctured Riemann spheres, the space of stable point configurations on $\mathbb{R}P^1$ (for an associated with $\mathcal{G}$ notion of stability), and with Bier spheres.

2. Preliminaries

2.1. Polygonal linkages: definitions and overview of the results. Given a vector $L = (l_1, ..., l_n) \in \mathbb{R}^n_+$ of n positive real numbers, consider n rigid bars of lengths $l_1, ..., l_n$ joined by revolving joints in a *closed chain*. Such a construction is called a *polygonal linkage*. By $M(L)$ we denote its *moduli space*, or *the space of planar configurations*:

$$M(L) := \{z_1, ..., z_n \in \mathbb{R}^2 : |z_i| = 1,\ \sum l_i z_i = 0\}/SO(2)$$
$$= \{z_1, ..., z_n \in \mathbb{R}^2 : |z_i| = 1,\ \sum l_i z_i = 0,\ z_1 = 1\}\ .$$

Denote by $[n]$ the set $\{1, ..., n\}$.

Definition 2.1. The length vector L is called *generic*, if there is no subset $J \subset [n]$ such that
$$\sum_{i\in J} l_i = \sum_{i\notin J} l_i.$$
Throughout the paper, we consider only generic length vectors L.

The hyperplanes
$$\sum_{i\in J} l_i = \sum_{i\notin J} l_i$$
called *walls* subdivide $\mathbb{R}^n_+$ into a collection of *chambers*.

Here is a (far from complete) summary of facts about $M(L)$:

- For a generic length vector, M(L) is a smooth manifold [8].
- The topological type of $M(L)$ depends only on the chamber of L [8].
- As it was shown in [11], $M(L)$ admits a structure of a regular cell complex. The combinatorics is very much related (but not equal) to the combinatorics of the permutohedron. The construction will be explained in details in Section 4.

Definition 2.2. For a generic length vector L, a subset $J \subset [n]$ is called *long*, if

$$\sum_{i \in J} l_i > \sum_{i \notin J} l_i.$$

Otherwise, J is called *short*. The set of all short sets we denote by $\mathcal{S}(L)$.

- Homology groups of $M(L)$ are free abelian groups. For a generic length vector L, the rank of the homology group $H_k(M(L))$ equals $a_k + a_{n-3-k}$, where a_i is the number of short subsets of size $i+1$ containing the longest edge (see [7]).

Cell structure on the configuration space of a linkage. Fix a generic length vector L. We remind that to describe a regular cell complex, it suffices to list all the (closed) cells ranged by dimension, and to describe incidence relations for closed cells.

Definition 2.3. A cyclically ordered partition $S_1, ..., S_k$ of $[n]$ into k non-empty subsets is called *admissible*, if every $S_i, 1 \leq i \leq k$, is a short set.

Theorem 2.4. *[11] The cell complex $\mathcal{K}(L)$ described below is a combinatorial manifold homeomorphic to the configuration space $M(L)$:*

1. *The k-cells of the complex are labeled by (all possible) admissible cyclically ordered partition of $[n]$ into $(n-k)$ non-empty subsets. Given a cell C, its label is denoted by $\lambda(C)$.*
2. *A closed cell C belongs to the boundary of another closed cell C' whenever the label $\lambda(C')$ is finer than the label $\lambda(C)$.* □

Let us explain in some more details how the cell structure appears. We put *labels* on the elements of the configuration space: according to definition, each configuration is a collection of unit vectors $\{z_i\}$. If the vectors are different, there is an induced cyclic ordering on $[n]$. If some of them coincide, there arises a cyclically ordered partition of $[n]$, whose parts correspond to coinciding sets of vectors. Clearly, all the labels are admissible partitions.

Two points from $M(L)$ (that is, two configurations) are *equivalent* if they have one and the same label. Equivalence classes of $M(L)$ are the *open cells*. The closure of an open cell in $M(L)$ is called a *closed cell*. For a cell

C, either closed or open, its label $\lambda(C)$ is defined as the label of (any) its interior point. The collection of open cells yields a structure of a regular cell complex which is dual to the complex $\mathcal{K}(L)$.

We stress that the complex $\mathcal{K}(L)$ depends only on the family of short subsets $\mathcal{S}(L)$. This hints that this construction can be extended to simple games.

