POSITROID VARIETIES AND CLUSTER ALGEBRAS

PAVEL GALASHIN AND THOMAS LAM

Abstract. We show that the coordinate ring of an open positroid variety coincides with the cluster algebra associated to a Postnikov diagram. This confirms conjectures of Postnikov, Muller–Speyer, and Leclerc, and generalizes results of Scott and Serhiyenko–Sherman-Bennett–Williams.

Positroid varieties are subvarieties of the Grassmannian that first appeared in the study of total positivity and Poisson geometry [Lus98, Pos06, BGY06, KLS13]. In this paper we establish the following result, see Theorem 3.5.

Theorem. *The coordinate ring* $\mathbb{C}[\mathring{\Pi}_{v,w}]$ *of an open positroid variety* $\mathring{\Pi}_{v,w}$ *is a cluster algebra.*

For the top-dimensional open positroid variety, this is due to Scott [Sco06], a result that motivated much of the subsequent work. Combinatorially, positroid varieties are parametrized by Postnikov diagrams, and each such diagram gives rise to a quiver whose vertices are labeled by Plücker coordinates on the Grassmannian, see [Pos06, Sco06]. This data gives rise to a cluster algebra of [FZ02] whose cluster variables are rational functions on the Grassmannian, and since the work of Scott, it has been expected that this cluster algebra coincides with the coordinate ring of $\mathring{\Pi}_{v,w}$. This conjecture was made explicit by Muller and Speyer [MS17, Remark 4.6], and was established recently in the special case of Schubert varieties by Serhiyenko–Sherman-Bennett–Williams [SSBW19]. Another closely related conjecture was given by Leclerc [Lec16], who constructed a cluster subalgebra of $\mathbb{C}[\mathring{\Pi}_{v,w}]$ using representations of preprojective algebras. We show (Corollary 3.7(i)) that these two cluster structures coincide. These cluster structures have also been compared in [SSBW19]; our work differs from theirs by switching from a left-sided to a right-sided quotient for the flag variety, i.e., from $B_-\backslash G$ to G/B_-, see Remark 3.2.

Leclerc's conjectures and results apply in the more general setting of open Richardson varieties. We hope to return to cluster structures of open Richardson varieties in the future [GL]. Some other closely related cluster structures include double Bruhat cells [BFZ05, GY16], partial flag varieties [GLS08], and unipotent groups [GLS07].

Combining our main result with the well-developed machinery of cluster algebras has many consequences for the structure of open positroid varieties, see e.g. the introduction of [SSBW19]. For instance, the existence of a green-to-red sequence [FS18], together with the constructions of [GHKK18] endow $\mathbb{C}[\mathring{\Pi}_{v,w}]$ with a basis of *theta functions* with positive structure constants. Additionally, the results of [LS16] imply that $H^*(\mathring{\Pi}_{v,w}, \mathbb{C})$ satisfies the

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curious Lefschetz property, which has implications for extension groups of certain Verma modules that we aim to explore in future work.

Finally, we show that the totally nonnegative part $\Pi_{v,w}^{>0}$ of $\mathring{\Pi}_{v,w}$ (as defined by [Lus98, Pos06]) is precisely the subset of $\mathring{\Pi}_{v,w}$ where all cluster variables take positive real values, see Corollary 4.4.

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Outline. We discuss the combinatorics of Le-diagrams in Section 1. The cluster algebra $\mathcal{A}(Q_D)$ coming from a Le-diagram D consists of some rational functions on the Grassmannian. As we discuss in Section 3.4, in order to prove our main result, one needs to show two inclusions: $\mathcal{A}(Q_D) \subseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$ and $\mathcal{A}(Q_D) \supseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$. For the first inclusion, we rely on the results of Leclerc [Lec16]. In particular, following ideas of [SSBW19], we show in Section 2 that the cluster algebra of [Lec16] is isomorphic to $\mathcal{A}(Q_D)$ (i.e., they have isomorphic quivers). We then prove the first inclusion $\mathcal{A}(Q_D) \subseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$ in Section 3, see Corollary 3.7(ii). We show the second inclusion $\mathcal{A}(Q_D) \supseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$ in Section 4, using the results of Muller–Speyer [MS17, MS16], of Muller [Mul13], and of Berenstein–Fomin–Zelevinsky [BFZ05].

Throughout the paper, we fix a positive integer n, and an integer $k \in [n] := \{1, 2, \dots, n\}$. For $a, b \in \mathbb{Z}$, we let $[a, b] := \{a, a+1, \dots, b\}$ if $a \leq b$, and $[a, b] := \emptyset$ otherwise.

1. Le-diagram cluster algebra

Let $W = S_n$ be the symmetric group on n letters. For $i \in [n-1]$, let $s_i \in W$ denote the simple transposition of i and $i+1$. Every permutation $w \in W$ can be written as a reduced word $w = s_{i_1} \cdots s_{i_m}$ (where $m = \ell(w)$ is the length of w). In this case, $\mathbf{w} := (i_1, \dots, i_m)$ is called a *reduced expression* for w. For $j \in [n]$ and $w = s_{i_1} \cdots s_{i_m}$, we let $w(j) := s_{i_1}(\dots(s_{i_m}(j))\dots)$, and for $A \subset [n]$, we denote $wA := \{w(a) \mid a \in A\}$.

Let $J = [n] \setminus \{k\}$, and denote by $W^J \subset W$ the set of *k-Grassmannian permutations*, i.e., permutations $w \in W$ satisfying $w(1) < \cdots < w(k)$ and $w(k+1) < \cdots < w(n)$. In other words, we have $w \in W^J$ if and only if $w = 1$ or each reduced word for w ends with s_k.

Let Q^J denote the set of pairs (v, w) where $w \in W^J$ and $v \leq w$ in the Bruhat order on W. The elements of Q^J label positroid varieties, see Section 3.1. By [MR04, Lemma 3.5], every reduced expression $\mathbf{w} = (i_1, \dots, i_m)$ for w contains a unique "rightmost" reduced subexpression $\mathbf{v}$ for v, called the *positive distinguished subexpression.* We let $J^\circ_{\mathbf{v}} \subset [m]$ denote the set of indices *not* used in $\mathbf{v}$.

1.1. Le-diagrams and subexpressions.

We use English notation for Young diagrams and label their boxes in matrix notation. A *Le-diagram* D is a Young diagram λ, contained in a $k \times (n-k)$ rectangle, together with a filling of some of its boxes with dots, satisfying the following condition: if a box b is both below a dot and to the right of a dot, then b must contain a dot.

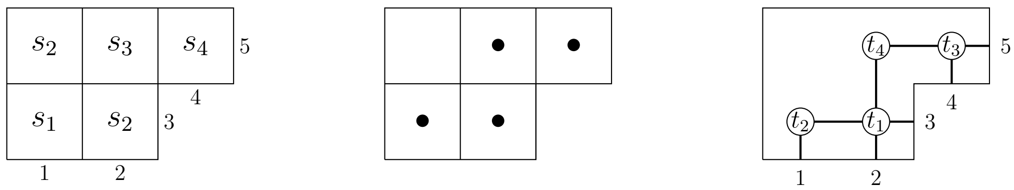


FIGURE 1. The Young diagram λ, Le-diagram D, and graph $G(D)$ corresponding to $(v, w) = (s_2, s_2s_1s_4s_3s_2)$.

We describe a well-known bijection [Pos06, Section 20] between elements of Q^J and Le-diagrams. First, Grassmannian permutations $w \in W^J$ are in bijection with Young diagrams $\lambda \subseteq k \times (n-k)$: placing s_{k+j-i} into the box (i, j) of λ, a reduced word for w is obtained by reading the boxes from right to left along each row, starting from the lowest row. The southeastern boundary edges of λ are labeled $1, 2, \ldots, n$ from bottom-left to top-right. Thus the southern boundary edges are labeled by the elements of $w[k+1, n]$.

Given $v \leq w$, we mark the letters *not* used by the positive distinguished subexpression for v with a dot, and this gives a Le-diagram denoted $D(v, w)$. For example, if $(v, w) = (s_2, s_2s_1s_4s_3s_2)$, we have the Young diagram $\lambda = (3, 2)$ and the Le-diagram $D(v, w)$ in Figure 1(left and middle). Note that $w[k+1, n] = w\{3, 4, 5\} = \{1, 2, 4\}$ are the labels of the southern boundary edges.

Throughout the paper, we assume $(v, w) \in Q^J$ and denote $D := D(v, w)$. We also fix a choice of $\mathbf{w} = (i_1, \ldots, i_m)$, $\mathbf{v}$, and $J^\circ_{\mathbf{v}}$ as above.

1.2. **The graph $G(D)$.** To a Le-diagram D we associate a planar graph $G(D)$ embedded into the disk. The boundary of λ is taken to be the boundary of the disk, and boundary vertices are placed at the east and south boundary steps of λ, labeled $1, 2, \ldots$ in counterclockwise order, starting from the southwest corner of λ. From each dot in D, we draw a hook: one line going eastward, and one line going southward, until they hit the boundary. The interior vertices of $G(D)$ correspond to the dots of D. Each dot of D corresponds to an element $r \in J^\circ_{\mathbf{v}}$, and we label the associated vertex of $G(D)$ by t_r. The edges of $G(D)$ are horizontal or vertical line segments connecting two dots. See Figure 1(right).

1.3. **Quiver.** A *quiver* Q is a directed graph without directed cycles of length 1 or 2. An *ice quiver* is a quiver Q such that each vertex of Q is declared to be either *frozen* or *mutable*. We always assume that an ice quiver contains no arrows between frozen vertices. In this section, we explain how to associate an ice quiver Q_D to a Le-diagram $D = D(v, w)$.

For each $r \in J^\circ_{\mathbf{v}}$, the vertex of $G(D)$ labeled by t_r is the northwestern corner of some face of $G(D)$, and we label this face by F_r. Label the remaining face of $G(D)$ (the one adjacent to the northwestern boundary of λ) by F_0. Thus the neighborhood of any vertex of $G(D)$ looks as in Figure 2.

Construct a quiver Q_D whose vertices are $\{F_r\}_{r \in J^\circ_{\mathbf{v}}}$, i.e., the faces of $G(D)$ excluding F_0. For each $r \in J^\circ_{\mathbf{v}}$, depending on the local structure of $G(D)$ near the vertex labeled t_r (cf. Figure 2), Q_D contains the arrows shown in Figure 3.
The boundary (resp., interior) faces of $G(D)$ are designated as frozen (resp., mutable) vertices of Q_D. We let $\partial J^\circ_{\mathbf{v}} \subset J^\circ_{\mathbf{v}}$ be the set of $r \in J^\circ_{\mathbf{v}}$ such that F_r labels a boundary face of $G(D)$, i.e., is a frozen vertex of Q_D. Some of the arrows in Figure 3 could connect frozen vertices, in which case we omit those arrows from Q_D.

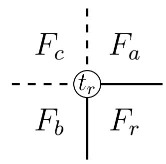


FIGURE 2. The neighborhood of a vertex in $G(D)$. The dashed lines may or may not be present, and F_a, F_b, F_c are the labels of the corresponding faces. Thus some of them may coincide: we may have either $c = a$, or $c = b$, or both.

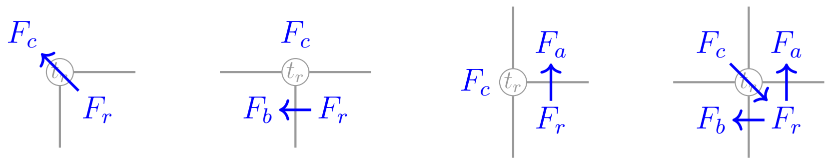


FIGURE 3. Local rules for constructing a quiver from a Le-diagram.

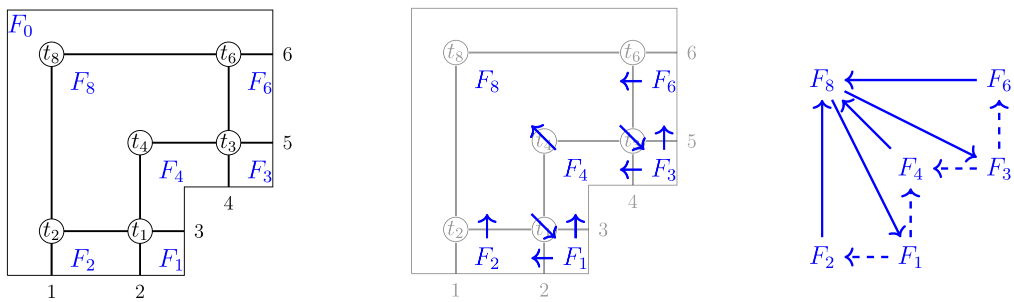


FIGURE 4. Constructing a quiver Q_D from a Le-diagram D. See Example 1.1.

