

LINEAR RECURRENCES FOR CYLINDRICAL NETWORKS

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ABSTRACT. We prove a general theorem that gives a linear recurrence for tuples of paths in every cylindrical network. This can be seen as a cylindrical analog of the Lindström-Gessel-Viennot theorem. We illustrate the result by applying it to Schur functions, plane partitions, and domino tilings.

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1. INTRODUCTION

Given a weighted directed graph N , the celebrated Lindström-Gessel-Viennot theorem [16, 21] gives a combinatorial interpretation for a certain determinant in terms of tuples of vertex-disjoint paths in N . Its applications in several different contexts have been studied extensively, such as:

- alternating sign matrices (equivalently, totally symmetric self-complementary plane partitions) [3, 29],

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- domino tilings of the Aztec diamond [4] (equivalently, states of the six-vertex model [11, 28], or monomials in the octahedron recurrence [23]),
- vicious walkers model [17, 18, 19],
- super-Schur functions [6], Q -Schur functions [26], etc.

In a series of papers [13, 14, 15], we studied Zamolodchikov phenomena associated with *finite* \boxtimes *finite*, *affine* \boxtimes *finite*, and *affine* \boxtimes *affine* quivers. In particular, in the affine \boxtimes finite case [14] we showed that the values of the T -system of type $\hat{A} \otimes A$ (equivalently, the values of the octahedron recurrence in a cylinder) satisfy a simple linear recurrence relation whose coefficients admit a nice combinatorial interpretation: they are sums over domino tilings of the cylinder with fixed *Thurston height*. Similarly, in [12] we show using a formula given by Carroll and Speyer [8] that the values of the cube recurrence in a cylinder satisfy a similar linear recurrence relation, where the coefficients now are sums over *groves*.

This led us to the formulation of a general theorem (Theorem 2.3) that applies to most of the networks mentioned above. We state two different versions of it, one for arbitrary *cylindrical networks* (see Definition 3.3) and one for *planar cylindrical networks* (see Definition 4.11), i.e., cylindrical networks that can be drawn in a cylinder without self intersections. The first version is more general, however, the second version gives a stronger result, that is, a shorter recurrence relation which is conjecturally minimal (see Conjecture 6.5). A closely related but different result on rationality of certain measurements taken in cylindrical networks has previously appeared in [20, Proposition 2.7].

In Section 2, we define cylindrical networks and related notions and state our main results, Theorem 2.1 and its generalization, Theorem 2.3. We then prove Theorem 2.1 in Section 3.3. After that, in Section 4 we give some background on plethysms of symmetric functions and the Lindström-Gessel-Viennot method which we then use to prove Theorem 2.3.

In Section 5, we give several applications of Theorem 2.3. We start with the case of Schur functions, see Section 5.1. We show how Theorem 2.3 gives a linear recurrence for sequences of Schur polynomials of the form $f(\ell) = s_{(\lambda+\ell\mu)}(x_1, \dots, x_n)$ where μ is of rectangular shape. This gives an alternative proof to a recent result of Alexandersson [1, 2] which however applies to more general sequences of the form $s_{(\lambda+\ell\mu)/(\kappa+\ell\nu)}$ where the partitions $\lambda, \mu, \nu, \kappa$ are not assumed to be

of rectangular shape. We also apply our results to lozenge tilings (see Section 5.2) and domino tilings (Section 5.3) in a strip.¹

Finally, we give two conjectures for planar cylindrical networks in Section 6. The first one states that the recurrence polynomials in the second part of Theorem 2.3 have nonnegative integer coefficients (Conjecture 6.2) and positive real roots (Conjecture 6.3). The second one (Conjecture 6.5) asks whether these polynomials are always *minimal* if the network is strongly connected and has algebraically independent edge weights.

2. MAIN RESULTS

We start by briefly introducing our main objects of study. More precise definitions are given in Sections 3.2 and 4.4.