Simple games. A family $\mathcal{G}$ of subsets of $[n] = \{1, ..., n\}$ is called *a simple game with constant sum*, or (in this paper) just a *simple game* for short, or an *Alexander self-dual complex* if it satisfies the following properties:

(1) $\mathcal{G}$ **contains all singletons**: for any $i \in [n]$, $\{i\} \in \mathcal{G}$.
(2) **Monotonicity**: if $S \in \mathcal{G}$, and $T \subset S$ then $T \in \mathcal{G}$.
(3) **Strong complementarity**: if $S \in \mathcal{G}$ then $([n] \setminus S) \notin \mathcal{G}$, and, conversely, if $S \notin \mathcal{G}$, then $([n] \setminus S) \in \mathcal{G}$.

In Game theory, the elements of $\mathcal{G}$ are called the *loosing coalitions*. One imagines that there are n players such that each subset (= each team) either beats its complement or looses. With this understanding the above axioms have a very natural meaning.

Assume that a simple game $\mathcal{G}$ is fixed.

Following the aforementioned motivation by polygonal linkages, we call any $S \in \mathcal{G}$ *a $\mathcal{G}$-short set*, or simply *a short set*, and any $S \notin \mathcal{G}$ *a long set.*

Remark 2.5. Each polygonal linkage L yields a simple game by the above defined short sets family $\mathcal{S}(L)$ (see Definition 2.1).

To the best of our knowledge it was D. Zvonkine [**?**] who observed the relation between polygonal linkages and simple games (he called the latter "voting schemes")

Definition 2.6. A simple game $\mathcal{G}$ is called *real*, if there exists a length vector L such that $\mathcal{S}(L) = \mathcal{G}$. Otherwise, $\mathcal{G}$ is called *imaginary.*

Here we list some additional properties that are true for real simple games, but in general may not hold for imaginary ones:

(1) **Comparability**: For any $A, B \in 2^{[n]}$, and any $i, j \notin A \cup B$, if $A \cup i$ is long, $A \cup j$ is short and $B \cup i$ is short then $B \cup j$ is also short. The property means that the edge i is in a sense "longer" than j.
(2) **Trade robustness**: Given k long subsets, there is no interchanging of the elements of these sets, which makes all of them short.

In [13] it was shown, that a simple game is a real game (a *weighted majority game*, in the terminology of [13]) if and only if it satisfies the trade robustness condition. Other characterizations of real simple games are given in [6, 14].

3. Simple games: tools and examples

3.1. **Small symmetric examples and non-examples.** Elementary case analysis shows that for $n \leq 5$ there are no imaginary simple games. However, for $n \geq 6$ there are many. We start with some symmetric examples of imaginary simple games in low dimensions.

Definition 3.1. We say that a simple game $\mathcal{G}$ is *symmetric* if for any $i, j \in [n]$ there exists an element σ of the symmetric group S_n such that:

(1) σ takes i to j, and
(2) σ takes short sets to short sets. (Equivalently, if σ takes long sets to long sets.)

Example 3.2. [14] Let $n = 6$. A symmetric simple game is defined by the following rules:

(1) All 2-element sets are short. (Equivalently, all 4-element sets are long.)
(2) The only ten short 3-element subsets are:

$$123, 124, 135, 146, 156, 236, 245, 256, 345, 346.$$

We give another example for $n = 7$, which is also symmetric:

Example 3.3. [14] A symmetric simple game for $n = 7$ is defined as follows:

(1) All 2-element subsets are short.
(2) The only seven 3-element long subsets are:

$$123, 145, 167, 257, 246, 347, 356.$$

Example 3.3 actually corresponds to Fano plane, and its automorphism group is known to be transitive, so this example is again symmetric.

Example 3.2 can be obtained in the following way: take an icosahedron and glue together all pairs of the opposite points. We get a simplicial complex with 6 vertices and 10 triangles which is equal to the one described in Example 3.2. It therefore corresponds to the 6-vertex triangulation of projective 2-plane, and can be generalized as vertex-minimal triangulation of projective space only in dimensions $4, 8, 16$, see [2, 3].