Example 1.1. Let $k = 3$, $n = 6$, and $(v, w) = (s_2 s_4, s_2 s_1 s_4 s_3 s_2 s_5 s_4 s_3)$. The graph $G(D)$ is shown in Figure 4(left), and the quiver Q_D is shown in Figure 4(right). The only mutable vertex of Q_D is F_8, the vertices F_1, F_2, F_3, F_4, F_6 are frozen. Dashed arrows connect frozen vertices and therefore are not present in Q_D.

1.4. **Cluster algebra.** Let Q be an ice quiver with vertex set V partitioned as $V = V_f \sqcup V_m$, where V_f (resp., V_m) denotes the set of frozen (resp., mutable) vertices. For each vertex $r \in V$, introduce a formal variable x_r, and let $\mathbf{x} := \{x_r\}_{r \in V}$. The *cluster algebra* $\mathcal{A}(Q)$ associated to Q is a certain $\mathbb{C}$-subalgebra of the ring $\mathbb{C}(\mathbf{x})$ of rational functions in the variables $\mathbf{x}$. Explicitly, $\mathcal{A}(Q)$ is the subalgebra generated (as a ring) by all *cluster variables* together with $\{x_r^{-1} \mid r \in V_f\}$, where the set of (in general infinitely many) cluster variables is constructed from the data $(Q, \mathbf{x})$ using combinatorial operations called *mutations*. Given a mutable vertex $r \in V_m$, a mutation at r changes the quiver Q in a certain way, and also replaces x_r with

$$x'_r := \frac{\prod_{i \to r} x_i + \prod_{r \to j} x_j}{x_r}, \tag{1.1}$$

where the products are taken over arrows in Q incident to r. We refer the reader to [FZ02] for further background on cluster algebras.

2. Leclerc's cluster algebra

For any $v \leq w$, Leclerc [Lec16] introduced another cluster algebra using representations of preprojective algebras. In this section, we recast his construction in elementary terms when $(v, w) \in Q^J$ and show that in this case, his cluster algebra coincides with the one from Section 1. The calculations in this section are very similar to those in [SSBW19, Section 5], to which we refer the reader for an accessible introduction to preprojective algebra representations in type A.

2.1. **Preprojective algebra representations from Young diagrams.** Let A_{n-1} be the quiver with vertex set $[n-1]$ and a pair of opposite arrows between i and $i+1$ for all $i \in [n-2]$. Recall that a *representation* of A_{n-1} is a collection $E_1, \dots, E_{n-1}$ of vector spaces together with linear maps $\vec{\psi}_i : E_i \to E_{i+1}$, $\overleftarrow{\psi}_{i+1} : E_{i+1} \to E_i$ for all $i \in [n-2]$. A Young diagram λ that fits inside a $k \times (n-k)$ rectangle gives rise to a representation of A_{n-1}: for each box (i, j) of λ, let $\mathrm{c}(i, j) := k + j - i$ (thus (i, j) is labeled by $s_{\mathrm{c}(i,j)}$ in Figure 1). Then for all $c \in [n-1]$, E_c has a basis $\{e_{i,j} \mid (i, j) \in \lambda : \mathrm{c}(i, j) = c\}$. Additionally, for each box (i, j), the values of the maps $\vec{\psi}_{\mathrm{c}(i,j)}$ and $\overleftarrow{\psi}_{\mathrm{c}(i,j)}$ on $e_{i,j}$ are given by

$$\vec{\psi}_{\mathrm{c}(i,j)}(e_{i,j}) = \begin{cases} e_{i,j+1}, & \text{if } (i, j+1) \in \lambda, \\ 0, & \text{otherwise;} \end{cases} \qquad \overleftarrow{\psi}_{\mathrm{c}(i,j)}(e_{i,j}) = \begin{cases} e_{i+1,j}, & \text{if } (i+1, j) \in \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Leclerc works not just with representations of A_{n-1}, but with representations of the associated *preprojective algebra* Λ. It is easy to see that each Young diagram λ contained inside a $k \times (n-k)$ rectangle yields a representation U_λ of Λ.

2.2. **Leclerc's representations.** Recall that we have fixed $(v, w) \in Q^J$. Leclerc associates a representation U_r of Λ to each $r \in J^\circ_{\mathbf{v}}$. Our goal is to define a family $\{\nu_{(r)}\}_{r \in J^\circ_{\mathbf{v}}}$ of Young diagrams such that each of them fits inside a $k \times (n-k)$ rectangle, and such that $U_r = U_{\nu_{(r)}}$ for all $r \in J^\circ_{\mathbf{v}}$.

For $r \in [m]$, we set

$$w_{(r)} := s_{i_1} \cdots s_{i_r}, \qquad v_{(r)} := s^{\mathbf{v}}_{i_1} \cdots s^{\mathbf{v}}_{i_r}, \quad \text{where} \quad s^{\mathbf{v}}_{i_r} := \begin{cases} s_{i_r}, & \text{if } r \notin J^\circ_{\mathbf{v}}, \\ 1, & \text{if } r \in J^\circ_{\mathbf{v}}; \end{cases}$$

$$w^{(r)} := w^{-1} \cdot w_{(r)} = s_{i_m} \cdots s_{i_{r+1}}, \qquad \text{and} \qquad v^{(r)} := v^{-1} \cdot v_{(r)} = s^{\mathbf{v}}_{i_m} \cdots s^{\mathbf{v}}_{i_{r+1}}.$$

For $a \in [n-1]$, let $\omega_a := \{1, 2, \dots, a\}$. For $u \in W$ and $a \in [n-1]$, the subset $u\omega_a$ can be identified with a Young diagram $\mu(u, a)$ fitting inside an $(n-a) \times a$ rectangle, such that if one places s_{a+i-j} inside each box $(i, j) \in \mu(u, a)$ and takes the product as in Section 1.1, the resulting element $\bar{u}$ satisfies $\bar{u}\omega_a = u\omega_a$. That is, $\bar{u}$ is the unique element of W^{J_a} satisfying $\bar{u}\omega_a = u\omega_a$, where $J_a = [n-1] \setminus \{a\}$, and $\mu(u, a)$ is the Young diagram associated to $\bar{u}$. Clearly, if $u' \leq u$, then $\mu(u', a) \subseteq \mu(u, a)$.

Since $w \in W^J$, we see that $(w^{(r-1)})^{-1} \in W^J$ so $\mu(w^{(r-1)}, i_r)$ is a rectangle for any r whose top left (resp., bottom right) box is labeled by s_a (resp., by s_k). Thus the skew shape $\mu(w^{(r-1)}, i_r)/\mu(v^{(r-1)}, i_r)$ is obtained from another Young diagram $\nu_{(r)}$ by a $180°$ rotation. We emphasize that $\nu_{(r)}$ is defined for all $r \in [m]$.

Example 2.1. Let $k = 6$, $n = 12$. Consider a Le-diagram in Figure 5(left). We have $J^\circ_{\mathbf{v}} = \{a, b, c, d\}$, and the diagrams $\nu_{(r)}$ for $r \in J^\circ_{\mathbf{v}}$ are shown in Figure 5(right). For instance, $i_b = 7$, $w^{(b-1)}\omega_{i_b} = s_6 s_7 s_8 s_9 s_5 s_6 s_7 s_8 s_4 s_5 s_6 s_7 \omega_7$, and thus $\mu(w^{(b-1)}, i_b)$ is a 3×4 rectangle.

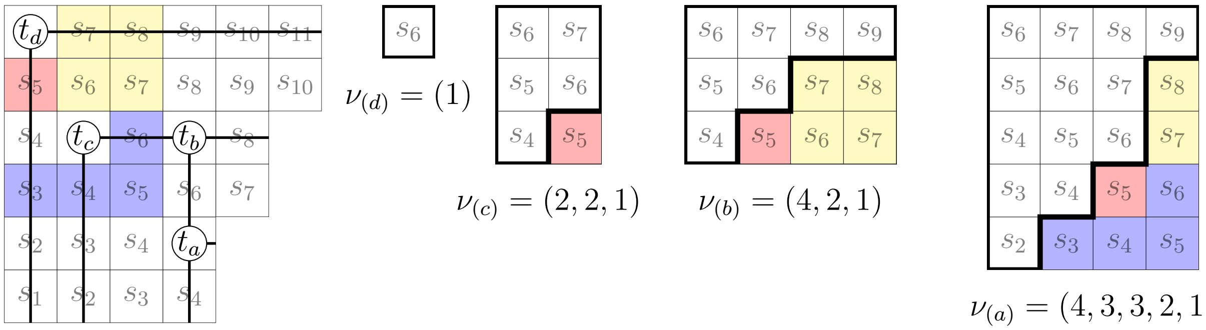


FIGURE 5. Constructing Young diagrams $\nu_{(r)}$ from a Le-diagram (cf. Example 2.1).

Similarly, $v^{(b-1)} = s_7s_8s_9s_5s_6s_7s_8s_4s_6\omega_7 = s_7s_8s_5s_6s_7\omega_7$. The shape $\mu(v^{(b-1)}, i_b) = (3, 2)$, rotated by 180°, consists of red and yellow squares (see Figure 5) labeled by s_7, s_8, s_5, s_6, s_7, and is the complement of $\nu_{(b)}$ inside a 3×4 rectangle. Thus $\nu_{(b)} = (4, 2, 1)$.

The next result follows from the definitions, see [Lec16, Proposition 4.3] and [SSBW19, Section 5].

Proposition 2.2. *For $r \in J^\circ_{\mathbf{v}}$, Leclerc's representation U_r of Λ coincides with the representation $U_{\nu_{(r)}}$ constructed from $\nu_{(r)}$ in Section 2.1.* □

We give an alternative description of the Young diagram $\nu_{(r)}$ using the combinatorics of Le-diagrams.

For integers $a, b \geq 1$, denote by $\mathrm{H}(a, b) = (a, 1^{b-1})$ the hook Young diagram whose first row contains a boxes and whose first column contains b boxes. We consider "Frobenius coordinates" for Young diagrams: we write $\mu = [(a_1, b_1), \dots, (a_d, b_d)]$ if $\mu = \{(l + i - 1, l + j - 1) \mid l \in [d], i \in [a_l], j \in [b_l]\}$. Thus the first row of μ has a_1 boxes, the first column of μ has b_1 boxes, etc.

For each box $(i, j) \in \lambda$ (not necessarily containing a dot), let $\mathrm{NW}(i, j) \in \lambda \sqcup \{(0, 0)\}$ be the box closest to (i, j) in the strictly northwest direction that is either $(0, 0)$ or contains a dot. Recall from Section 1.1 that the boxes of λ correspond to the terms in the reduced word $w = s_{i_1} \cdots s_{i_m}$ for w, i.e., to the elements of $[m]$. For $r \in [m]$, we denote by $(i^{(r)}, j^{(r)}) \in \lambda$ the corresponding box of λ (thus $i_r = k + j^{(r)} - i^{(r)}$).

Proposition 2.3. *For $r \in J^\circ_{\mathbf{v}}$, the Frobenius coordinates of $\nu_{(r)}$ are given by*

$$\nu_{(r)} = \left[(i^{(r)}, j^{(r)}), \mathrm{NW}(i^{(r)}, j^{(r)}), \dots, \mathrm{NW}^p(i^{(r)}, j^{(r)})\right],$$

where $p \geq 0$ is such that $\mathrm{NW}^{p+1}(i^{(r)}, j^{(r)}) = (0, 0)$.

Proof. Recall that $\nu_{(r)}$ is defined for all $r \in [m]$, not just for $r \in J^\circ_{\mathbf{v}}$. We will also show that for $r \notin J^\circ_{\mathbf{v}}$, we have $\nu_{(r)} = \left[\mathrm{NW}(i^{(r)}, j^{(r)}), \dots, \mathrm{NW}^p(i^{(r)}, j^{(r)})\right]$, where $p \geq 0$ satisfies $\mathrm{NW}^{p+1}(i^{(r)}, j^{(r)}) = (0, 0)$. If $i^{(r)} = 1$ or $j^{(r)} = 1$ then the result is clear. Otherwise, assume that we have shown the result for the box $(i^{(r)} - 1, j^{(r)} - 1)$, and let $r' \in [m]$ be such that $(i^{(r')}, j^{(r')}) = (i^{(r)} - 1, j^{(r)} - 1)$. It is easy to see that $\mu(w^{(r-1)}, i_r)$ is an $i^{(r)} \times j^{(r)}$ rectangle while $\mu(w^{(r'-1)}, i_{r'})$ is an $(i^{(r)} - 1) \times (j^{(r)} - 1)$ rectangle. If $r \notin J^\circ_{\mathbf{v}}$ then clearly $\mu(v^{(r-1)}, i_r)$ is a disjoint union of $\mathrm{H}(i^{(r)}, j^{(r)})$ and $\mu(v^{(r'-1)}, i_{r'})$, so $\nu_{(r)} = \nu_{(r')}$. If $r \in J^\circ_{\mathbf{v}}$ then $\mu(v^{(r-1)}, i_r) = \mu(v^{(r'-1)}, i_{r'})$, so $\nu_{(r)}$ is a disjoint union of $\mathrm{H}(i^{(r)}, j^{(r)})$ and $\nu_{(r')}$. □

Example 2.4. Continuing Example 2.1, the Frobenius coordinates of $\nu_{(r)}$ for $r \in J^\circ_{\mathbf{v}}$ are given as follows:

$$\nu_{(d)} = [(1,1)], \quad \nu_{(c)} = [(2,3),(1,1)], \quad \nu_{(b)} = [(4,3),(1,1)], \quad \nu_{(a)} = [(4,5),(2,3),(1,1)].$$

We see that $\nu_{(a)}$ is a union of $\mathrm{H}(i^{(a)}, j^{(a)}) = \mathrm{H}(4,5)$ and $\nu_{(c)}$, which is a union of $\mathrm{H}(i^{(c)}, j^{(c)}) = \mathrm{H}(2,3)$ and $\nu_{(d)} = \mathrm{H}(1,1)$. Similarly, $\nu_{(b)}$ is a union of $\mathrm{H}(i^{(b)}, j^{(b)}) = \mathrm{H}(4,3)$ and $\nu_{(d)}$.