Consider an acyclic directed graph \tilde{N} drawn in some horizontal strip $\mathcal{S} = \{(x, y) \in \mathbb{R}^2 \mid A \leq y \leq B\}$ in the plane such that its vertex set \tilde{V} and edge set \tilde{E} are invariant with respect to the shift by some horizontal vector $\tilde{g} = (m, 0) \in \mathbb{R}^2$. Suppose in addition that we are given a shift-invariant function $\text{wt} : \tilde{E} \rightarrow K$ assigning weights from some field K of characteristic zero² to the edges of \tilde{N} . We call such a weighted directed graph a *cylindrical network*. We also define in an obvious way the *projection* N of \tilde{N} to the cylinder $\mathcal{O} = \mathcal{S}/\mathbb{Z}\tilde{g}$. Thus N is a weighted directed graph drawn in the cylinder. We require that all the vertices of \tilde{N} have finite degree and that for every directed path in \tilde{N} connecting a vertex \tilde{v} to its shift $\tilde{v} + \ell\tilde{g}$ we have $\ell > 0$.

We say that a cycle C in N is *simple* if it passes through each vertex of N at most once. For a simple cycle C in N , we define its *winding number* $\text{wind}(C)$ to be the unique integer ℓ such that any lift of C to a path in \tilde{N} connects a vertex \tilde{v} to $\tilde{v} + \ell\tilde{g}$. Thus for any cylindrical network \tilde{N} , *any cycle in N has a positive winding number*. An example of a cylindrical network can be found in Figure 1.

An r -cycle $\mathbf{C} = (C_1, \dots, C_r)$ in N is an r -tuple of pairwise vertex disjoint simple cycles in N . We set $\text{wt}(\mathbf{C}) = \text{wt}(C_1) \cdots \text{wt}(C_r)$ where $\text{wt}(C_i)$ is the product of the edges of C_i . We put

$$\text{wind}(\mathbf{C}) = \text{wind}(C_1) + \text{wind}(C_2) + \cdots + \text{wind}(C_r).$$

The set of all r -cycles in N is denoted by $\mathcal{C}^r(N)$.

¹We thank Per Alexandersson for pointing out his paper [2] to us.

²For us K will be either \mathbb{R} or the field of rational functions in several variables.

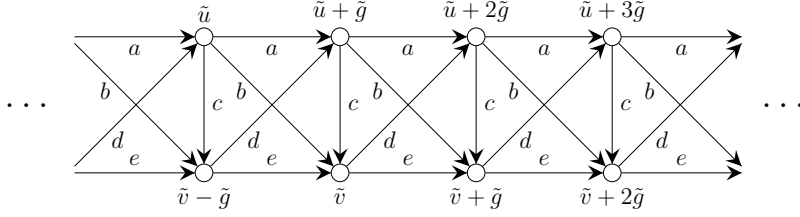


FIGURE 1. An example of a cylindrical network \tilde{N} . The edge weights are a, b, c, d, e . The network N has two vertices u and v .

We now define the polynomial $Q_N(t) \in K[t]$ by

$$(2.1) \quad Q_N(t) = \sum_{r=0}^d (-1)^{d-r} \sum_{\mathbf{C} \in \mathcal{C}^r(N)} t^{d-\text{wind}(\mathbf{C})} \text{wt}(\mathbf{C}).$$

Here the degree d of $Q_N(t)$ is the maximal winding number of an r -cycle in N for $r \geq 0$. It is clear that d is finite. Thus $Q_N(t)$ is a monic polynomial in t of degree d . For instance, for the network \tilde{N} in Figure 1, we get $Q_N(t) = t^2 - (a + e + cd)t + (ae - bd)$, see Example 3.5.

Given two vertices $\tilde{u}, \tilde{v} \in \tilde{V}$, define

$$\tilde{N}(\tilde{u}, \tilde{v}) = \sum_{\tilde{P}} \text{wt}(\tilde{P}),$$

where the sum is taken over all paths \tilde{P} in \tilde{N} that start at \tilde{u} and end at \tilde{v} , and the weight $\text{wt}(\tilde{P})$ of a path is the product of weights of its edges. We are ready to state our first result:

Theorem 2.1. *Consider a cylindrical network \tilde{N} . Let \tilde{u} and \tilde{v} be any two vertices of \tilde{N} . For $\ell \geq 0$, let $\tilde{v}_\ell = \tilde{v} + \ell\tilde{g} \in \tilde{V}$ be the shift of \tilde{v} , and define a sequence $f : \mathbb{N} \rightarrow K$ by $f(\ell) := \tilde{N}(\tilde{u}, \tilde{v}_\ell)$. Then for all but finitely many values of ℓ , the sequence f satisfies a linear recurrence with characteristic polynomial $Q_N(t)$.*