Lemma 3.4. (1) *If n is odd, there exists exactly one symmetric real simple game. It assigns equal lengths to all the edges. Equivalently, a set is short whenever its size is smaller than $n/2$.*
(2) *If n is even, there exists no symmetric simple game.*

Proof. Fix any symmetric real simple game $\mathcal{G}$ with length vector L. For $j \in [n]$ and $k \in \mathbb{N}$, denote by $a_k(j)$ the number of short subsets of size $k+1$ containing j. By symmetry assumption, $a_k(j)$ does not depend on j. Now assume that $l_i < l_j$ for some $i, j \in [n]$. Take a set $A \subset [n]$ such that $i, j \notin A$. If $A \cup j$ is short, then $A \cup i$ is also short. If $A \cup i$ is short and $A \cup j$ is long, then $a_{|A|}(i) > a_{|A|}(j)$, which contradicts the symmetry assumption. Therefore $A \cup j$ is short if and only if $A \cup i$ is short for any $i, j \in [n]$. This means that for any k, all the k-element subsets of $[n]$ are either simultaneously short or simultaneously long. This immediately implies the result of the lemma. □

Corollary 3.5. *Examples 3.2 and 3.3 present imaginary simple games.*

Proposition 3.6. *For $n = 8$, there is no symmetric simple game (neither real, no imaginary).*

Proof. There are $\binom{8}{4} = 70$ four-element subsets of $[n]$. For any simple game, exactly 35 of them are long, and 35 of them are short. By symmetry, any of the 8 elements of $[n]$ should be contained in the same number of short 4-element subsets, therefore $35 \cdot 4$ should be divisible by 8, but it is not. □

3.2. **Flips of simple games.**

Definition 3.7. Let $\mathcal{G}$ be a simple game and let T be a maximal (by inclusion) subset of $[n]$ such that $T \in \mathcal{G}$. Define the *flip* $F_T(\mathcal{G})$ as follows:

$$F_T(\mathcal{G}) := (\mathcal{G} \setminus \{T\}) \cup \{([n] \setminus T)\}$$

In other words, a flip is an operation that makes the $\mathcal{G}$-short set T long, and its complement short, leaving all the other sets unchanged.

Proposition 3.8. *$F_T(\mathcal{G})$ is again a simple game.*

Proof. The strong complementarity property obviously holds for $F_T(\mathcal{G})$, so it remains to check monotonicity for $F_T(\mathcal{G})$. Assume that $S \subset S' \subset [n]$, and $S' \in F_T(\mathcal{G})$. We need to prove that $S \in F_T(\mathcal{G})$. If $S' \neq \overline{T} := ([n] \setminus T)$ then every proper subset of S' is $\mathcal{G}$-short and is not equal to T by maximality, so the only remaining case is $S' = \overline{T}$. But every proper subset of $\overline{T}$ is $\mathcal{G}$-short, again, by maximality of T, so the proposition is proven. □

Example 3.9. Take the length vector $L = (l_1, ..., l_6)$ with

$$l_1 = l_2 = l_3 = 1 + \varepsilon,\ l_4 = l_5 = l_6 = 1.$$

It corresponds to a real simple game $\mathcal{S}(L)$. Now take the (maximal short) set $T = \{4, 5, 6\}$ and make a flip $\mathcal{G} := F_T(\mathcal{S}(L))$. This simple game is imaginary, because it violates the comparability condition: $\{4, 5, 6\}$ is $\mathcal{G}$-long, while $\{1, 5, 6\}$ is $\mathcal{G}$-short, so 4 must be longer than 1, but, from the other hand, $\{1, 3, 5\}$ is $\mathcal{G}$-long, while $\{4, 3, 5\}$ is $\mathcal{G}$-short.

This example differs from Example 3.2. One more example of an imaginary simple game arises from the below proposition.

Proposition 3.10. *Any flip of an imaginary simple game $\mathcal{G}$ from Example 3.2 is again imaginary.*

Proof. Because of the total symmetry of $\mathcal{G}$, it does not matter what set we will choose to be flipped, so we can choose $T := \{1, 2, 3\}$. But the simple game $\mathcal{F} := F_T(\mathcal{G})$ still violates the comparability condition: the sets $\{1, 2, 4\}$ and $\{3, 4, 5\}$ are $\mathcal{F}$-short while the sets $\{3, 2, 4\}$ and $\{1, 4, 5\}$ are $\mathcal{F}$-long, so 1 and 3 are not comparable. □

Proposition 3.11. *For a fixed n, any two simple games are connected by a sequence of flips.*