2.3. **Leclerc's quiver.** Now that we have constructed Young diagrams $\nu_{(r)}$ for $r \in J^\circ_{\mathbf{v}}$, we can analyze the quiver $\tilde{Q}$ that Leclerc associates to $(v,w) \in Q^J$. The vertex set[1] of $\tilde{Q}$ is just $J^\circ_{\mathbf{v}}$. The frozen vertices of $\tilde{Q}$ correspond to the Young diagrams obtained from $\mu(w^{-1}, i_a)/\mu(v^{-1}, i_a)$ (for $a \in [n-1]$) by a 180° rotation. It is easy to see that these are precisely the Young diagrams $\nu_{(r)}$ such that F_r is a boundary face of $G(D)$ that contains the part of the boundary of λ between boundary vertices a and $a+1$. Thus the map $r \mapsto F_r$ sends the vertices of $\tilde{Q}$ bijectively to the vertices of the quiver Q_D from Section 1.3, preserving the partition into frozen and mutable vertices.

The arrows of $\tilde{Q}$ can be described in terms of morphisms of Young diagrams. Given a skew shape λ/μ for $\mu \subset \lambda$, we say that their set-theoretic difference λ/μ is an *order ideal of* λ. For a Young diagram λ and an integer $p \geq 0$, we denote $\mathrm{shift}^p(\lambda) := \{(i+p, j+p) \mid (i,j) \in \lambda\}$. For another Young diagram μ, we write $\lambda \xrightarrow{p} \mu$ if the set $\mathrm{shift}^p(\lambda) \cap \mu$ is an order ideal of μ. In this case, we say that $\lambda \xrightarrow{p} \mu$ is a *morphism from* λ *to* μ. The morphism $\lambda \xrightarrow{0} \lambda$ is considered trivial, and a morphism $\lambda \xrightarrow{p} \mu$ is the zero morphism if $\mathrm{shift}^p(\lambda) \cap \mu = \emptyset$. Morphisms can be composed: if $\lambda \xrightarrow{p} \mu$ and $\mu \xrightarrow{q} \nu$ then $\lambda \xrightarrow{p+q} \nu$. For $r, r' \in J^\circ_{\mathbf{v}}$, a nonzero and nontrivial morphism $\nu_{(r)} \xrightarrow{p} \nu_{(r')}$ is *irreducible* if it is not a composition of non-trivial morphisms $\nu_{(r)} \xrightarrow{p''} \nu_{(r'')} \xrightarrow{p'} \nu_{(r')}$ for some $r'' \in J^\circ_{\mathbf{v}}$. Leclerc's quiver $\tilde{Q}$ contains an arrow $r \to r'$ for $r \neq r' \in J^\circ_{\mathbf{v}}$ if and only if at least one of r, r' is mutable and there is an irreducible morphism $\nu_{(r)} \xrightarrow{p} \nu_{(r')}$ for some $p \geq 0$.

Remark 2.5. We explain the relation between our morphisms of Young diagrams and the morphisms of representations of Λ from the original definition [Lec16] of $\tilde{Q}$. Let U_λ and U_μ be the two (indecomposable) Λ-modules associated to Young diagrams λ and μ as in Section 2.1. Since U_λ is generated (as a Λ-module) by the vector $e_{1,1}$, a morphism $f: U_\lambda \to U_\mu$ is uniquely determined by $f(e_{1,1}) \in U_\mu$, which must be a linear combination of $e_{1,1}, e_{2,2}, \dots, e_{d,d} \in U_\mu$, where d is the length of μ in Frobenius coordinates.

Associated to each morphism $\lambda \xrightarrow{p} \mu$ of Young diagrams is the *elementary morphism* $U_\lambda \xrightarrow{p} U_\mu$ of Λ-modules sending $e_{1,1}$ to $e_{p,p}$. (The condition that $\mathrm{shift}^p(\lambda) \cap \mu$ is an order ideal of μ corresponds exactly to the condition that $e_{1,1} \mapsto e_{p,p}$ defines a morphism of Λ-modules, see [SSBW19, Remark 5.15].) Any morphism $f: U_\lambda \to U_\mu$ is thus a linear combination of the elementary morphisms $U_\lambda \xrightarrow{p} U_\mu$ of Young diagrams.

A morphism $f: U_{\nu_{(r)}} \to U_{\nu_{(r')}}$ is *irreducible* if it is nonzero, not an isomorphism, and cannot be factored nontrivially within the category $\mathrm{add}(U)$ whose objects are isomorphic to direct sums of the $U_{\nu_{(s)}}$ for $s \in J^\circ_{\mathbf{v}}$. Leclerc's quiver $\tilde{Q}$ has no loops, i.e., arrows from r to r, see [Lec16, Definition 3.9(d)]. For $r \neq r'$, the number of arrows in $\tilde{Q}$ from r

[1] More precisely, the vertex set of $\tilde{Q}$ consists of irreducible factors of functions $\{f_r\}_{r \in J^\circ_{\mathbf{v}}}$ defined in (3.2). However, when $(v,w) \in Q^J$, each f_r is irreducible by Corollary 3.4 below, thus we may label the vertices of $\tilde{Q}$ by elements of $J^\circ_{\mathbf{v}}$.

to r' is equal to the *dimension of the space of irreducible morphisms* from $U_{\nu_{(r)}}$ to $U_{\nu_{(r')}}$. This dimension is defined [BIRS09, Sch14] to be equal to the dimension of the space of all morphisms $U_{\nu_{(r)}} \to U_{\nu_{(r')}}$ modulo the subspace consisting of reducible morphisms. If $\lambda \xrightarrow{p} \mu$ is a morphism, then (see the proof of Proposition 2.6) $\lambda \xrightarrow{q} \mu$ is a reducible morphism for all $q > p$. Thus any morphism $f : U_{\nu_{(r)}} \to U_{\nu_{(r')}}$ is equal modulo reducible morphisms to a scalar multiple of some elementary morphism $U_\lambda \xrightarrow{p} U_\mu$. It follows that the dimension of the space of irreducible morphisms from $U_{\nu_{(r)}}$ to $U_{\nu_{(r')}}$ is equal to 1 or 0, depending on whether there is an elementary morphism $U_{\nu_{(r)}} \xrightarrow{p} U_{\nu_{(r')}}$ that is irreducible as a morphism of Λ-modules. Furthermore, when considering the irreducibility of an elementary morphism $U_{\nu_{(r)}} \xrightarrow{p} U_{\nu_{(r')}}$, we only need to check if it factors nontrivially as a product of elementary morphisms. Thus the irreducibility of $U_{\nu_{(r)}} \xrightarrow{p} U_{\nu_{(r')}}$ in the sense of Λ-modules agrees with the notion of irreducibility we defined for a morphism $\nu_{(r)} \xrightarrow{p} \nu_{(r')}$ of Young diagrams.

Proposition 2.6. *The map $r \mapsto F_r$ gives a quiver isomorphism between Leclerc's quiver $\tilde{Q}$ and the Le-diagram quiver Q_D.*

Proof. If the Frobenius coordinates of a Young diagram λ are given by $\lambda = [(a_1, b_1), \dots, (a_d, b_d)]$, we set $\lambda^1 := (a_1, b_1), \dots, \lambda^d := (a_d, b_d)$, and $\lambda^{d+1} = \lambda^{d+2} = \cdots = (0, 0)$. Let us write $(a, b) \geq (a', b')$ if $a \geq a'$ and $b \geq b'$.

Let $\lambda \xrightarrow{p} \mu$ be a morphism. Since its image is an order ideal of μ, a morphism $\lambda \xrightarrow{p} \mu$ exists if and only if $\lambda^1 \geq \mu^{p+1}$, $\lambda^2 \geq \mu^{p+2}$, etc. Moreover, if $p < q$ and we have a morphism $\lambda \xrightarrow{p} \mu$ then the morphism $\lambda \xrightarrow{q} \mu$ is not irreducible: it factors through $\lambda \xrightarrow{p} \mu \xrightarrow{q-p} \mu$. Also note that $\lambda \xrightarrow{0} \mu$ if and only if $\mu \subset \lambda$. We write $\lambda \xhookrightarrow{p} \mu$ if the morphism $\lambda \xrightarrow{p} \mu$ exists and is *injective* (i.e., $\mathrm{shift}^p(\lambda) \subset \mu$ is an order ideal of μ). This is equivalent to $\lambda^1 = \mu^{p+1}$, $\lambda^2 = \mu^{p+2}$, etc.

By Proposition 2.3, for each $r \in J^\circ_{\mathbf{v}}$, we have $\nu^1_{(r)} = (i^{(r)}, j^{(r)})$. Thus all Young diagrams $\{\nu_{(r)}\}_{r \in J^\circ_{\mathbf{v}}}$ are different. Observe that the morphism $\lambda \xhookrightarrow{0} \mu$ exists if and only if $\lambda = \mu$, in which case it is a trivial morphism.

Let $r \in J^\circ_{\mathbf{v}}$ and suppose that the neighboring faces F_a, F_b, F_c, F_r of t_r in $G(D)$ are labeled as in Figure 2. It follows from Proposition 2.3 that we have morphisms $\nu_{(r)} \xrightarrow{0} \nu_{(a)}$, $\nu_{(r)} \xrightarrow{0} \nu_{(b)}$, $\nu_{(c)} \xhookrightarrow{1} \nu_{(r)}$.

We first show that for $r, r' \in J^\circ_{\mathbf{v}}$, if there is no arrow $F_r \to F_{r'}$ in Q_D then there is no arrow $r \to r'$ in $\tilde{Q}$. Indeed, let $\nu := \nu_{(r)}$ and $\nu' := \nu_{(r')}$, and suppose that there is no arrow $F_r \to F_{r'}$ in Q_D but we have a morphism $\nu \xrightarrow{p} \nu'$ for some $p \geq 0$. Assume that the regions around $t_{r'}$ in $G(D)$ are labeled by $F_{a'}, F_{b'}, F_{c'}, F_{r'}$ as in Figure 2. If $p \geq 1$, then $\nu \xrightarrow{p} \nu'$ factors through $\nu \xrightarrow{p-1} \nu_{(c')} \xrightarrow{1} \nu'$. Therefore such a morphism is not irreducible unless $p = 1$ and the morphism $\nu \xrightarrow{p-1} \nu_{(c')}$ is trivial, i.e., $r = c'$. Since there is no arrow $F_r \to F_{r'}$ in Q_D, we must have either $c' = a'$, or $c' = b'$, or both. Without loss of generality, assume that $c' = a' (= r)$. If both r and r' are frozen then $\tilde{Q}$ contains no arrow between them. Thus at least one of them must be mutable, so the horizontal edge of $G(D)$ between r and r' must have another vertex to the right of $t_{r'}$. Let that vertex be labeled by t_q, then we have morphisms $\nu \xrightarrow{1} \nu_{(q)} \xrightarrow{0} \nu'$, thus we see that indeed our morphism $\nu \xrightarrow{p} \nu'$ is not irreducible when $p \geq 1$.

Assume now that $p = 0$, which implies that $(i^{(r)}, j^{(r)}) \geq (i^{(r')}, j^{(r')})$ and $(i^{(r)}, j^{(r)}) \neq (i^{(r')}, j^{(r')})$. From the definition of the Le-diagram quiver Q_D, we see that there exists a path from F_r to $F_{r'}$ that consists of arrows all going up, left, or up-left. The composition of the corresponding morphisms gives the morphism $\nu \xrightarrow{0} \nu'$, which shows that it is not irreducible if there is no arrow $F_r \to F_{r'}$ in Q_D.