We say that \tilde{N} is a *planar cylindrical network* if \tilde{N} is drawn in the strip \mathcal{S} without self-intersections. More specifically, we require that the edges are drawn in a way that is shift-invariant so that N is also drawn in the cylinder \mathcal{O} without self-intersections. For planar cylindrical networks, it is easy to see that every simple cycle in N has winding number 1. Thus the formula (2.1) simplifies as follows:

$$(2.2) \quad Q_N(t) = \sum_{r=0}^d (-t)^{d-r} \sum_{\mathbf{C} \in \mathcal{C}^r(N)} \text{wt}(\mathbf{C}).$$

Therefore for a planar cylindrical network N , the coefficient of $(-t)^{d-r}$ in $Q_N(t)$ is a polynomial in the edge weights with nonnegative coefficients.

We now pass to the ‘‘Lindström-Gessel-Viennot’’ part of our results. A few more definitions are in order.

Definition 2.2. An r -vertex $\tilde{\mathbf{v}} = (\tilde{v}_1, \dots, \tilde{v}_r)$ in \tilde{N} is an r -tuple of distinct vertices of \tilde{N} . An r -path $\tilde{\mathbf{P}} = (\tilde{P}_1, \dots, \tilde{P}_r)$ is an r -tuple of paths in \tilde{N} that are pairwise vertex disjoint, and we set $\text{wt}(\tilde{\mathbf{P}}) = \text{wt}(\tilde{P}_1) \cdots \text{wt}(\tilde{P}_r)$. If for $1 \leq i \leq r$, the path \tilde{P}_i starts at \tilde{u}_i and ends at \tilde{v}_i then $\tilde{\mathbf{u}} = (\tilde{u}_1, \dots, \tilde{u}_r)$ and $\tilde{\mathbf{v}} = (\tilde{v}_1, \dots, \tilde{v}_r)$ are called the *start* and the *end* of $\tilde{\mathbf{P}}$. We denote by $\tilde{\mathcal{P}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ the collection of all r -paths in \tilde{N} that start at $\tilde{\mathbf{u}}$ and end at $\tilde{\mathbf{v}}$, and we set

$$\tilde{N}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) := \sum_{\tilde{\mathbf{P}} \in \tilde{\mathcal{P}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})} \text{wt}(\tilde{P}).$$

Given an r -vertex $\tilde{\mathbf{v}} = (\tilde{v}_1, \dots, \tilde{v}_r)$ and a permutation $\sigma \in \mathfrak{S}_r$ of $[r]$, we denote by $\sigma\tilde{\mathbf{v}} = (\tilde{v}_{\sigma(1)}, \dots, \tilde{v}_{\sigma(r)})$ the action of σ on $\tilde{\mathbf{v}}$.

For each $r \geq 1$, we introduce certain polynomials $Q_N^{[h_r]}(t)$ and $Q_N^{[e_r]}(t)$ of degrees $\binom{d}{r}$ and $\binom{d}{r}$ respectively. To give a precise definition, suppose that we are given the roots $\gamma_1, \gamma_2, \dots, \gamma_d \in \overline{K}$ of $Q_N(t)$, where \overline{K} denotes the algebraic closure of K . Thus we can write

$$Q_N(t) = (t - \gamma_1)(t - \gamma_2) \cdots (t - \gamma_d).$$

For $r \geq 1$, we set

$$(2.3) \quad \begin{aligned} Q^{[h_r]}(t) &= \prod_{1 \leq i_1 < i_2 < \cdots < i_r \leq d} (t - \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_r}); \\ Q^{[e_r]}(t) &= \prod_{1 \leq i_1 < i_2 < \cdots < i_r \leq d} (t - \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_r}). \end{aligned}$$

For example, $Q^{[h_1]}(t) = Q^{[e_1]}(t) = Q(t)$ and $Q^{[e_d]}(t) = t - \alpha_d$, where α_d denotes the constant term of $Q(t)$. On the other hand, $Q^{[h_d]}(t)$ is a polynomial of degree $\binom{d}{d} = \binom{2d-1}{d}$. It is clear from (2.3) that $Q^{[h_r]}(t)$ is always divisible by $Q^{[e_r]}(t)$ in $\overline{K}[t]$. It is non-trivial to show that $Q^{[h_r]}(t)$, $Q^{[e_r]}(t)$, and their ratio all belong to $K[t]$. Their coefficients are polynomial expressions in the coefficients of $Q_N(t)$ described explicitly in terms of plethysms of symmetric functions, see Section 4.1.