Proof. Take an arbitrary simple game $\mathcal{G}$, and take any maximal short set $T \subset [n]$ such that $1 \in T$. Apply the flip $F_T(\mathcal{G})$, take any other maximal short set containing 1, and make it long by another flip, and so on. After a finite number of steps we get a simple game $\mathcal{G}'$ such that the set S is $\mathcal{G}'$-long if and only if it contains 1. This simple game corresponds to the real simple game $\mathcal{S}(L)$ for the length vector $L = (1, \varepsilon, \varepsilon, ..., \varepsilon)$. □

Definition 3.12. (Freezing for simple games) Assume that $S_1, ..., S_k$ is a (non-ordered) partition of $[n]$ into k non-empty short sets. We build a new simple game $FREEZE(\mathcal{G})$ on the set $[k]$ by the rule:

$$J \subset [k] \text{ is short iff } \bigcup_{i \in J} S_i \text{ is short.}$$

4. Manifold associated to a simple game

Assume that a simple game, or, equivalently, a quasilinkage $\mathcal{G}$ is fixed. Although the quasilinkage in general has no configurations, we can literally repeat the construction of the cell complex for the configuration.

Definition 4.1. A cyclically ordered partition $S_1, ..., S_k$ of $[n]$ into k non-empty subsets is called $\mathcal{G}$-*admissible*, if every $S_i, 1 \leq i \leq k$, is $\mathcal{G}$-short.

Definition 4.2. For a simple game $\mathcal{G}$ its *configuration space* $\mathcal{K}(\mathcal{G})$ is the cell complex defined as follows:

(1) The k-cell of the complex are labeled by (all possible) admissible cyclically ordered partition of $[n]$ into $(n - k)$ non-empty subsets. Given a cell C, its label is denoted by $\lambda(C)$.
(2) A closed cell C belongs to the boundary of another closed cell C' whenever the label $\lambda(C')$ is finer than $\lambda(C)$.

The complex is a combinatorial manifold, which is locally isomorphic to the complex $\mathcal{K}(L)$ of some real linkage:

Theorem 4.3. (1) *For every vertex v of cell complex $\mathcal{K}(\mathcal{G})$, there exists a length vector L_v such that the star of the vertex v is combinatorially isomorphic to the star of some vertex of $\mathcal{K}(L)$.*

(2) *For every cell σ of cell complex $\mathcal{K}(\mathcal{G})$, there exists a length vector L_σ such that the star of the cell σ is combinatorially isomorphic to the star of some vertex of $\mathcal{K}(L_\sigma)$.*

(3) *For every simple game $\mathcal{G}$, the complex $\mathcal{K}(\mathcal{G})$ is a combinatorial manifold.*

Proof. (1) Fix a vertex v of $\mathcal{K}(\mathcal{G})$. By construction, it is labeled by some $\mathcal{G}$-admissible cyclically ordered partition of $[n]$ into n short non-empty subsets, that is, by a cyclic ordering on $[n]$. Without loss of generality we may assume that v is labeled by the partition

$$\lambda(v) = \{1\}, \{2\}, ..., \{n\}.$$

The partition p should be viewed as numbers $1, ..., n$ placed on the circle counterclockwise.

We need the following observation: let σ be a k-cell of $M(\mathcal{G})$ labeled by a partition $\lambda = S_1, ..., S_{n-k}$. Then σ is incident to v if and only if each of the sets S_i is of the form $\{a, a+1, ..., a+b\}$ for some natural numbers a and b (the sums are taken modulo n). It is true because otherwise the partition $\lambda(v)$ would not be a refinement of S. Let us call the sets of the form $\{a, a+1, ..., a+b\}$ *the segments of the partition* $\lambda(v)$.

Now the statement (1) follows from the lemma:

Lemma 4.4. *In the above notation, there exists a length vector L_v (depending on the vertex v) such that for any segment T of the partition $\lambda(v)$, the set T is $\mathcal{G}$-short if and only if T is L_v-short.*

Proof of the lemma.

To construct such a length vector, we will need some additional observations. Recall that $\lambda(v)$ is viewed as numbers $1, ..., n$ placed on the circle. There are n ways to break the circle into a line: $(1, 2, ..., n)$, $(2, 3, ..., n, 1)$, etc. Each such way will be called *a separator position*.

Define a positive number $q(s)$ for each of the separators as follows. Assume for example that $s = (2, ..., n, 1)$. The value $q = q(s) \in [n]$ is uniquely defined by the conditions: (1) the set $\{2, 3, ..., q-1\}$ is short, and (2) the set $\{2, 3, ..., q\}$ is long.

