

EXTENSIONS OF ISOMETRIC EMBEDDINGS OF PSEUDO-EUCLIDEAN METRIC POLYHEDRA

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ABSTRACT. We extend the results of B. Minemyer by showing that any indefinite metric polyhedron (either compact or not) with the vertex degree bounded from above admits an isometric simplicial embedding into a Minkowski space of the lowest possible dimension. We provide a simple algorithm of constructing such embeddings. We also show that every partial simplicial isometric embedding of such space in general position extends to a simplicial isometric embedding of the whole space.

1. INTRODUCTION

The question about isometrical simplicial embeddings of indefinite metric polyhedra into a Minkowski space was recently considered in [5]. We show that the results from [5] hold for non-compact indefinite metric polyhedra as well, and give an explicit construction. We also show that every partial simplicial isometric embedding of indefinite metric polyhedra into a Minkowski space such that the images of the vertices are in d -general position extends to a simplicial isometric embedding of the whole space.

1.1. Definitions. An *indefinite metric polyhedron* \mathcal{X} is a simplicial complex endowed with a bilinear form attached to every simplex. The forms must agree on the intersections of the simplices, and are not assumed to be positive-definite or even non-degenerate. For any edge (1-simplex) of this complex the value of the bilinear form on this edge is called its *squared length*. The squared lengths of the edges can be arbitrary (including negative) real numbers, and these numbers completely determine the bilinear forms.

A *simplicial map* of an indefinite metric polyhedron into a vector space \mathbb{R}^n is a map that is affine on every simplex. Every simplicial map is completely determined by the images of the vertices.

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A *simplicial isometric map* of an indefinite metric polyhedron into a Minkowski space \mathbb{R}_q^p with signature $(\underbrace{+, \dots, +}_p, \underbrace{-, \dots, -}_q)$ is a simplicial map such that

the bilinear form of every simplex equals to the pull-back of the inner product of \mathbb{R}_q^p along this map. In particular, such maps preserve the squared lengths of edges. Conversely, every simplicial map that preserves squared lengths of edges is isometric, because a bilinear form is completely determined by its values on the edges of a non-degenerate simplex.

A *simplicial isometric embedding* of a metric polyhedron into a Minkowski space is an injective simplicial isometric map.

For an indefinite metric polyhedron, by \mathcal{V} we denote its vertex set. For every vertex v of this set we define its *degree* $\deg(v)$ as the number of edges of the simplicial complex containing v .

In the sequel we restrict our attention only to finite or countable indefinite metric polyhedra.

The special case of embedding of Euclidean polyhedra into a Euclidean space has been studied in details in [1, 2, 4]. Note that in general a Euclidean polyhedron does not admit a simplicial embedding into a Euclidean space. However, every polyhedron can be subdivided in such a way that the resulting polyhedron admits such an embedding, see, for example, [6]. In contrast, in the pseudo-Euclidean setting one does not need to subdivide the triangulation. In particular, a Euclidean polyhedron (of bounded vertex degree) admits a simplicial isometric embedding into a (non-Euclidean) Minkowski space of an appropriate dimension.

Now we are ready to cite a theorem from [5]:

Theorem (B. Minemyer, 2012, [5], Theorem 1.1, Corollary 3.4). *Let \mathcal{X} be a compact n -dimensional indefinite metric polyhedron with vertex set \mathcal{V} . Then*

- (1) *There exists a simplicial isometric map of \mathcal{X} into \mathbb{R}_d^d , where $d = \max\{\deg(v) \mid v \in \mathcal{V}\}$.*
- (2) *There exists a simplicial isometric embedding of \mathcal{X} into \mathbb{R}_q^q , where $q = \max\{d, 2n + 1\}$.*

The dimension of \mathbb{R}_d^d in the first part of the theorem is shown in [5] to be optimal. The idea is that the 1-skeleton of the standard d -dimensional simplex cannot be isometrically embedded into \mathbb{R}_p^p with $p < d$.

This theorem is followed in [5] by two generalizations. One of them provides an algorithm to explicitly construct such extensions, and the other one applies to non-compact polyhedra. Both of these upgrades result in a significant increase of the target space dimension. For non-compact polyhedra, the target space is \mathbb{R}_p^p , where $p = 2q(d^3 - d^2 + d + 1)$, and d and q are the same as in the theorem.