It remains to show that if we have an arrow $F_r \to F_{r'}$ in Q_D then the morphism $\nu \to \nu'$ is irreducible. Let the regions around t_r (resp., $t_{r'}$) be labeled by F_a, F_b, F_c, F_r (resp., $F_{a'}, F_{b'}, F_{c'}, F_{r'}$) as above. First, suppose that $r = c'$, in which case our morphism is $\nu \overset{1}{\hookrightarrow} \nu'$. If it factors as $\nu \xrightarrow{p} \nu'' \xrightarrow{q} \nu'$ then the injectivity of $\nu \overset{1}{\hookrightarrow} \nu'$ forces $\nu \xrightarrow{p} \nu''$ to be injective, and thus we must have $\nu \overset{1}{\hookrightarrow} \nu'' \xrightarrow{0} \nu'$. But then ν is obtained from ν'' by removing a hook, so by Proposition 2.3, $\nu \overset{1}{\hookrightarrow} \nu''$ must be one of the down-right arrows in Q_D. In this case, the hooks $(\nu'')^1$ and $(\nu')^1$ have to be incomparable, contradicting $\nu'' \xrightarrow{0} \nu'$. Next, suppose that r' equals to a or b. Then we have a morphism $\nu \xrightarrow{0} \nu'$. It is clear from the definition of Q_D that if $F_r \to F_{r'}$ is an arrow of Q_D then Q_D contains no directed path from F_r to $F_{r'}$ of length more than 1. If the morphism $\nu \xrightarrow{0} \nu'$ is not irreducible then there must be such a directed path from r to r' in $\tilde{Q}$. But we have already shown that each arrow of $\tilde{Q}$ appears as an arrow in Q_D, thus $\nu \xrightarrow{0} \nu'$ must be irreducible. □

3. Clusters and positroid varieties

Proposition 2.6 shows that the two (abstract) cluster algebras $\mathcal{A}(Q_D)$ and $\mathcal{A}(\tilde{Q})$ are isomorphic. In this section, we further connect them by showing that the conjectural cluster structures they define on $\mathbb{C}[\mathring{\Pi}_{v,w}]$ coincide.

3.1. **Background on positroid varieties.** Let $G = \mathrm{SL}_n(\mathbb{C})$ and B, B_-, N, N_- denote the upper- and lower-triangular Borel subgroups, and their unipotent parts. For $i \in [n-1]$, denote by $\dot{s}_i \in G$ a (signed) permutation matrix representing $s_i \in W$ that has a 2×2 block equal to $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in rows and columns $i, i+1$. Given a reduced word $w = s_{i_1} \cdots s_{i_m}$ for $w \in W$, we let $\dot{w} \in G$ denote the (signed) permutation matrix given by $\dot{w} := \dot{s}_{i_1} \cdots \dot{s}_{i_m}$. For $v \leq w$, we define the *open Richardson variety* $\mathring{R}_{v,w}$ to be the image of $B\dot{v}B_- \cap B_-\dot{w}B_-$ in G/B_-. Thus $\mathring{R}_{v,w}$ is a smooth affine subvariety of G/B_-.

Recall that $J = [n] \setminus \{k\}$. Let $P_-^J \supset B_-$ denote the J-parabolic subgroup such that the projection $\pi_J : G \to G/P_-^J \simeq \mathrm{Gr}(n-k, n)$ is given by sending a $n \times n$ matrix g to the column span of its last $n-k$ columns. We sometimes denote $\pi_J(g)$ by gP_-^J. For an $(n-k)$-element subset I of $[n]$, we denote by Δ_I the corresponding *Plücker coordinate* on $\mathrm{Gr}(n-k, n)$, i.e., the maximal $(n-k) \times (n-k)$ minor of g with row set I and column set $[k+1, n]$.

For $(v, w) \in Q^J$, the *open positroid variety* is the image $\mathring{\Pi}_{v,w} := \pi_J(\mathring{R}_{v,w}) \subset \mathrm{Gr}(n-k, n)$, see [Lus98, KLS13]. It is isomorphic to $\mathring{R}_{v,w}$, and is a smooth affine subvariety of $\mathrm{Gr}(n-k, n)$. For other descriptions of $\mathring{\Pi}_{v,w}$, see [Pos06, BGY06].

3.2. **Leclerc's functions.** Fix $v \leq w$. Following [Lec16, Section 2], denote $N'(v) = N \cap (\dot{v}^{-1}N\dot{v}) \subset N$ and $N_{v,w} := N'(v) \cap \dot{v}^{-1}B_-\dot{w}B_-$. We will be interested in the variety

$$\dot{v}N_{v,w} = \dot{v}N \cap N\dot{v} \cap B_-\dot{w}B_-. \tag{3.1}$$

Lemma 3.1 ([BGY06, Theorem 2.3], [Lec16, Lemma 2.2])**.** *The map $\dot{v}N_{v,w} \to G/B_-$ given by $g \mapsto gB_-$ gives an isomorphism $\dot{v}N_{v,w} \xrightarrow{\sim} \mathring{R}_{v,w}$.*

Remark 3.2. Both [Lec16] and [SSBW19] work with the left-sided flag variety $B_-\backslash G$. In particular, Leclerc shows that the map $g \mapsto B_-g$ gives an isomorphism $\dot{v}N_{v,w} \xrightarrow{\sim} B_-\backslash(B_-\dot{v}B \cap B_-\dot{w}B_-)$. Lemma 3.1 follows from this statement by replacing g, v, w with their inverses. However, since in either case one works with *rightmost* positive distinguished subexpressions, switching from $B_-\backslash G$ to G/B_- has a drastic effect on the combinatorics of Leclerc's quivers, as one can see by comparing Section 2 with [Lec16, Section 7] or [SSBW19]. In fact, when working with $B_-\backslash G$, Leclerc's cluster structure does not in general coincide with (either source or target labeled versions of) the cluster structure coming from Postnikov diagrams, see [SSBW19, Appendix B].

For $u \in W$ and $a \in [n-1]$, let ω_a and $\mu(u,a)$ be as in Section 2.2. Leclerc [Lec16] considers a family of functions on the unipotent group N: for each $r \in J^\circ_{\mathbf{v}}$, the corresponding function is

$$f_r := \Delta_{v^{(r-1)}\omega_{i_r}, w^{(r-1)}\omega_{i_r}} : N \to \mathbb{C} \tag{3.2}$$

where $\Delta_{A,B}$ is the minor whose rows and columns are indexed by A and B respectively. The functions f_r restrict to functions on $N_{v,w}$. Leclerc proves that the irreducible (as elements of $\mathbb{C}[N]$) factors of $\{f_r \mid r \in J^\circ_{\mathbf{v}}\}$ form the initial cluster variables of a cluster subalgebra of $\mathbb{C}[N_{v,w}]$.

Lemma 3.3. *For $u' \le u \in W$, if the skew shape $\mu(u,a)/\mu(u',a)$ is connected then the minor $\Delta_{u'\omega_a, u\omega_a}$ is an irreducible element of $\mathbb{C}[N]$.*

Proof. The polynomial $\Delta_{u'\omega_a, u\omega_a}(g)$ is homogeneous with $\deg(g_{i,j}) = j - i$. Restricting to the subspace of N consisting of matrices constant along diagonals, we see that the result is implied by the Jacobi-Trudi formula combined with the irreducibility [BRvW09, Theorem 1] of skew Schur functions indexed by connected skew shapes. □

Suppose now that $(v, w) \in Q^J$. As we have established in Section 2.2, for all $r \in J^\circ_{\mathbf{v}}$, the skew shape $\mu(w^{(r-1)}, i_r)/\mu(v^{(r-1)}, i_r)$ is a 180° rotation of a Young diagram $\nu_{(r)}$, thus we have shown the following.

Corollary 3.4. *For $(v, w) \in Q^J$ and all $r \in J^\circ_{\mathbf{v}}$, the function f_r defined in* (3.2) *is an irreducible element of $\mathbb{C}[N]$.*

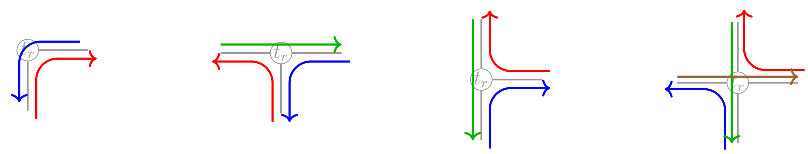
3.3. **Face labels.** So far the faces of $G(D)$ have been labeled by an abstract set $\{F_r\}_{r \in J^\circ_{\mathbf{v}} \sqcup \{0\}}$. We now identify each face F_r with an $(n-k)$-element subset of $[n]$, so that it would correspond to a Plücker coordinate on $\mathrm{Gr}(n-k, n)$.

The graph $G(D)$ has n distinguished paths $p_1, p_2, \ldots, p_n$ connecting boundary vertices, called *strands.* For $a \in [n]$, the strand p_a starts[2] at the boundary vertex labeled a, and then

[2]This is called the *source-labeling* of strands. For the other convention, called *target-labeling*, the path p_a *ends* at vertex a.

travels along the edges of $G(D)$, making turns at each vertex t_r according to the following "rules of the road" (cf. [Pos06, Figure 20.2]):

(3.3)



In other words, the strand p_a zig-zags in the northwest direction until it hits the north or west boundary, after which it goes straight southward or straight eastward until it arrives at the boundary again. If there is no edge of $G(D)$ incident to the boundary vertex a then p_a is taken to be a small clockwise or counterclockwise loop depending on whether a is on a vertical or horizontal edge of λ. Every face F_r of $G(D)$ is labeled by an $(n-k)$-element subset of $[n]$, consisting of those a such that F_r lies to the right of p_a. See Figure 6 for the labeling of the Le-diagram from Example 1.1. From now on, we identify F_r with the corresponding subset, and write Δ_{F_r} for the corresponding Plücker coordinate on $\mathrm{Gr}(n-k,n)$.

It is known that F_0 coincides with the lexicographically maximal $(n-k)$-element subset $S \subset [n]$ such that Δ_S is not identically zero on $\mathring{\Pi}_{v,w}$. Moreover, we have $\Delta_{F_r}(x) \neq 0$ for any $x \in \mathring{\Pi}_{v,w}$ and any $r \in \partial J^\circ_{\mathbf{v}} \sqcup \{0\}$. Since the image of the Plücker embedding lies in the projective space, we always assume that the Plücker coordinates are rescaled ("gauge fixed") so that $\Delta_{F_0}(x) = 1$ for all $x \in \mathring{\Pi}_{v,w}$.

3.4. **Main result.** Recall from Section 1.4 that the cluster algebra $\mathcal{A}(Q_D)$ is a subring of the field of rational functions in the variables $\{x_{F_r}\}_{r \in J^\circ_{\mathbf{v}}}$. The following result is explicitly conjectured in [MS17, Remark 4.6]; the statement may be considered implicitly conjectured in [Pos06, Sco06].

Theorem 3.5. *For all $(v,w) \in Q^J$, the map sending $x_{F_r} \mapsto \Delta_{F_r}$ for each $r \in J^\circ_{\mathbf{v}}$ induces a ring isomorphism $\eta : \mathcal{A}(Q_D) \xrightarrow{\sim} \mathbb{C}[\mathring{\Pi}_{v,w}]$ (with $\Delta_{F_0} = 1$ on $\mathring{\Pi}_{v,w}$).*

When $v = 1$ and w is the maximal element of W^J (i.e., when $D(v,w)$ is a $k \times (n-k)$ rectangle filled with dots), $\mathring{\Pi}_{v,w}$ is the top-dimensional positroid variety in $\mathrm{Gr}(n-k,n)$, in which case Theorem 3.5 was shown by Scott [Sco06].

Recall that $\partial J^\circ_{\mathbf{v}} \subset J^\circ_{\mathbf{v}}$ is the set of $r \in J^\circ_{\mathbf{v}}$ such that F_r labels a boundary face of $G(D)$. Theorem 3.5 is equivalent to the following two explicit statements for Plücker coordinates on $\mathring{\Pi}_{v,w}$:

(1) We have $\eta(\mathcal{A}(Q_D)) \subseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$, that is, the image $\eta(x)$ of every cluster variable $x \in \mathcal{A}(Q_D)$ is a regular function on $\mathring{\Pi}_{v,w}$. Equivalently, $\eta(x)$ can be written as a polynomial in the Plücker coordinates divided by a monomial in $\{\Delta_{F_r}\}_{r \in \partial J^\circ_{\mathbf{v}}}$.

(2) We have $\eta(\mathcal{A}(Q_D)) \supseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$, that is, the images of cluster variables generate $\mathbb{C}[\mathring{\Pi}_{v,w}]$ *as a ring.*

In general, both of these statements are non-obvious. We will deduce (1) from Leclerc's results in the next subsection. The non-trivial part of (2) is that unlike in the case of the top-dimensional positroid variety [Sco06], not every Plücker coordinate is the image of a cluster variable. But Theorem 3.5 implies that every Plücker coordinate can be written as a polynomial in the images of cluster variables divided by a monomial in $\{\Delta_{F_r}\}_{r \in \partial J^\circ_{\mathbf{v}}}$. We will prove this in Section 4.