Theorem 2.3. *Let \tilde{N} be a cylindrical network and let $\tilde{\mathbf{u}} = (\tilde{u}_1, \dots, \tilde{u}_r)$ and $\tilde{\mathbf{v}} = (\tilde{v}_1, \dots, \tilde{v}_r)$ be two r -vertices in \tilde{N} . For $\ell \geq 0$, let $\tilde{\mathbf{v}}_\ell = \tilde{\mathbf{v}} + \ell\tilde{\mathbf{g}} = (\tilde{v}_1 + \ell\tilde{g}_1, \dots, \tilde{v}_r + \ell\tilde{g}_r)$. Define the sequence $f : \mathbb{N} \rightarrow K$ by*

$$f(\ell) = \sum_{\sigma \in \tilde{\mathfrak{S}}_r} \text{sgn}(\sigma) \tilde{N}(\tilde{\mathbf{u}}, \sigma\tilde{\mathbf{v}}_\ell),$$

where $\text{sgn}(\sigma)$ denotes the sign of σ .

- (1) *For any cylindrical network \tilde{N} , the sequence f satisfies a linear recurrence with characteristic polynomial $Q_N^{[h_r]}(t)$.*
- (2) *If \tilde{N} is a planar cylindrical network, the sequence f satisfies a linear recurrence with characteristic polynomial $Q_N^{[e_r]}(t)$.*

In both of the cases, f satisfies the recurrence for all but finitely many values of ℓ .

3. LINEAR RECURRENCES FOR SINGLE PATHS

In this section, we first describe some standard material on linear recurrences in Section 3.1, then we define cylindrical networks and related notions rigorously in Section 3.2. We then proceed to the proof of Theorem 2.1 in Section 3.3.

3.1. Background on linear recurrences and characteristic polynomials. We briefly recall some well known algebraic facts, mostly following [25]. Let K be a field of characteristic zero and $f : \mathbb{N} \rightarrow K$ be a sequence of elements of K . We say that f *satisfies a linear recurrence* if there exist some elements $\alpha_1, \alpha_2, \dots, \alpha_d \in K$ such that for all but finitely many values of $n \in \mathbb{N}$ we have

$$f(n+d) - \alpha_1 f(n+d-1) + \dots + (-1)^d \alpha_d f(n) = 0.$$

The *characteristic polynomial* of this linear recurrence is the polynomial $Q(t) \in K[t]$ defined by

$$Q(t) = t^d - \alpha_1 t^{d-1} + \dots + (-1)^d \alpha_d.$$

Proposition 3.1 ([25, Theorem 4.1.1]). *A sequence $f : \mathbb{N} \rightarrow K$ satisfies a linear recurrence with characteristic polynomial $Q(t)$ if and only if there exists a polynomial $P(t) \in K[t]$ such that the generating function for f is a rational formal power series:*

$$\sum_{n \in \mathbb{N}} f(n)t^n = \frac{P(t)}{R(t)},$$

where the polynomials $R(t)$ and $Q(t)$ are palindromic images of each other:

$$t^d Q(1/t) = R(t).$$

The following is an obvious fact which we will repeatedly use:

Proposition 3.2. *Suppose that $f_1, f_2, \dots, f_r : \mathbb{N} \rightarrow K$ each satisfy a linear recurrence with characteristic polynomial $Q(t)$. Then for any $c_1, \dots, c_r \in K$, the sequence $f = (c_1 f_1 + c_2 f_2 + \dots + c_r f_r) : \mathbb{N} \rightarrow K$ defined by*

$$f(\ell) = c_1 f_1(\ell) + c_2 f_2(\ell) + \dots + c_r f_r(\ell)$$

satisfies a linear recurrence with the same characteristic polynomial $Q(t)$.