We are now ready to define the length vector. For any $j \in [n]$ set

$$l_j := 1 + |q^{-1}(j)|,$$

or, equivalently,

$$l_j := 1 + \frac{1}{2}\Big|\{S \subset [n] : S \text{ is a short segment of } \lambda(v); S \cup \{j\} \text{ is a long segment of } \lambda(v)\}\Big|.$$

It is clear that $\sum l_i = 2n$. We need to prove that the segment S of $\lambda(v)$ is short iff $\sum_{j\in S} l_j < n$. Note that $\sum_{j\in S} l_j = |S| + |q^{-1}(S)|$.

Take arbitrary short segment S of $\lambda(v)$. If s is a separator position adjacent to some element of S (there are $|S|+1$ such separator positions), then it is obvious that $q(s) \notin S$. Therefore $|q^{-1}(S)| \leq n - |S| - 1$, because the total number of separator positions equals to n. So for a short segment S of $\lambda(v)$ we conclude that $\sum_{j\in S} l_j = |S| + |q^{-1}(S)| \leq n-1$. Lemma is proven. □

(2) The star of a cell can be reduced to the case (1) by freezing technique. Indeed, for a cell σ labeled by $\lambda(\sigma) = S_1, S_2, ..., S_k$, we freeze all the entries in each of the sets S_i, and arrive at a simple game on the set $[k]$.

(3) Follows directly from (1), (2), and Theorem 2.4. □

The below construction gives an analysis of the vertex links of the complex $M(\mathcal{G})$.

Assume that a simple game $\mathcal{G}$ and a vertex v of $M(\mathcal{G})$ are fixed. Theorem 4.3 assigns to v a length vector $L_v = (l_1, ..., l_n)$. Without loss of generality we may assume that $l_1 + ... + l_n = 2\pi$ and that v is labeled by the cyclical ordering $\lambda(v) = (1, 2, ..., n)$.

Decompose the (metric) circle S^1 centered at the origin 0 into a union of arches of lengths $l_1, .., l_n$. The endpoints of the arches give the *Gale diagram* (see [15]) of some convex polytope $K = K(F, v) \subset \mathbb{R}^{n-3}$.

Proposition 4.5. *The link of the vertex v is combinatorially dual to boundary complex of the above defined convex polytope K.*

Proof. The vertices of K correspond to partitions of $[n]$ into $n-1$ short subsets, and, equivalently, to the short pairs of the form $(i, i+1)$ (this pair is represented by the vector u_i. By a property of Gale diagrams, the vertices of the set $I \subset [n]$ form a facet if and only if the convex hull $conv(\{u_i | i \in ([n] \backslash I)\}$ contains the origin 0 the in its relative interior. This means that the angle between every two succeeding vectors of the set $([n] \setminus I)\}$ is smaller than π. Let the indices $i_1, i_2 \notin I$ be such that for any $i_1 < i < i_2$, we have $i \in I$. Then the angle between u_{i_1} and u_{i_2} is equal to the sum $\sum_{i_1 < i \leq i_2} l_i$. So the vertices of the set I form a facet if and only if I gives a refinement of partition $\lambda(v)$ into short subsets. This corresponds to the cell incident to v, which completes the proof of the proposition. □

Theorem 4.6. *For any simple game $\mathcal{G}$, the complex $\mathcal{K}(\mathcal{G})$ admits a PL structure.*

Proof. The proof is literally the same as the proof of the analogous theorem for real linkages from [11].

For the proof we need some important property of the *standard permutohedron* which is defined as the convex hull of the set of points obtained by all possible permutations of coordinates of the point $(1, 2, ..., n) \in \mathbb{R}^n$. Its crucial property is that all faces of standard permutohedron are metric Cartesian products of standard permutohedra of smaller n. This follows almost straightforwardly from zonoid representation of the standard permutohedron.

With this knowledge let us come back to our cell complex. Each of its cells is combinatorially equivalent to a Cartesian product of permutohedra. We metrically realize each of the cells by the Cartesian product of standard permutohedra. Due to the above property, this metric realization is consistent on a cell and on its faces. □

The next proposition gives us information about what happens to the configuration space of after a flip.