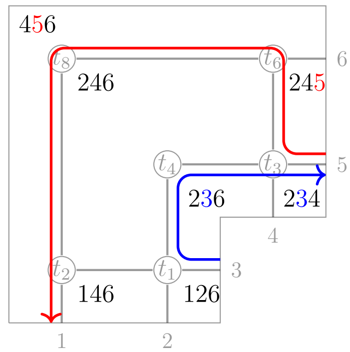


FIGURE 6. Labeling the faces of a Le-diagram by subsets. The strand p_5 is shown in red and p_3 is shown in blue. Here we abbreviate $\{a, b, c\}$ as abc.

3.5. **Converting Leclerc's functions into Plücker coordinates.** Let $v \cdot f_r \in \mathbb{C}[\dot{v}N_{v,w}]$ denote the image of f_r under the isomorphism $\mathbb{C}[N_{v,w}] \simeq \mathbb{C}[\dot{v}N_{v,w}]$. Explicitly, we have $v \cdot f_r := \Delta_{v_{(r)}\omega_{i_r}, w^{(r-1)}\omega_{i_r}}$. Recall that the map $g \mapsto gB_-$ gives an isomorphism $\dot{v}N_{v,w} \xrightarrow{\sim} \mathring{R}_{v,w}$, while the map $\pi_J : G/B_- \to G/P_-^J$ restricts to an isomorphism $\mathring{R}_{v,w} \xrightarrow{\sim} \mathring{\Pi}_{v,w}$. For a function $v \cdot f \in \mathbb{C}[\dot{v}N_{v,w}]$, denote by $\overline{v \cdot f} \in \mathbb{C}[\mathring{\Pi}_{v,w}]$ the image of $v \cdot f$ under the composition of these isomorphisms.

Lemma 3.6. *Let $(v, w) \in Q^J$ and gauge-fix $\Delta_{F_0} = 1$ on $\mathring{\Pi}_{v,w}$. Then for all $r \in J^\circ_{\mathbf{v}}$, the regular functions $\overline{v \cdot f}_r, \Delta_{F_r} \in \mathbb{C}[\mathring{\Pi}_{v,w}]$ agree on $\mathring{\Pi}_{v,w}$. Equivalently, we have*

$$f_r(g) = \frac{\Delta_{F_r}(\dot{v}gP_-^J)}{\Delta_{F_0}(\dot{v}gP_-^J)} \quad \text{for all } g \in N_{v,w} \text{ and } r \in J^\circ_{\mathbf{v}}. \tag{3.4}$$

Proof. We will prove (3.4) more generally for all $g \in N$. Observe that $F_0 = v[k + 1, n]$ (see [GKL19, Example 9.5]). Therefore for any $g \in N$, the submatrix of g with row set $v^{-1}F_0 = [k+1, n]$ and column set $[k+1, n]$ is an $(n-k) \times (n-k)$ upper-triangular unipotent matrix, thus $\Delta_{F_0}(\dot{v}g) = 1$. It remains to show that $f_r(g) = \Delta_{F_r}(\dot{v}g)$ for all $r \in J^\circ_{\mathbf{v}}$.

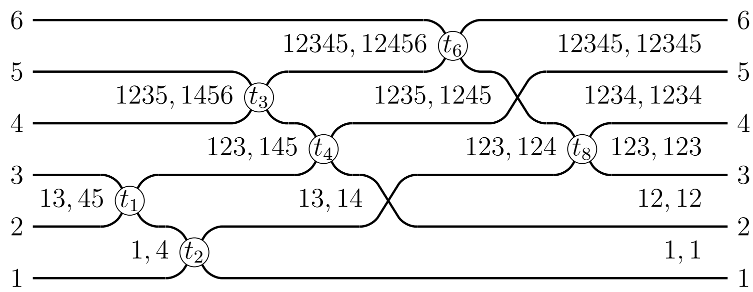
Fix $r \in J^\circ_{\mathbf{v}}$. We have $f_r = \Delta_{A_r,B_r}$, where

$$A_r = v^{(r-1)}\omega_{i_r} = s^{\mathbf{v}}_{i_m} \cdots s^{\mathbf{v}}_{i_r}\omega_{i_r} \quad \text{and} \quad B_r = w^{(r-1)}\omega_{i_r} = s_{i_m} \cdots s_{i_r}\omega_{i_r}. \tag{3.5}$$

Recall from Section 2.2 that we have two Young diagrams $\mu(v^{(r-1)}, i_r) \subset \mu(w^{(r-1)}, i_r)$ that fit inside an $(n - i_r) \times i_r$ rectangle, and moreover, $\mu(w^{(r-1)}, i_r)$ is itself a rectangle. Thus there exist integers $a_r \in [0, k]$ and $b_r \in [k, n]$ such that $B_r = [1, a_r] \sqcup [k+1, b_r]$, so $a_r + b_r - k = |B_r| = i_r$. And because $\mu(v^{(r-1)}, i_r) \subset \mu(w^{(r-1)}, i_r)$, we find that $[1, a_r] \subset A_r$ and $A_r \cap [b_r + 1, n] = \emptyset$. Let us define $C_r := (A_r \setminus [1, a_r]) \sqcup [b_r + 1, n]$, thus $|C_r| = n - k$. (See Figure 7(b) for an example.) It is clear that the functions $f_r = \Delta_{A_r,B_r}$ and $\Delta_{C_r,[k+1,n]}$ agree on N. Therefore it suffices to show

$$F_r = vC_r. \tag{3.6}$$

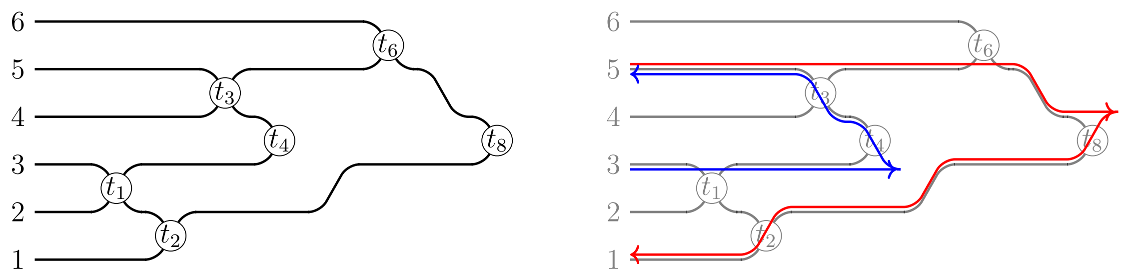
Let us first give a pictorial description of A_r and B_r using *wiring diagrams*. It is analogous to [MR04, Section 9]. Draw a wiring diagram $\mathrm{WD}(w)$ for w, and for each $r \in J^\circ_{\mathbf{v}}$, place a dot labeled t_r at the crossing that corresponds to s_{i_r}. Denote this *dotted wiring diagram* by $\mathrm{WD}^\bullet(v, w)$, cf. Figure 7(a). A wiring diagram $\mathrm{WD}(v)$ for v is obtained from $\mathrm{WD}^\bullet(v, w)$



(a) Labeling the regions of $\mathrm{WD}^{\bullet}(v,w)$ by (A_r, B_r).

r	A_r	B_r	C_r
1	13	45	136
2	1	4	156
3	1235	1456	235
4	123	145	236
6	12345	12456	345
8	123	124	356

(b) A_r, B_r, C_r for $r \in J^{\circ}_{\mathbf{v}}$.



(c) The graph $\overline{\mathrm{WD}^{\bullet}}(v,w)$.

(d) Strands $\overline{p}_4$ (red) and $\overline{p}_2$ (blue) in $\overline{\mathrm{WD}^{\bullet}}(v,w)$.

FIGURE 7. Constructing the graphs $\mathrm{WD}^{\bullet}(v,w)$ and $\overline{\mathrm{WD}^{\bullet}}(v,w)$ from the proof of Lemma 3.6. Here $k=3$, $n=6$, and (v,w) is as in Example 1.1.

by "uncrossing" each dot, i.e., replacing each crossing of $\mathrm{WD}^{\bullet}(v,w)$ that has a dot by a pair of parallel wires. Label each wire in $\mathrm{WD}(w)$ and $\mathrm{WD}(v)$ by its right endpoint (the right endpoints are labeled $1,\dots,n$ from bottom to top). To each chamber R of $\mathrm{WD}(w)$ we associate a set B_R of wires that are below R in $\mathrm{WD}(w)$. Similarly, we introduce a set $A_{R'}$ of wires that are below each chamber R' of $\mathrm{WD}(v)$. Any chamber R of $\mathrm{WD}(w)$ is contained inside a unique chamber R' of $\mathrm{WD}(v)$, so we label the corresponding chamber R of $\mathrm{WD}^{\bullet}(v,w)$ by the pair (A_R, B_R), where $A_R := A_{R'}$, see Figure 7(a). For each $r \in J^{\circ}_{\mathbf{v}}$, let R_r be the chamber of $\mathrm{WD}^{\bullet}(v,w)$ that is immediately to the left of the dot labeled by t_r. It is straightforward to check (see also [MR04, Section 9]) that

$$A_r = A_{R_r} \quad \text{and} \quad B_r = B_{R_r} \quad \text{for all } r \in J^{\circ}_{\mathbf{v}}, \tag{3.7}$$

where A_r, B_r are as in (3.5).

Let us now introduce a certain planar graph $\overline{\mathrm{WD}^{\bullet}}(v,w)$ drawn in a disk, with boundary vertices labeled by $1,\dots,n$ and interior vertices labeled by t_r for $r \in J^{\circ}_{\mathbf{v}}$. The same construction appears in [Kar16, Figure 5]. The graph $\overline{\mathrm{WD}^{\bullet}}(v,w)$ is obtained from $\mathrm{WD}^{\bullet}(v,w)$ by removing the *redundant part* of each wire, that is, the part to the right of the rightmost dot that is placed on an intersection involving this wire, see Figure 7(c). (The redundant part of each wire is common to $\mathrm{WD}(v)$ and $\mathrm{WD}(w)$.)

Observe that there is a simple isomorphism between the graphs $G(D)$ and $\overline{\mathrm{WD}^{\bullet}}(v,w)$ that preserves the labels of the vertices (i.e., $1,\dots,n$ for boundary vertices and $\{t_r\}_{r\in J^{\circ}_{\mathbf{v}}}$

for interior vertices). This isomorphism can be obtained by reflecting $G(D)$ along the line $y = 2x$ in the xy-plane. For example, compare Figure 7(c) with Figure 4(left).

For each $a \in [n]$, we introduce a path $\overline{p}_a$ in $\overline{\mathrm{WD}^\bullet}(v,w)$ that starts at $v^{-1}(a)$ on the left and ends at $w^{-1}(a)$ on the left. First, consider a path $\overline{p}'_a$ in $\mathrm{WD}^\bullet(v,w)$ that starts at $v^{-1}(a)$ on the left, goes right following the strands of $\mathrm{WD}(v)$ (i.e., ignores all intersections that have dots on them) until it reaches a on the right, and then goes left following the strands of $\mathrm{WD}(w)$ until it reaches $w^{-1}(a)$ on the left. The path $\overline{p}'_a$ in $\mathrm{WD}^\bullet(v,w)$ travels right and then left along the redundant part of the wire whose right endpoint is a. We define $\overline{p}_a$ to be the path in $\overline{\mathrm{WD}^\bullet}(v,w)$ obtained from $\overline{p}'_a$ by removing this redundant part. See Figure 7(d) for an example. Comparing the "rules of the road" (3.3) with the definition of the paths $\overline{p}_a$ in $\overline{\mathrm{WD}^\bullet}(v,w)$, we find that for each $a \in [n]$, our graph isomorphism $G(D) \cong \overline{\mathrm{WD}^\bullet}(v,w)$ sends the path p_a in $G(D)$ to the path $\overline{p}_{v(a)}$ in $\overline{\mathrm{WD}^\bullet}(v,w)$. For example, compare Figure 7(d) with Figure 6.

Note that for each $r \in J^\circ_{\mathbf{v}}$, the chamber R_r of $\mathrm{WD}^\bullet(v,w)$ is contained inside a unique chamber (also denoted R_r) of $\overline{\mathrm{WD}^\bullet}(v,w)$. We claim that for each $a \in [n]$ and $r \in J^\circ_{\mathbf{v}}$,

$$a \text{ belongs to } C_r \text{ if and only if the chamber } R_r \text{ is to the } \textit{left} \text{ of the path } \overline{p}_a. \tag{3.8}$$

To show this, suppose first that $a \leq k$. Then

$$a \in C_r \iff a \in A_r \setminus [1, a_r] \iff a \in A_r \setminus B_r \qquad (\text{for } a \leq k).$$

On the other hand, $a \leq k$ implies $v^{-1}(a) \leq w^{-1}(a)$, so a belongs to $A_r \setminus B_r$ if and only if the wire labeled a in $\mathrm{WD}(v)$ (resp., in $\mathrm{WD}(w)$) is below (resp., above) the chamber R_r, which is equivalent to R_r being to the left of the path $\overline{p}_a$.