In this paper we give an elementary proof of an extension theorem for such isometric embeddings of indefinite metric polyhedra. As a corollary, we get completely constructive versions of the theorems from [5] for both compact and non-compact polyhedra, and the dimension of the target Minkowski space always remains optimal.

1.2. Preliminaries and notation. We denote an indefinite metric polyhedron by a triple $(\mathcal{X}, \mathcal{T}, g)$, where $(\mathcal{X}, \mathcal{T})$ is the simplicial complex, and g is the function $g : E(\mathcal{T}) \rightarrow \mathbb{R}$ associating a squared length to every edge. Here $E(\mathcal{T})$ is the set of edges of \mathcal{T} .

A *Minkowski space* of signature (p, q) denoted by \mathbb{R}_q^p is a vector space $\mathbb{R}^{p+q} = \{v = (v^+, v^-) | v^+ \in \mathbb{R}^p, v^- \in \mathbb{R}^q\}$ endowed with the following pseudoscalar product:

$$\langle v, u \rangle_{\mathbb{R}_q^p} := \langle v^+, u^+ \rangle_{\mathbb{R}^p} - \langle v^-, u^- \rangle_{\mathbb{R}^q},$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^d}$ denotes the standard scalar product in \mathbb{R}^p or \mathbb{R}^q . We restrict our attention to Minkowski spaces of signature (d, d) for some d .

Let $Q \subset \mathbb{R}_d^d$ be a collection of points. We say that the points of Q are in d -general position if and only if any subcollection of at most $d + 1$ points of Q forms an affinely independent set.

Let $(\mathcal{X}, \mathcal{T}, g)$ be an indefinite metric polyhedron with vertex set \mathcal{V} and let $\mathcal{V}' \subset \mathcal{V}$ be an arbitrary subset of \mathcal{V} . A map $m : \mathcal{V}' \rightarrow \mathbb{R}_d^d$ is called a *partial simplicial isometric map* if for every simplex S of \mathcal{X} whose vertices belong to \mathcal{V}' , m preserves the bilinear form of S . Equivalently, for every edge l with endpoints $a, b \in \mathcal{V}'$, the squared length of l equals $\langle m(a) - m(b), m(a) - m(b) \rangle_{\mathbb{R}_d^d}$.

For a positive integer k , a finite or countable graph is called *k -degenerate* if every its subgraph has a vertex of degree at most k . Equivalently speaking, a graph is k -degenerate if and only if its vertices can be ordered so that each vertex has at most k neighbors that are earlier in the ordering. Clearly, a graph with maximal degree d is d -degenerate.

Similarly, we say that an indefinite metric polyhedron is *d -degenerate* if its 1-dimensional skeleton is a d -degenerate graph.

1.3. Main results. All the indefinite metric polyhedra are assumed to be countable or finite.

We begin with a constructive version of theorems from [5] for non-compact indefinite metric polyhedra:

Theorem 1. *Let $(\mathcal{X}, \mathcal{T}, g)$ be a (possibly non-compact) n -dimensional d -degenerate indefinite metric polyhedron. Then*

- (1) *There exists a simplicial isometric map of \mathcal{X} into \mathbb{R}_d^d .*

- (2) In addition, assume that $d \geq 2n + 1$. Then there exists a simplicial isometric embedding of \mathcal{X} into \mathbb{R}_d^d .

In Section 2.1 we provide a simple algorithm to construct such maps.

Our main theorem states that every partial simplicial isometric map of an indefinite metric polyhedron with uniformly bounded vertex degrees extends to a simplicial isometric map of the whole polyhedron.

Theorem 2. *Let $G = (\mathcal{X}, \mathcal{T}, g)$ be an indefinite n -dimensional metric polyhedron with vertex set \mathcal{V} , and let $d := \max\{\deg(v) | v \in \mathcal{V}\}$.*

- (1) *Let $\tau' : \mathcal{V}' \rightarrow \mathbb{R}_d^d$ be a partial simplicial isometric map of G into a Minkowski space such that the points $\tau'(\mathcal{V}')$ are in d -general position. Then there exists a simplicial isometric map $\tau : \mathcal{V} \rightarrow \mathbb{R}_d^d$ of G such that $\tau|_{\mathcal{V}'} = \tau'$ and the points $\tau(\mathcal{V})$ are also in d -general position.*
- (2) *If in addition $d \geq 2n + 1$, then both τ and τ' are automatically injective, so τ is a simplicial isometric embedding.*

Theorem 1 is a special case of Theorem 2 with $\mathcal{V}' = \emptyset$, but we provide a separate short proof for this important case.