Proposition 4.7. *Let $\mathcal{G}$ be a simple game and let T be any maximal $\mathcal{G}$-short subset of $[n]$. Then the configuration space of the flipped simple game $M(F_T(\mathcal{G}))$ differs from $M(\mathcal{G})$ by a Morse surgery of index $(n - |T| - 1)$.*

Proof. Consider the cell complex $M(\mathcal{G})$. The flip deletes from the complex some of the cells and adds some new cells. Assume that a cell labeled by some partition $S = (S_1, ..., S_k)$ gets deleted. This means that $T \subseteq S_i$ for some i. Since T is a maximal $\mathcal{G}$-short set, we have $T = S_i$. Therefore, all the $(n-k)$-cells which are deleted during the flip are labeled by all possible partitions of type $(T, S_1, S_2, ..., S_{k-1})$. Thus we arrive at the cell structure of the boundary of the permutohedron (see [15]) $\Pi_{n-|T|} \subset \mathbb{R}^{n-|T|-1}$ multiplied by a disk. The cell structure of $M(\mathcal{G})$ converts this disk to the permutohedron $\Pi_{|T|}$. So, we cut out a cell subcomplex $(\partial\Pi_{n-|T|}) \times \Pi_{|T|}$ and then we patch instead the cell complex $\Pi_{n-|T|} \times \partial\Pi_{|T|}$ along the identity mapping on their common boundary $\partial\Pi_{n-|T|} \times \partial\Pi_{|T|}$. This operation is the Morse surgery of index $(n - |T| - 1)$. □

Remark 4.8. Propositions 4.7 and 3.11 give an alternative proof of Theorem 4.3.

5. Stable point configurations

There is an important relationship between configuration space of a polygonal linkage and moduli space of stable point configurations on S^1. The relationship almost automatically extends to simple games. We stress that the below is a combination of the classical construction borrowed from [8] with the cell decomposition approach from [11].

Assume that a simple game $\mathcal{G}$ is fixed.

Definition 5.1. A configuration of n (not necessarily distinct) marked points $p_1, ..., p_n$ on the unit circle $\mathbb{S}^1$ is called *$\mathcal{G}$-stable* if the following holds:

If the points $\{p_i\}_{i\in I}$ coincide, then the set $I \subset [n]$ is $\mathcal{G}$-short.

We identify $\mathbb{S}^1$ with the real projective line $\mathbb{R}P^1$, which enables us to speak of diagonal action of the group $PSL(2,\mathbb{R})$ on the space of all stable configurations. We introduce the quotient space

$$M_{st}(\mathcal{G}) = \{\text{space of } \mathcal{G}\text{-stable configurations}\}/PSL(2,\mathbb{R}).$$

Theorem 5.2. *Given a simple game $\mathcal{G}$,*

(1) *$M_{st}(\mathcal{G})$ is a $(n-3)$-dimensional manifold.*
(2) *$M_{st}(\mathcal{G})$ is homeomorphic to $M(\mathcal{G})$.*
(3) *The stratification of the space $M_{st}(\mathcal{G})$ by combinatorial types is a regular cell complex dual to the cell complex $M(\mathcal{G})$.*

Proof. We label each point configuration by its *combinatorial type* – the cyclically ordered partition of the set $[n]$. The labels do not change under the action of the group $PSL(2,\mathbb{R})$. Equivalence classes are open balls of different dimensions, and can be considered as open cells of some cell decomposition.

We arrive at the cell complex on $M_{st}(\mathcal{G})$ defined as follows:

(1) The k-cell of the complex are labeled by (all possible) admissible cyclically ordered partition of $[n]$ into $k+3$ non-empty subsets. Given a cell C, its label is denoted by $\lambda(C)$.
(2) A closed cell C belongs to the boundary of another closed cell C' whenever the label $\lambda(C')$ is finer than $\lambda(C')$.

This cell decomposition is obviously combinatorially dual to the cell complex $\mathcal{K}(\mathcal{G})$. ∎

6. Concluding remarks

We conclude the paper by a survey-type paragraph indicating relationships of our construction with already existing objects.