Suppose now that $a \geq k+1$. Then

$$a \in C_r \iff a \in A_r \sqcup [b_r + 1, n] \iff a \in (A_r \cap B_r) \text{ or } a \notin (A_r \cup B_r) \qquad (\text{for } a \geq k+1).$$

On the other hand, $a \geq k+1$ implies $v^{-1}(a) \geq w^{-1}(a)$, so a belongs to $A_r \cap B_r$ if and only if the chamber R_r is above both wires of $\overline{p}'_a$, in which case R_r is to the left of $\overline{p}_a$. The only other case when R_r is to the left of $\overline{p}_a$ is when R_r is below both wires of $\overline{p}'_a$, and this corresponds precisely to $a \notin (A_r \cup B_r)$. This shows (3.8). Combining (3.8) with the rule for face labels in Section 3.3, we obtain a proof of (3.6). ☐

In view of Corollary 3.4, Leclerc's result [Lec16, Theorem 4.5] implies in the case $(v,w) \in Q^J$ that the map $x_r \mapsto \overline{v \cdot f_r}$ extends to an injective ring homomorphism $\mathcal{A}(\tilde{Q}) \hookrightarrow \mathbb{C}[\mathring{\Pi}_{v,w}]$. He conjectured that this map is actually an isomorphism. Thus Theorem 3.5 confirms his conjecture in the case $(v,w) \in Q^J$.

Combining Proposition 2.6 and Lemma 3.6, we have the following result.

Corollary 3.7. *Let $(v,w) \in Q^J$ and assume $\Delta_{F_0} = 1$ on $\mathring{\Pi}_{v,w}$.*

(i) *The cluster structure of* [Lec16] *coincides with that of Theorem 3.5.*
(ii) *We have an injection $\eta : \mathcal{A}(Q_D) \hookrightarrow \mathbb{C}[\mathring{\Pi}_{v,w}]$ sending x_{F_r} to Δ_{F_r} for all $r \in J^\circ_{\mathbf{v}}$.*

4. Surjectivity

In view of Corollary 3.7(ii), in order to complete the proof of Theorem 3.5, it suffices to show that the map $\eta : \mathcal{A}(Q_D) \hookrightarrow \mathbb{C}[\mathring{\Pi}_{v,w}]$ is surjective.

4.1. **Paths in Le-diagrams.** The space $\mathring{R}_{v,w}$ contains a distinguished torus of the same dimension, called the *open Deodhar stratum*. We describe a parametrization of this torus following [MR04].

For $\mathbf{t} = (t_r)_{r\in J^\circ_{\mathbf{v}}} \in (\mathbb{C}^*)^{|J^\circ_{\mathbf{v}}|}$, define an element

$$\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = g_1 \cdots g_m \in N\dot{v} \cap B_-wB_- \qquad \text{where} \qquad g_r = \begin{cases} \dot{s}_{i_r} & \text{if } r \notin J^\circ_{\mathbf{v}}, \\ x_{i_r}(t_r) & \text{if } r \in J^\circ_{\mathbf{v}}. \end{cases} \tag{4.1}$$

The map $(\mathbb{C}^*)^{J^\circ_{\mathbf{v}}} \to \mathring{R}_{v,w}$ given by $\mathbf{t} \mapsto \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})B_-$ is an isomorphism onto its image, the open Deodhar stratum in $\mathring{R}_{v,w}$.

Let $\mathring{X}^w = (B_-\dot{w}B_-)/B_-$ be a *Schubert cell* inside G/B_-. We have an isomorphism $N_-\dot{w} \cap \dot{w}N \xrightarrow{\sim} \mathring{X}^w$ sending $g \mapsto gB_-$. Let $\phi_w : \mathring{X}^w \to N_-\dot{w} \cap \dot{w}N$ denote the inverse of this isomorphism. Since $\mathring{R}_{v,w} \subset \mathring{X}^w$, for each $\mathbf{t} \in (\mathbb{C}^*)^{J^\circ_{\mathbf{v}}}$, we have a unique $h := \phi_w(\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})B_-) \in N_-\dot{w} \cap \dot{w}N$ satisfying $hB_- = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})B_-$. When $(v,w) \in Q^J$, computing the matrix h amounts to computing the column-echelon form of $\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})P^J_- \in \mathrm{Gr}(n-k,n)$. Our goal is to describe the entries of h in terms of the variables $\mathbf{t}$. The answer essentially coincides with the *boundary measurement map* of [Pos06, Definition 4.7].

Let $\vec{G}(D)$ be obtained from $G(D)$ by orienting every vertical edge down and every horizontal edge left. Suppose that $i \in w[k+1,n]$ (resp., $j \in w[k]$) labels a horizontal (resp., vertical) boundary edge of λ. For $r \in J^\circ_{\mathbf{v}}$ and a directed path P in $\vec{G}(D)$, we write $r \in P$ if P passes through the vertex labeled t_r, and let $\mathrm{wt}_P(\mathbf{t}) := \prod_{r\in P} t_r^{-1}$. Denote $\mathrm{Meas}_{i,j}(\mathbf{t}) := \sum_P \mathrm{wt}_P(\mathbf{t})$, where the sum is taken over all directed paths in $\vec{G}(D)$ connecting i to j. Finally, for $i,j \in [n]$, set $\mathrm{inv}_{i,j} := \#\{j' > j : w(j') < i\}$, so that when $i = w(j)$, the (i,j)-th entry of $\dot{w}$ equals $(-1)^{\mathrm{inv}_{i,j}}$. The following result can be deduced from [TW13, Theorem 5.10]. We include a proof here for completeness.

Proposition 4.1. *Let $h = \phi_w(\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})B_-)$, where $\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})$ is as in* (4.1)*. For $i \in w[k]$ and $j \in [k+1,n]$, the (i,j)-th entry of h equals* $(-1)^{\mathrm{inv}_{i,j}} \mathrm{Meas}_{i,w(j)}(\mathbf{t})$.

Proof. Because $w \in W^J$ and $h \in N_-\dot{w} \cap \dot{w}N$, the left k columns of h coincide with the left k columns of $\dot{w}$, so we are interested in the right $n-k$ columns of h, which contain the identity submatrix with row set $w[k+1,n]$. Let us denote by $|h]$ the submatrix of h with column set $[k+1,n]$.

We proceed by induction on the length $m = \ell(w)$ of w. The case $m = 0$ is clear: the matrix $|h]$ has 0-s in all entries except for the identity matrix in the rows $k+1, k+2, \dots, n$. Suppose the result is known for (v',w'), where $w' = s_iw < w$, $v' = s_i^{\mathbf{v}}v \le v$, and $i := i_1$. Then $D = D(v,w)$ is obtained from $D' := D(v',w')$ by adding a box (which may or may not contain a dot), whose horizontal and vertical boundary edges are labeled by i and $i+1$. Let $\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t}) = g_2g_3\cdots g_m$ and $h' := \phi_{w'}(\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t})B_-)$.

Suppose that $g_1 = \dot{s}_i$. Then $v' = s_iv < v$, $\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = \dot{s}_i\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t})$, $h = \dot{s}_ih'$, and the extra box of D does not contain a dot. The definition of a Le-diagram implies that either there are no paths involving i or no paths involving $i+1$ in D'. The paths in D are thus in bijection with the paths in D' with the roles of i and $i+1$ swapped. This agrees with $h = \dot{s}_ih'$, and the signs of the entries change in accordance with $(-1)^{\mathrm{inv}_{i,j}}$.

Suppose that $g_1 = x_i(t_1)$. Then $v' = v$, $\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) = x_i(t_1)\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t})$, and we have $|h] = x_i(t_1)|h']dy$, where $d = \mathrm{diag}(d_{k+1},\dots,d_n)$ is an $(n-k)\times(n-k)$ diagonal matrix and

$y = (y_{ab})_{k+1\leq a,b\leq n}$ is an $(n-k)\times(n-k)$ lower-triangular unipotent matrix given by

$$d_a = \begin{cases} 1/t_1, & \text{if } a = w^{-1}(i), \\ 1, & \text{otherwise;} \end{cases} \qquad y_{ab} = \begin{cases} -h'_{i,b} & \text{if } a = w^{-1}(i) \text{ and } k+1 \leq b < a, \\ 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

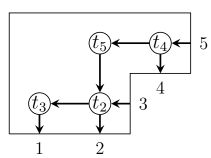
Since dy is lower-triangular, we have $hB_- = \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})B_-$. Multiplying by dy on the right "kills off" all nonzero entries corresponding to non-inversions of w and yields $h \in N_-\dot{w} \cap \dot{w}N$, thus $h = \phi_w(\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}))$. Note also that the extra box of D contains a dot labeled by t_1. Thus the matrix entries of h correspond again exactly to paths in $G(D)$, and the sign of each entry agrees with $(-1)^{\mathrm{inv}_{i,j}}$. □

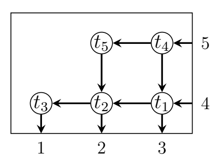
Example 4.2. Let $(v', w') = (s_2, s_2s_1s_4s_3s_2)$. We find

$$\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t}) = x_2(t_2)x_1(t_3)x_4(t_4)x_3(t_5)\dot{s}_2 = \begin{bmatrix} 1 & 0 & t_3 & 0 & 0 \\ 0 & -t_2 & 1 & t_2t_5 & 0 \\ 0 & -1 & 0 & t_5 & 0 \\ 0 & 0 & 0 & 1 & t_4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad h' = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{t_2t_3} & \frac{1}{t_2} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -1 & \frac{1}{t_2t_3t_4t_5} & -\frac{1}{t_2t_4t_5} & \frac{1}{t_4} \end{bmatrix}.$$

The matrices $\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t})$ and $h' = \phi_{w'}\left(\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t})P_-^J\right)$ represent the same element of G/B_-, which can be checked by comparing their right-justified *flag minors*: for each $j \in [n]$, the linear span of the last j columns of $\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t})$ equals that of h'.

Let now $v = v'$, and $w = s_3w'$. Thus $\vec{G}(D')$ and $\vec{G}(D)$ are given by

(4.2) $\vec{G}(D') =$ 

, $\vec{G}(D) =$ 

.

We see that the entries of h' are indeed expressed as sums over paths in $\vec{G}(D')$. Next, temporarily denoting the non-trivial entries of h' by a, b, c, d, e (with $a := \frac{1}{t_2t_3}, \dots, e := \frac{1}{t_4}$), the calculation of $h = \phi_w\left(x_3(t_1)\mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t})P_-^J\right)$ in the proof of Proposition 4.1 proceeds as follows:

$$|h'| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & b & 0 \\ 0 & 0 & 1 \\ c & -d & e \end{bmatrix}, \quad x_3(t_1)|h'| = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & b & t_1 \\ 0 & 0 & 1 \\ c & -d & e \end{bmatrix}, \quad x_3(t_1)|h'|d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & b & 1 \\ 0 & 0 & \frac{1}{t_1} \\ c & -d & \frac{e}{t_1} \end{bmatrix},$$

$$|h| = x_3(t_1)|h'|dy = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{a}{t_1} & -\frac{b}{t_1} & \frac{1}{t_1} \\ c+\frac{ae}{t_1} & -\left(d+\frac{be}{t_1}\right) & \frac{e}{t_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{t_1t_2t_3} & -\frac{1}{t_1t_2} & \frac{1}{t_1} \\ \frac{1}{t_2t_3t_4t_5}+\frac{1}{t_1t_2t_3t_4} & -\left(\frac{1}{t_2t_4t_5}+\frac{1}{t_1t_2t_4}\right) & \frac{1}{t_1t_4} \end{bmatrix}.$$

We indeed see that the entries of h are given by sums over directed paths in $\vec{G}(D)$.

4.2. **Muller–Speyer twist.** Fix $(v, w) \in Q^J$. Muller and Speyer [MS17, Section 1.8] have defined a right twist isomorphism $\tau : \mathring{\Pi}_{v,w} \xrightarrow{\sim} \mathring{\Pi}_{v,w}$. We shall not recall the definition here, however, see Example 4.6 below. For $r \in J^\circ_{\mathbf{v}}$, we let $q_r := \Delta_{F_r} \circ \tau \in \mathbb{C}[\mathring{\Pi}_{v,w}]$ denote the *twisted minor* indexed by the corresponding face label of $G(D)$.