3.2. Cylindrical networks. Recall that $\mathcal{O} = \mathcal{S}/\mathbb{Z}\tilde{g}$ denotes the cylinder and denote the canonical projection map by $\text{proj} : \mathcal{S} \rightarrow \mathcal{O}$. Let $h : [0, 1] \rightarrow \mathcal{O}$ be a loop defined by $h(t) = \text{proj}(\tilde{g}t)$ for $t \in [0, 1]$. Thus the element $[h] \in \pi_1(\mathcal{O})$ generates the fundamental group $\pi_1(\mathcal{O})$. For a loop $p : [0, 1] \rightarrow \mathcal{O}$, define its *winding number* $\text{wind}(p)$ to be the unique integer ℓ such that $[h]^\ell = [p]$ in $\pi_1(\mathcal{O})$.

Definition 3.3. A triple $\tilde{N} = (\tilde{V}, \tilde{E}, \text{wt})$ is a *cylindrical network* if:

- (1) the *vertex set* \tilde{V} of \tilde{N} is a discrete subset of \mathcal{S} that is invariant with respect to the shift by \tilde{g} : $\tilde{V} = \tilde{V} + \tilde{g}$;
- (2) the *edge set* $\tilde{E} \subset (\tilde{V} \times \tilde{V})$ is a set of ordered pairs (\tilde{u}, \tilde{v}) of elements $\tilde{u}, \tilde{v} \in \tilde{V}$ such that if $\tilde{e} = (\tilde{u}, \tilde{v}) \in \tilde{E}$ then $\tilde{e} + \tilde{g} := (\tilde{u} + \tilde{g}, \tilde{v} + \tilde{g}) \in \tilde{E}$;
- (3) $\text{wt} : \tilde{E} \rightarrow K$ is an assignment of *weights* from some field K of characteristic zero to the edges of \tilde{N} satisfying $\text{wt}(\tilde{e} + \tilde{g}) = \text{wt}(\tilde{e})$;
- (4) for every vertex $\tilde{v} \in \tilde{V}$ of \tilde{N} , the sets $\text{out}_{\tilde{N}}(\tilde{v})$ and $\text{in}_{\tilde{N}}(\tilde{v})$ of outgoing and incoming edges of \tilde{v} in \tilde{N} are finite.
- (5) if for some $\ell \in \mathbb{Z}$ there exists a directed path in \tilde{N} from $\tilde{v} \in \tilde{V}$ to $\tilde{v} + \ell\tilde{g}$ then $\ell > 0$. In particular, \tilde{N} is acyclic.

We do not allow multiple edges in \tilde{N} , however, replacing several edges with the same endpoints by a single edge whose weight equals the sum of their weights does not change any of the quantities we are interested in.

We view every element $\tilde{e} = (\tilde{u}, \tilde{v}) \in \tilde{E}$ as a linear path $\tilde{e} : [0, 1] \rightarrow \mathcal{S}$ starting at \tilde{u} and ending at \tilde{v} defined by $\tilde{e}(t) = (1-t)\tilde{u} + t\tilde{v}$. Note that the paths corresponding to different edges may intersect each other.

We let $N = (V, E, \text{wt})$ be the *projection* of \tilde{N} defined by $V = \text{proj}(\tilde{V})$, $E = \text{proj}(\tilde{E})$, and $\text{wt} : E \rightarrow K$ is defined via $\text{wt}(\text{proj}(\tilde{e})) = \text{wt}(\tilde{e})$. Since \tilde{V} is discrete, the set V is finite, and the set E is finite since the degrees of the vertices of \tilde{N} are finite.

For $\tilde{u}, \tilde{v} \in \tilde{V}$, define

$$\tilde{N}(\tilde{u}, \tilde{v}) = \sum_{\tilde{P} \in \tilde{\mathcal{P}}(\tilde{u}, \tilde{v})} \text{wt}(\tilde{P}),$$

where the sum is taken over the set $\tilde{\mathcal{P}}(\tilde{u}, \tilde{v})$ of all directed paths \tilde{P} from \tilde{u} to \tilde{v} in \tilde{N} . It is easy to see that for any two vertices $\tilde{u}, \tilde{v} \in \tilde{V}$, the set $\tilde{\mathcal{P}}(\tilde{u}, \tilde{v})$ is finite.

Recall that if C is a directed cycle in N , we say that C is *simple* if each vertex of N occurs in it at most once. Thus C has a positive winding number $\text{wind}(C) \geq 1$.