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2. PROOF OF THEOREM 1

A linear subspace H of \mathbb{R}_d^d is called *isotropic* if any vector $h \in H$ has zero squared length, i.e. $\langle h, h \rangle = 0$.

Fix two arbitrary complementary isotropic d -dimensional subspaces Σ, Δ of \mathbb{R}_d^d . This can be easily achieved, for example, by putting

$$\Sigma := \{(v^+, v^-) | v^+ - v^- = 0\}$$

and

$$\Delta := \{(v^+, v^-) | v^+ + v^- = 0\}.$$

We denote by P_Σ and P_Δ the projection operators on these subspaces with respect to the direct sum decomposition $\mathbb{R}_d^d = \Sigma \oplus \Delta$. That is, for any $v \in \mathbb{R}_d^d$, $P_\Sigma(v) \in \Sigma$, $P_\Delta(v) \in \Delta$ and $P_\Sigma(v) + P_\Delta(v) = v$.

The following lemma is used in the proofs of Theorems 1 and 2:

Lemma 3. *Let $\mathbb{R}_d^d = \Sigma \oplus \Delta$, where Σ, Δ are two complementary isotropic d -dimensional subspaces of \mathbb{R}_d^d . Let v_0 be a point in Δ , let $k \leq d$ be a positive integer, and let u_1, \dots, u_k be a set of points in \mathbb{R}_d^d such that $v_0, P_\Delta(u_1), \dots, P_\Delta(u_k)$ are affinely independent in Δ . Then for any sequence c_1, \dots, c_k of real numbers there is a point $u_0 \in \mathbb{R}_d^d$ such that*

- (1) for every $1 \leq i \leq k$, $\langle u_0 - u_i, u_0 - u_i \rangle_{\mathbb{R}_d^d} = c_i$,
(2) $P_\Delta(u_0) = v_0$.

Proof. It is easy to see that for any vector $v \in \mathbb{R}_d^d$, one has

$$\langle v, v \rangle_{\mathbb{R}_d^d} = 2\langle P_\Delta(v), P_\Sigma(v) \rangle_{\mathbb{R}_d^d},$$

because the subspaces Δ and Σ are required to be isotropic. Note that $\langle \cdot, \cdot \rangle_{\mathbb{R}_d^d}|_{\Delta \times \Sigma}$ is a non-degenerate bilinear pairing between Δ and Σ .

System (1) is equivalent to the system

$$2\langle P_\Delta(u_0 - u_i), P_\Sigma(u_0 - u_i) \rangle_{\mathbb{R}_d^d} = c_i, 1 \leq i \leq k.$$

Denote $P_\Delta(u_i)$ by v_i and $P_\Sigma(u_i)$ by h_i for $1 \leq i \leq k$. We construct u_0 in the form $u_0 = v_0 + h_0$ where $h_0 \in \Sigma$, so the system now takes the form

$$(1) \quad 2\langle v_0 - v_i, h_0 - h_i \rangle_{\mathbb{R}_d^d} = c_i, i = 1..k.$$

We need to find h_0 , and all the other vectors are fixed.

Clearly, this is a linear equation system, which is non-degenerate, because the vectors v_i are affinely independent and $\langle \cdot, \cdot \rangle_{\mathbb{R}_d^d}|_{\Delta \times \Sigma}$ is a non-degenerate pairing, so the required vector h_0 (and, therefore, u_0) exists. \square

2.1. The algorithm. Fix two arbitrary complementary isotropic d -dimensional subspaces Σ, Δ of \mathbb{R}_d^d , and fix an infinite sequence of points $v_0, v_1, \dots \in \Delta$ in d -general position (for example, these points can lie on the Moment Curve). Let $G = (\mathcal{X}, \mathcal{T}, g)$ be a d -degenerate n -dimensional indefinite metric polyhedron with vertex set $\mathcal{V} = \{t_0, t_1, \dots\}$, where each vertex t_i is connected to at most d vertices from the set $\{t_0, \dots, t_{i-1}\}$. We need to find a simplicial isometric map $\tau : \mathcal{X} \rightarrow \mathbb{R}_d^d$ of G into \mathbb{R}_d^d . Since any simplicial map is completely determined by images of vertices, it is enough to define τ on the vertex set \mathcal{V} . First we choose $\tau(t_0)$, then $\tau(t_1)$, and so on.