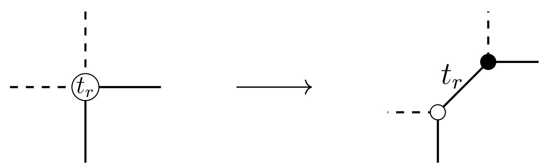
Let $r \in J^\circ_{\mathbf{v}}$, and suppose that the faces of $G(D)$ adjacent to t_r are labeled by F_a, F_b, F_c, F_r as in Figure 2.

Proposition 4.3. *Let $x := \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})$ be given by* (4.1)*. Then with $\Delta_{F_0} = 1$,*

$$t_r = \frac{q_a(x) q_b(x)}{q_c(x) q_d(x)} \qquad \text{and} \qquad q_r(x) = \prod_{r'} \frac{1}{t_{r'}}, \tag{4.3}$$

where the product is taken over all $r' \in J^\circ_{\mathbf{v}}$ such that the vertex of $G(D)$ labeled $t_{r'}$ is northwest of F_r.

Proof. We associate to the vertex-labeled graph $G(D)$ a planar, bipartite, edge-weighted graph $N(\mathbf{t})$ via the following local substitution at each vertex t_r of $G(D)$:



Here the horizontal (resp., vertical) dashed edge is present in $N(\mathbf{t})$ if and only if it is present in $G(D)$, and the weights of all horizontal and vertical edges in $N(\mathbf{t})$ are set to 1. We make the following observations concerning $N(\mathbf{t})$:

(a) $N(\mathbf{t})$ is a *reduced plabic graph* in the language of [Pos06]. It satisfies the assumptions of [MS17, Section 3.1].
(b) The strands from [Pos06, MS17] agree with the strands described in Section 3.3.
(c) The point $\mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t}) P_-^J \in \mathrm{Gr}(n-k, n)$ is equal to the image of $N(\mathbf{t})$ in $\mathrm{Gr}(n-k, n)$ under the *boundary measurement map* (denoted $\mathbb{D}$ in [MS17]). This has been verified in e.g. [TW13] or [Kar16], or can be easily checked using Le-diagram induction directly from the setup of [MS17].
(d) The *downstream wedge* ([MS17, Section 5]) of an edge of weight t_r in $N(\mathbf{t})$ consists precisely of the faces $F_{r'}$ to the southeast of the vertex labeled t_r in $G(D)$.

The formula for $q_r(x)$ in (4.3) follows from (d) and [MS17, Proposition 5.5 and Theorem 7.1]: they denote this monomial transformation by $\overrightarrow{\mathbb{M}}$. The formula for t_r in (4.3) is then obtained by expressing the values $q_a(x)$, $q_b(x)$, $q_c(x)$, $q_d(x)$ in $\mathbf{t}$ using the above monomial transformation. □

We have the following relationship between the cluster structure and the *totally nonnegative Grassmannian* studied in [Lus98, Pos06].

Corollary 4.4. *Assume $\Delta_{F_0} = 1$ on $\mathring{\Pi}_{v,w}$. Then the following subsets of $\mathring{\Pi}_{v,w}$ coincide:*

(1) the positroid cell $\Pi^{>0}_{v,w} := \left\{ x \in \mathring{\Pi}_{v,w} \mid \Delta_I(x) \in \mathbb{R}_{\geq 0} \text{ for all } I \subset [n] \text{ of size } n-k \right\}$*;*
(2) the subset of $\mathring{\Pi}_{v,w}$ where all cluster variables of $\mathcal{A}(Q_D)$ take positive real values.

Proof. Since $\{\Delta_{F_r}\}_{r \in J^\circ_{\mathbf{v}}}$ is the image (under η) of a single cluster of $\mathcal{A}(Q_D)$, and since the mutation rule (1.1) is subtraction-free, the subset in (2) equals $\left\{ x \in \mathring{\Pi}_{v,w} \mid \Delta_{F_r}(x) \in \mathbb{R}_{>0}\ \forall r \in J^\circ_{\mathbf{v}} \right\}$. But then applying the twist of [MS17], we see that this set coincides with the image of the boundary measurement map $\mathbb{D}$ applied to the graph $N(\mathbf{t})$ when $\mathbf{t}$ takes values in $\mathbb{R}^{J^\circ_{\mathbf{v}}}_{>0}$, and this set coincides with $\Pi^{>0}_{v,w}$ by either [Pos06] or [MR04, Section 12]. □

Remark 4.5. For arbitrary $v \leq w \in W$, there exists a simple automorphism $\tau_{v,w} : \mathring{R}_{v,w} \xrightarrow{\sim} \mathring{R}_{v,w}$ which gives a common generalization of the twist maps of [BFZ05] (when $v = 1$)

and [MS17] (when $(v,w) \in Q^J$), and shares many properties with them. For example, $\tau_{v,w}$ preserves the positive part $R^{>0}_{v,w}$ of $\mathring{R}_{v,w}$ and satisfies a generalization of the *chamber ansatz* (4.3). The map $\tau_{v,w}$ will be studied in a separate paper [GL].

Example 4.6. Let $(v',w') = (s_2, s_2s_1s_4s_3s_2)$ and $x := \mathbf{g}'_{\mathbf{v}',\mathbf{w}'}(\mathbf{t})$ be as in Example 4.2. Then x and its twist $\tau(x)$ are represented by the following $n \times (n-k)$ matrices:

$$|x| = \begin{bmatrix} t_3 & 0 & 0 \\ 1 & t_2t_5 & 0 \\ 0 & t_5 & 0 \\ 0 & 1 & t_4 \\ 0 & 0 & 1 \end{bmatrix}, \quad |\tau(x)| = \begin{bmatrix} \frac{1}{t_3} & -\frac{1}{t_2t_3t_5} & \frac{1}{t_2t_3t_4t_5} \\ 1 & 0 & 0 \\ 0 & \frac{1}{t_5} & -\frac{1}{t_4t_5} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.4}$$

For each $i \in [n]$, the i-th row of $|\tau(x)|$ is orthogonal to the $(i+1)$-th row of $|x|$, and has scalar product 1 with the i-th row of $|x|$, in agreement with [MS17, Section 1.8]. For $G(D')$ as in (4.2), we have $F_0 = \{2,4,5\}$, and we see that the matrix $|\tau(x)|$ above is gauge-fixed to have $\Delta_{F_0} = 1$. The values of $q_r(x) = \Delta_{F_r}(\tau(x))$ for $r \in J^\circ_{\mathbf{v}} \sqcup \{0\}$ are given by:

(4.5)

r	0	2	3	4	5
F_r	$\{2,4,5\}$	$\{1,2,5\}$	$\{1,4,5\}$	$\{2,3,4\}$	$\{2,3,5\}$
$q_r(x)$	1	$\frac{1}{t_2t_3t_5}$	$\frac{1}{t_3}$	$\frac{1}{t_4t_5}$	$\frac{1}{t_5}$

This agrees with (4.3).

4.3. **Proof of Theorem 3.5.** Our approach is similar to that of [BFZ05], who gave an upper cluster algebra structure on double Bruhat cells.

Fix $(v,w) \in Q^J$. Let $\eta^{(\tau)} : \mathcal{A}(Q_D) \hookrightarrow \mathbb{C}[\mathring{\Pi}_{v,w}]$ be obtained by composing the map η from Corollary 3.7(ii) with the twist isomorphism $\tau : \mathring{\Pi}_{v,w} \xrightarrow{\sim} \mathring{\Pi}_{v,w}$. Explicitly, for $r \in J^\circ_{\mathbf{v}}$, $\eta^{(\tau)}$ sends $x_r \in \mathcal{A}(Q_D)$ to the element $q_r \in \mathbb{C}[\mathring{\Pi}_{v,w}]$ from Section 4.2.

By Proposition 4.3, this injection is induced by the invertible monomial transformation (4.3) between $\mathbf{t} := \{t_r\}_{r\in J^\circ_{\mathbf{v}}}$ and $\mathbf{q} := \{q_r\}_{r\in J^\circ_{\mathbf{v}}}$. By Section 3.1, we have an injection $(\mathbb{C}^*)^{J^\circ_{\mathbf{v}}} \hookrightarrow \mathring{R}_{v,w}$ sending $\mathbf{t} \mapsto \mathbf{g}_{\mathbf{v},\mathbf{w}}(\mathbf{t})B_-$, whose image is an open Zariski dense subset of $\mathring{R}_{v,w}$. This gives rise to an injection

$$\mathbb{C}[\mathring{\Pi}_{v,w}] \hookrightarrow \mathbb{C}[\mathbf{t}^{\pm1}] = \mathbb{C}[\mathbf{q}^{\pm1}], \quad \text{where} \quad \mathbf{t}^{\pm1} := \{t_r^{\pm1}\}_{r\in J^\circ_{\mathbf{v}}} \quad \text{and} \quad \mathbf{q}^{\pm1} := \{q_r^{\pm1}\}_{r\in J^\circ_{\mathbf{v}}}. \tag{4.6}$$

For $r \in J^\circ_{\mathbf{v}}$, let $q'_r := \eta^{(\tau)}(x'_r)$ (where x'_r is given in (1.1)), and let $\mathbf{q}'_r := \{q_a\}_{a\in J^\circ_{\mathbf{v}}\setminus\{r\}} \sqcup \{q'_r\}$.

Lemma 4.7. *For each $r \in J^\circ_{\mathbf{v}}$, we have an injection $\mathbb{C}[\mathring{\Pi}_{v,w}] \hookrightarrow \mathbb{C}[(\mathbf{q}'_r)^{\pm1}]$. In other words, every element of $\mathbb{C}[\mathring{\Pi}_{v,w}]$ can be written as a Laurent polynomial in the variables $\mathbf{q}'_r$.*

Proof. Let $T := \mathrm{Spec}(\mathbb{C}[(\mathbf{q})^{\pm1}])$ denote the initial cluster torus, and let $T'_r := \mathrm{Spec}(\mathbb{C}[(\mathbf{q}'_r)^{\pm1}])$ denote the mutated cluster torus in the r-th direction. The intersection $T \cap T'_r$ is Zariski dense in T'_r, thus the map (4.6) gives a rational map $T'_r \dashrightarrow \mathring{\Pi}_{v,w}$. (This is also the map induced by the inclusion $\eta^{(\tau)} : \mathcal{A}(Q_D) \hookrightarrow \mathbb{C}[\mathring{\Pi}_{v,w}]$.) The required statement is equivalent to showing that this rational map $T'_r \dashrightarrow \mathring{\Pi}_{v,w}$ is in fact an inclusion $T'_r \hookrightarrow \mathring{\Pi}_{v,w}$. Each q_i and q'_r is a regular function on $\mathring{\Pi}_{v,w}$ and hence we have a regular map $\mathring{\Pi}_{v,w} \to \mathrm{Spec}(\mathbb{C}[\mathbf{q}'_r]) \simeq \mathbb{C}^{J^\circ_{\mathbf{v}}}$. It thus suffices to show that the rational map $T'_r \dashrightarrow \mathring{\Pi}_{v,w}$ is a regular map on the torus T'_r. (Indeed, in this case, the composition $T'_r \to \mathring{\Pi}_{v,w} \to \mathrm{Spec}(\mathbb{C}[\mathbf{q}'_r])$ is a regular map whose restriction to the open dense subset $T'_r \cap T$ agrees with the inclusion map $T'_r \hookrightarrow \mathrm{Spec}(\mathbb{C}[\mathbf{q}'_r])$.

Therefore this composition coincides with the identity map on T'_r, and in particular the map $T'_r \to \mathring{\Pi}_{v,w}$ is automatically injective.)

We begin by showing that the map $T \cap T'_r \hookrightarrow \mathrm{Gr}(n-k,n)$ given by $\mathbf{t} \mapsto \mathbf{g_{v,w}}(\mathbf{t})P_-^J$ extends to a regular map $\theta_r : T'_r \to \mathrm{Gr}(n-k,n)$. It suffices to write each matrix entry of $|h]$ (where $h = \phi_w(\mathbf{g_{v,w}}(\mathbf{t})B_-)$ as in Section 4.1) as an element of $\mathbb{C}[(\mathbf{q}'_r)^{\pm 1}]$. By Proposition 4.1, each such matrix entry is a sum of $\mathrm{wt}_P(\mathbf{t})$ over paths P in $\vec{G}(D)$. We may restrict our attention to paths P such that the monomial $\mathrm{wt}_P(\mathbf{t})$ contains q_r in the denominator. Let r_1 (resp., r_2) be the bottom-left (resp., top-right) vertex of the face of $G(D)$ labeled by F_r. Then $\mathrm{wt}_P(\mathbf{t})$ contains q_r in the denominator precisely when P passes through both r_1 and r_2, and either contains the top-left or the bottom-right boundary of the face F_r. We may group such paths into pairs (P_1, P_2) where P_1 contains the top-left boundary of F_r, while P_2 contains the bottom-right boundary of F_r, and otherwise P_1 and P_2 agree. By Proposition 4.1, the contribution of such a pair is

$$\mathrm{wt}_{P_1}(\mathbf{t}) + \mathrm{wt}_{P_2}(\mathbf{t}) = \frac{M}{q_r} \cdot \left(\frac{q_c}{q_a q_b} + \frac{\prod_{F_r \to F_j : j \neq a,b} q_j}{\prod_{F_i \to F_r : i \neq c} q_i} \right) = \frac{M}{q_r} \cdot \left(\frac{q_r q'_r}{q_a q_b \prod_{F_i \to F_r : i \neq c} q_i} \right),$$

where F_a, F_b, F_c, F_r are the labels of the faces adjacent to t_r as in Figure 2, M is a monomial in $\{q_a\}_{a \in J^\circ_{\mathbf{v}} \setminus \{r\}}$, and the products in the second term are taken over the arrows of the quiver Q_D not involving F_a, F_b, F_c. The common factor q_r cancels, and we have constructed our desired map $\theta_r : T'_r \to \mathrm{Gr}(n-k,n)$.