Deligne-Mumford-Knudsen compactification of $M_{0,n}(\mathbb{R})$. The space $M_{0,n}(\mathbb{C})$ of Riemann spheres with n distinct labeled punctures plays an important role in many respects and has been studied extensively. In the present paper we are interested in the space of its *real points* $M_{0,n}(\mathbb{R})$, e.g. points that are fixed under complex conjugation. It equals the space of configurations of n labeled distinct points on $S^1 = \mathbb{R}P^1$ (modulo projective transforms). The spaces $M_{0,n}(\mathbb{C})$ and $M_{0,n}(\mathbb{R})$ are obviously non-compact, since the points cannot collide. People were looking for nice ways to compactify them. Probably the most remarkable is the Deligne-Mumford-Knudsen compactification $M_{0,n}(\mathbb{C}) \hookrightarrow \overline{M_{0,n}}(\mathbb{C})$, which adds a normal crossing divisor and yields a smooth variety of complex dimension $n-3$. This gives rise to the compactification $M_{0,n}(\mathbb{R}) \hookrightarrow \overline{M_{0,n}}(\mathbb{R})$, which can be described as a series of blow-ups coming from De Concini and Procesi wonderful compactification [4] for the braid arrangement, see [9]. This compactification comes automatically with a cell decomposition into a number of associahedra.

Very informally, in this setting the blow-ups mean that when two points come close to each other, the limit is not a point, but a new branch, containing the two points.

One more compactification comes from flexible polygons. As the Deligne-Mumford-Knudsen compactification, it also comes with some cellulation. But now the cells are Cartesian products of permutohedra.

In fact, this in not "one", but a series of different compactifications, provided by the diversity of flexible polygons. For all of them, the points are allowed to collide, but not all the collisions are admissible. The specification of admissible collisions comes from a *flexible polygon*, or (in other terminology) from a *polygonal linkage*.

Let us fix a flexible polygon $L = (l_1, ..., l_n)$.

A configuration of (not necessarily distinct) points $p_1, ..., p_n$ on $S^1 = \mathbb{R}P^1$ is *L-stable* if whenever for a set $I \subset [n] = \{1, ..., n\}$ we have

$$p_i = p_j \ \ \forall \ i, j \in I \ ,$$

then the following condition should hold:

$$\sum_{i \in I} l_i < \sum_{i \notin I} l_i.$$

In other words, if we think of the points as of weighted ones, the condition says that the weight of the colliding points should not exceed one half of the total weight of all the points.

It is known (see [8]) that the configuration space is diffeomorphic to the quotient of the space of all stable configurations by the diagonal action of the group $PSL(2, \mathbb{R})$:

$$M(L) \cong \{L\text{-stable configurations }\}/PSL(2, \mathbb{R}).$$

As we have conclude from Section 5, yet another compactification $\mathcal{K}(\mathcal{G})$ comes from simple games. Its cells are again Cartesian products of permutohedra, see Section 4.

Bier spheres. In 1992 Thomas Bier explained how to cook up a simplicial $(n-2)$-spheres on $2n$ vertices out of a simplicial complex on n vertices by taking the *deleted join* of the complex with its Alexander dual. In particular, each simple game gives an associated Bier sphere. In turn, our construction assigns a combinatorial $(n-3)$-manifold (generically, not necessarily a sphere) with each simple game.

We try to compare the two constructions and indicate some elementary similarities. However, it remains an open problem whether there exists a deeper relation between these two objects.

For a simplicial complex Δ its *Alexander dual* simplicial complex is defined as $\Delta^* := \{F \in [n] : [n] \setminus F \notin \Delta\}$.

Right from the definition it follows that a simple game is an Alexander self-dual complex, and vise versa. The *deleted join* of the complex with its Alexander dual is called the Bier sphere associated to Δ. The definition implies that the simplices of the deleted join are labeled by ordered partitions of the set $[n]$ into three parts (A, B, C) such that:

(1) $A \in \Delta$,
(2) $C \in \Delta^*$, that is $[n] \setminus C \notin \Delta$.
(3) B is just any; however, one concludes that B is never empty.
(4) A and B may be empty, but not both of them at the same time.

The incidence relations in the complex are the following: a simplex (A, B, C) belongs to (A', B', C') whenever

$$A \subseteq A', \quad B \subseteq B'.$$

A Bier sphere is indeed a combinatorial sphere. In [10] it is proven inductively, by showing that adding a simplex to Δ corresponds to a *bistellar move*, that is, to cutting off a triangulated $(n-2)$-ball and replacing it by another triangulation of the same ball.

Therefore, a flip applied to a simple game $\mathcal{G}$ amounts to a pair of bistellar moves with disjoint supports on the associated Bier sphere.

In turn, we have seen that a flip amounts to a Morse surgery on the manifold $\mathcal{K}(\mathcal{G})$.

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