For every $i \geq 0$, there are at most d points in $\{t_j\}_{j < i}$ connected to t_i by an edge. If this set is empty, then put $\tau(t_i)$ to be any point of \mathbb{R}_d^d such that $P_\Delta(\tau(t_i)) = v_i$. If this set is non-empty, apply Lemma 3 to its points by solving the system (1) and obtain $\tau(t_i)$ such that $P_\Delta(\tau(t_i)) = v_i$.

This completes the first part of Theorem 1. Now the second part of Theorem 1 is a consequence of the following lemma from [5]:

Lemma 4. *Let $(\mathcal{X}, \mathcal{T}, g)$ be an n -dimensional metric polyhedron, let $f : \mathcal{X} \rightarrow \mathbb{R}^N$ be a simplicial map with respect to \mathcal{T} , and let \mathcal{V} be the vertex set of \mathcal{X} . If $f(\mathcal{V})$ is in $(2n + 1)$ -general position then f is an embedding.*

Proof. Indeed, if the images of two non-intersecting simplices intersect, then their vertices cannot be in general position. \square

If the points $P_\Delta(\tau(t_i))$ lie in d -general position then the points $\tau(t_i)$ also lie in d -general position, so if $d \geq 2n + 1$ then this lemma implies the desired result. \square

3. PROOF OF THEOREM 2

Proof. We assume that $\mathcal{V}', \mathcal{V}, d, g, \tau'$ are the same as in the statement of Theorem 2. Since our extension can be performed one point at a time, it suffices to consider the case $\mathcal{V} = \mathcal{V}' \cup \{t_0\}$. Let $H := \{v_1, \dots, v_m\}$ be the set of images of vertices adjacent to t_0 , so $m \leq d$. By the assumption of Theorem 2, v_1, \dots, v_m are affinely independent. Let Σ and Δ be two fixed complementary isotropic d -dimensional subspaces of \mathbb{R}_d^d , for example, we can again put

$$\Sigma := \{(v^+, v^-) | v^+ - v^- = 0\}$$

and

$$\Delta := \{(v^+, v^-) | v^+ + v^- = 0\}.$$

If the points $P_\Delta(v_1), \dots, P_\Delta(v_m)$ are also affinely independent, then we can take almost any $v_0 \in \Delta$ and apply Lemma 3 to H to obtain the point $\tau(t_0)$ with the required properties. Unfortunately, this is not always the case: the points $P_\Delta(v_1), \dots, P_\Delta(v_m)$ can easily be affinely dependent even if the points v_1, \dots, v_m are not.

But instead of having a fixed pair of Σ and Δ , for every new vertex we can choose its own pair of isotropic subspaces depending on the set of neighbors of that vertex. And this additional freedom actually allows us to apply Lemma 3, because of the following observation:

Lemma 5. *Let H be a set of at most $d + 1$ affinely independent points in \mathbb{R}_d^d . Then there exist two complementary isotropic d -dimensional subspaces Σ_H, Δ_H such that the points of $P_{\Delta_H}(H)$ are also affinely independent.*

Proof. The case $|H| < d + 1$ clearly follows from the case $|H| = d + 1$, so we assume $|H| = d + 1$. Without loss of generality we may assume that H contains the origin.

Let U denote the linear subspace of \mathbb{R}_d^d spanned by the points of H . Let $P_+ : \mathbb{R}_d^d \rightarrow \mathbb{R}_d^d$ denote a projection operator on the positive coordinate component. We may assume that $P_+(U)$ is the coordinate subspace spanned by the vectors e_1, \dots, e_k , where the vectors e_1, \dots, e_k are the first k vectors of the standard basis of the positive coordinate component, otherwise standard basis of the positive coordinate component should be substituted by another one which satisfies this condition. It is possible to choose a basis u_1, \dots, u_d of U such that $P_+(u_i) = e_i$ for $i = 1..k$, and $P_+(u_i) = 0$ for $i = k + 1..d$.