The intersection $T \cap T'_r$ is dense in T'_r, and $\theta_r(T \cap T'_r) \subseteq \mathring{\Pi}_{v,w}$, so we must have $\theta_r(T'_r) \subseteq \Pi_{v,w}$, where $\Pi_{v,w}$ is the Zariski closure of $\mathring{\Pi}_{v,w}$. By [MS17, Equation (9)], we have $\Delta_{F_r} = \frac{1}{q_r}$ for all $r \in \partial J^\circ_{\mathbf{v}} \sqcup \{0\}$. (For example, compare the minors $\Delta_{F_r}(x)$ and $\Delta_{F_r}(\tau(x))$ for $x, \tau(x)$ from (4.4) and F_r from (4.5).) Thus Δ_{F_r} is nonzero on $\theta_r(T'_r)$ for any $r \in \partial J^\circ_{\mathbf{v}} \sqcup \{0\}$. By [KLS13, Section 5], $\mathring{\Pi}_{v,w}$ is exactly the locus in $\Pi_{v,w}$ where the Plücker variables indexed by the *Grassmann necklace* are nonvanishing, and this Grassmann necklace is precisely the collection $\{F_r\}_{r \in \partial J^\circ_{\mathbf{v}} \sqcup \{0\}}$, see [Pos06] and [MS17, Proposition 4.3]. We conclude that we have a regular map $\theta_r : T'_r \to \mathring{\Pi}_{v,w}$. □

Example 4.8. Let $k = 3$, $n = 6$, $(v, w) = (s_2 s_4, s_2 s_1 s_4 s_3 s_2 s_5 s_4 s_3)$ as in Example 1.1. Using Figure 4 and Proposition 4.3, we find

$$t_8 = \frac{1}{q_8}, \quad t_6 = \frac{q_8}{q_6}, \quad t_4 = \frac{q_8}{q_4}, \quad t_3 = \frac{q_4 q_6}{q_3 q_8}, \quad t_2 = \frac{q_8}{q_2}, \quad t_1 = \frac{q_2 q_4}{q_1 q_8}, \tag{4.7}$$

and the mutation rule (1.1) gives $q'_8 = \frac{q_2 q_4 q_6 + q_1 q_3}{q_8}$. We now express $h = \phi_w(\mathbf{g_{v,w}}(\mathbf{t})B_-)$ both in terms of $\mathbf{t}$ and in terms of $\mathbf{q}$ using Proposition 4.1 and Equation (4.7):

$$|h] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{t_1 t_2} & \frac{1}{t_1} & 0 \\ 0 & 0 & 1 \\ \frac{1}{t_1 t_2 t_3 t_4} & -\frac{1}{t_1 t_3 t_4} & \frac{1}{t_3} \\ \frac{t_1 t_3 t_4 + t_8}{t_1 t_2 t_3 t_4 t_6 t_8} & -\frac{1}{t_1 t_3 t_4 t_6} & \frac{1}{t_3 t_6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{q_1}{q_4} & \frac{q_1 q_8}{q_2 q_4} & 0 \\ 0 & 0 & 1 \\ \frac{q_1 q_3}{q_4 q_6} & -\frac{q_1 q_3 q_8}{q_2 q_4 q_6} & \frac{q_3 q_8}{q_4 q_6} \\ \frac{q_2 q_4 q_6 + q_1 q_3}{q_4 q_8} & -\frac{q_1 q_3}{q_2 q_4} & \frac{q_3}{q_4} \end{bmatrix}.$$

Thus the only entry of h that has q_8 in the denominator is

$$\frac{t_1 t_3 t_4 + t_8}{t_1 t_2 t_3 t_4 t_6 t_8} = \frac{q_2 q_4 q_6 + q_1 q_3}{q_4 q_8} = \frac{q'_8}{q_4}.$$

In particular, all matrix entries of h can be written as Laurent polynomials in the cluster $\mathbf{q}'_8 = \{q_1, q_2, q_3, q_4, q_6, q'_8\}$, in agreement with Lemma 4.7.

Before we finish the proof, we need one more technical statement. Given an ice quiver Q with vertex set $V = V_f \sqcup V_m$ partitioned as in Section 1.4, the *extended exchange matrix* $\tilde{B}(Q) = (b_{r,r'})_{r\in V, r'\in V_m}$ of Q has rows indexed by the vertices of Q and columns indexed by the mutable vertices of Q. We have $b_{r,r'} \in \{1, -1, 0\}$, depending on whether there is an arrow $r \to r'$, or an arrow $r' \to r$, or no arrows between r and r' (assuming no two vertices of Q are connected by more than one arrow).

Proposition 4.9. *The extended exchange matrix $\tilde{B}(Q_D)$ is of full rank (i.e., has rank $|V_m|$).*

Proof. We proceed by induction on the size of the Young diagram λ of D, the case $|\lambda| = 0$ being trivial. Suppose $|\lambda| > 0$. Let D' be obtained from D by removing a box (i, j) adjacent to the boundary of λ. If D does not contain a dot inside the box (i, j) then $\tilde{B}(Q_D) = \tilde{B}(Q_{D'})$ so the result holds by induction. Thus assume that D contains a dot labeled t_r inside the box (i, j). Then F_r is a boundary face. If either the row or the column of (i, j) contains no other dots, then $\tilde{B}(Q_D)$ is obtained from $\tilde{B}(Q_{D'})$ by removing the row indexed by F_r, and this row is 0; the result holds by induction. Finally, suppose that both the row and the column of (i, j) contains another dot. Let F_a, F_b, F_c, F_r be the labels of faces adjacent to t_r as in Figure 2. Thus F_a and F_b are boundary faces. If F_c is also a boundary face, then again $\tilde{B}(Q_D)$ and $\tilde{B}(Q_{D'})$ differ by a 0 row. So assume that F_c is an interior face, then F_c becomes a boundary face in $G(D')$. The matrix $\tilde{B}(Q_D)$ satisfies $b_{F_r,F_c} = -1$, and this is the only nonzero entry in the row indexed by F_r. The matrix $\tilde{B}(Q_{D'})$ is obtained from $\tilde{B}(Q_D)$ by deleting the row of F_r and the column of F_c. It is clear that $\tilde{B}(Q_D)$ has full rank if and only if $\tilde{B}(Q_{D'})$ has full rank, so the result again holds by induction. □

Proof of Theorem 3.5. By Proposition 4.9 and [BFZ05, Corollary 1.9], the intersection of Laurent polynomial rings (called the *upper bound* in [BFZ05])

$$\mathbb{C}[(\mathbf{q})^{\pm 1}] \cap \left(\bigcap_r \mathbb{C}[(\mathbf{q}'_r)^{\pm 1}] \right)$$

is equal to the *upper cluster algebra* $\overline{\mathcal{A}}(Q_D)$, defined to be the intersection of Laurent polynomial rings for *all* clusters of $\mathcal{A}(Q_D)$. By Lemma 4.7, we have $\mathbb{C}[\mathring{\Pi}_{v,w}] \subseteq \eta^{(\tau)}(\overline{\mathcal{A}}(Q_D))$. By [MS16], $\mathcal{A}(Q_D)$ is a locally acyclic cluster algebra, and by [Mul13, Theorem 4.1], we have $\mathcal{A}(Q_D) = \overline{\mathcal{A}}(Q_D)$. Recall that $\tau : \mathring{\Pi}_{v,w} \xrightarrow{\sim} \mathring{\Pi}_{v,w}$ is an isomorphism and $\eta^{(\tau)} = \tau \circ \eta$. By Corollary 3.7(ii), we have $\eta^{(\tau)}(\mathcal{A}(Q_D)) \subseteq \mathbb{C}[\mathring{\Pi}_{v,w}]$, and therefore $\eta^{(\tau)}(\mathcal{A}(Q_D)) = \mathbb{C}[\mathring{\Pi}_{v,w}]$. We conclude that $\eta(\mathcal{A}(Q_D)) = \mathbb{C}[\mathring{\Pi}_{v,w}]$, completing the proof. □

References

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[BGY06] K. A. Brown, K. R. Goodearl, and M. Yakimov. Poisson structures on affine spaces and flag varieties. I. Matrix affine Poisson space. *Adv. Math.*, 206(2):567–629, 2006.

[BIRS09] A. B. Buan, O. Iyama, I. Reiten, and J. Scott. Cluster structures for 2-Calabi-Yau categories and unipotent groups. *Compos. Math.*, 145(4):1035–1079, 2009.

[BRvW09] Farzin Barekat, Victor Reiner, and Stephanie van Willigenburg. Corrigendum to “Coincidences among skew Schur functions” [Adv. Math. 216 (1) (2007) 118–152]. *Adv. Math.*, 220(5):1655–1656, 2009.

[FS18] Nicolas Ford and Khrystyna Serhiyenko. Green-to-red sequences for positroids. *J. Combin. Theory Ser. A*, 159:164–182, 2018.

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[GHKK18] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31(2):497–608, 2018.

[GKL19] Pavel Galashin, Steven N. Karp, and Thomas Lam. Regularity theorem for totally nonnegative flag varieties. `arXiv:1904.00527v2`, 2019.

[GL] Pavel Galashin and Thomas Lam. In preparation.

[GLS07] Christof Geiss, Bernard Leclerc, and Jan Schröer. Cluster algebra structures and semicanonical bases for unipotent groups. `arXiv:math/0703039v4`, 2007.

[GLS08] Christof Geiss, Bernard Leclerc, and Jan Schröer. Partial flag varieties and preprojective algebras. *Ann. Inst. Fourier (Grenoble)*, 58(3):825–876, 2008.

[GY16] K. R. Goodearl and M. T. Yakimov. The Berenstein-Zelevinsky quantum cluster algebra conjecture. *J. Eur. Math. Soc. (JEMS)*, to appear. `arXiv:1602.00498v3`, 2016.

[Kar16] Rachel Karpman. Bridge graphs and Deodhar parametrizations for positroid varieties. *J. Combin. Theory Ser. A*, 142:113–146, 2016.

[KLS13] Allen Knutson, Thomas Lam, and David E. Speyer. Positroid varieties: juggling and geometry. *Compos. Math.*, 149(10):1710–1752, 2013.

[Lec16] B. Leclerc. Cluster structures on strata of flag varieties. *Adv. Math.*, 300:190–228, 2016.

[LS16] Thomas Lam and David E. Speyer. Cohomology of cluster varieties. I. Locally acyclic case. `arXiv:1604.06843v1`, 2016.

[Lus98] G. Lusztig. Total positivity in partial flag manifolds. *Represent. Theory*, 2:70–78, 1998.

[MR04] R. J. Marsh and K. Rietsch. Parametrizations of flag varieties. *Represent. Theory*, 8:212–242, 2004.

[MS16] Greg Muller and David E. Speyer. Cluster algebras of Grassmannians are locally acyclic. *Proc. Amer. Math. Soc.*, 144(8):3267–3281, 2016.

[MS17] Greg Muller and David E. Speyer. The twist for positroid varieties. *Proc. Lond. Math. Soc. (3)*, 115(5):1014–1071, 2017.

[Mul13] Greg Muller. Locally acyclic cluster algebras. *Adv. Math.*, 233:207–247, 2013.

[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, `http://math.mit.edu/~apost/papers/tpgrass.pdf`, 2006.

[Sch14] Ralf Schiffler. *Quiver representations*. CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, Cham, 2014.

[Sco06] J. S. Scott. Grassmannians and cluster algebras. *Proc. London Math. Soc. (3)*, 92(2):345–380, 2006.

[SSBW19] K. Serhiyenko, M. Sherman-Bennett, and L. Williams. Cluster structures in Schubert varieties in the Grassmannian. `arXiv:1902.00807v2`, 2019.

[TW13] Kelli Talaska and Lauren Williams. Network parametrizations for the Grassmannian. *Algebra Number Theory*, 7(9):2275–2311, 2013.

Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA

Email address: `galashin@mit.edu`

Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA

Email address: `tfylam@umich.edu`