Let

$$\Sigma := \{(v^+, v^-) | v^+ - v^- = 0\}$$

$$\Delta := \{(v^+, v^-) | v^+ + v^- = 0\}.$$

We are going to find an isometry f of \mathbb{R}_d^d , such that the subspaces $\Sigma_H := f^{-1}\Sigma$ and $\Delta_H := f^{-1}\Delta$ meet the requirements of the lemma, namely, f has to be such that the points $P_{f^{-1}\Delta}(H)$ are affinely independent, or, equivalently, $P_\Delta(fU) = \Delta$.

Now we explicitly describe the matrix of the desired Lorentz transformation f . Let M^+ and M^- denote positive and negative components of \mathbb{R}_d^d . We are going to find f in the form $f = (f^+, f^-)$, where $f^+ : M^+ \rightarrow M^+$ and $f^- : M^- \rightarrow M^-$ are two linear isometries. We put $f^+ := \text{id}$. Now we are going to find the matrix F^- of f^- . Let S be the $2d \times d$ matrix whose columns are coordinates of u_i , it can be split into two $d \times d$ matrices S^+ and S^- . The subspace $P_\Delta(f(U))$ has dimension d if and only if

$$\det(S^+ - F^- S^-) \neq 0,$$

because for any vector $v = (v^+, v^-)$ one has $P_\Delta(v) = \frac{1}{2}(v^+ - v^-, v^- - v^+)$.

By construction, the matrix S^+ has the first k diagonal entries equal to 1, while all the other entries of S^+ are zeroes. Hence the columns of S^- with numbers $k + 1, \dots, d$ have to be linearly independent, because $\dim U = d$.

Now we need to make the following standard decomposition of the matrix S^- called *QL-decomposition*, see, for example, [3, §5.2]. Namely, if A is a $d \times d$ -matrix, then there exist an orthogonal matrix Q and a lower-triangular matrix L with non-negative diagonal entries, such that $A = QL$. The proof is a slight modification of the Gram–Schmidt orthogonalization process. Note that if the last $d - k$ columns of A were linearly independent, then the last $d - k$ diagonal entries of L become strictly positive, because the columns remain linearly independent after the multiplication by Q from the left.

So, let $-S^- = Q_- L_-$ be such QL-decomposition of $-S_-$. Put $F^- := Q_-^T$, where by T we denote the matrix transpose. Then $S^+ - F^- S^- = S_+ + L_-$. The matrix on the right-hand side is clearly lower-triangular and has strictly positive diagonal entries, so its determinant is also strictly positive, which concludes the proof of the lemma. \square

Now we have the following situation: we are given a set $H := \{u_1, \dots, u_m\}$ of points in \mathbb{R}_d^d which are the images of the vertices t_1, \dots, t_m adjacent to t_0 , so $m \leq d$. Then Lemma 5 provides us with a pair Σ_H, Δ_H of complementary isotropic d -dimensional subspaces of \mathbb{R}_d^d such that the points of $P_{\Delta_H}(H)$ are affinely independent. We want to choose a point v_0 in such a way that the points of $\{v_0\} \cup P_{\Delta_H}(H)$ are affinely independent. If we manage to do that, then we can apply Lemma 3, taking each c_i to be the squared length of the edge connecting t_0 and t_i . Lemma 3 returns a point u_0 such that $P_{\Delta_H}(u_0) = v_0$ and if we put $\tau(t_0) := u_0$, then τ will preserve the squared length of every edge. This almost completes the proof, but there is also a

requirement in the statement of the theorem that the points $\tau(\mathcal{V}' \cup \{t_0\})$ must be in d -general position, so we need to be slightly more careful when we choose v_0 . Namely, if for every d -tuple of vertices $t_1, \dots, t_d \in \mathcal{V}$ the point v_0 does not lie in the P_{Δ_H} -image of the affine span of $\tau(t_1), \dots, \tau(t_d)$, then the points $\tau(\mathcal{V}' \cup \{t_0\})$ are in d -general position. But this forbids v_0 to lie in a union of only a countable number of hyperplanes in Δ_H , therefore such a point v_0 exists. Thus, all one needs to do now is to apply Lemma 3 to the set H and extend τ to the vertex t_0 . \square

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