3.3. Linear recurrences for single paths. In this section, we prove Theorem 2.1.

For each vertex $v \in V$, choose its *lift* $l(v) \in \tilde{V}$ arbitrarily so that $\text{proj}(l(v)) = v$. Let $L = \{l(v) \mid v \in V\}$ and denote by $\tilde{y}_1, \dots, \tilde{y}_p$ the elements of L in some order.

Recall that the map wt takes values in some field K . For simplicity, we assume in this section that $K = \mathbb{Q}(\mathbf{x})$ is the field of rational functions in variables $\mathbf{x} = (x_e)_{e \in E}$ and $\text{wt}(\tilde{e}) = x_{\text{proj}(\tilde{e})}$. Define a $p \times p$ matrix $B(t)$ with entries $b_{ij}(t) \in \mathbb{Z}[\mathbf{x}; t^{\pm 1}]$ as follows:

$$(3.1) \quad b_{ij}(t) = \sum_{\tilde{e} = (\tilde{y}_i, \tilde{y}_j + \ell \tilde{g}) \in \tilde{E}} t^\ell \text{wt}(\tilde{e}),$$

where the sum is taken over all edges $\tilde{e} \in \tilde{E}$ that connect \tilde{y}_i to some vertex $\tilde{y}_j + \ell \tilde{g}$ in $\tilde{y}_j + \mathbb{Z}\tilde{g}$. By part (4) of Definition 3.3, there are only finitely many such edges in \tilde{N} .

We denote

$$(3.2) \quad Q_L(t) = \det(\text{Id} - B(t)); \quad A(t) = (\text{Id} - B(t))^{-1} = \sum_{r \geq 0} B(t)^r.$$

Thus $Q_L(t)$ is a certain polynomial in $\mathbb{Z}[\mathbf{x}; t^{\pm 1}]$ and $A(t)$ is a matrix with entries in the field of rational functions $\mathbb{Q}(\mathbf{x}; t)$. It follows that the (i, j) -th entry $a_{ij}(t)$ of $A(t)$ equals

$$a_{ij}(t) = \sum_{\tilde{P}} t^\ell \text{wt}(\tilde{P}),$$

where the sum is taken over all paths that start at \tilde{y}_i and end at $\tilde{y}_j + \ell \tilde{g}$ for some $\ell \in \mathbb{Z}$. Note also that the series expansion (3.2) of $A(t)$ is a well-defined power series because all the cycles in N have positive winding number and thus $B(t)^r$ is divisible by large powers of t for large values of r . This also shows that $\text{Id} - B(t)$ is an invertible matrix

because its inverse $A(t)$ is well defined. In fact, one can write down a formula for the determinant $Q_L(t)$ of $\text{Id} - B(t)$ explicitly:

Proposition 3.4. *The polynomial $Q_L(t)$ does not depend on the choice of L . We have*

$$(3.3) \quad Q_L(t) = (-t)^d Q_N(1/t),$$

where the polynomial $Q_N(t)$ is given by (2.1).

Proof. By definition, we have

$$Q_L(t) = \det(\text{Id} - B(t)) = \sum_{\sigma \in \mathfrak{S}_n} \text{sgn}(\sigma) H_\sigma,$$

where $H_\sigma = \prod_{i=1}^p (\text{Id} - B(t))_{i, \sigma(i)}$. Decomposing σ into cycles immediately yields (3.3). \square

Example 3.5. Consider the network \tilde{N} in Figure 1 together with a choice of $L = \{\tilde{y}_1 = \tilde{u}, \tilde{y}_2 = \tilde{v}\}$.

We have

$$B(t) = \begin{pmatrix} at & b + ct^{-1} \\ dt^2 & et \end{pmatrix}; \quad \text{Id} - B(t) = \begin{pmatrix} 1 - at & -b - ct^{-1} \\ -dt^2 & 1 - et \end{pmatrix}.$$

Thus

$$Q_L(t) = \det(\text{Id} - B(t)) = 1 - (a + e + cd)t + (ae - bd)t^2.$$

We see that in (2.1), there is one (empty) 0-cycle with weight t^2 , four 1-cycles with weights $-at$, $-et$, $-cdt$, $-bd$, and one 2-cycle with weight ae , and thus

$$Q_N(t) = t^2 - (a + e + cd)t + (ae - bd).$$

Proof of Theorem 2.1. We can choose L so that $\tilde{y}_i = \tilde{u}$ and $\tilde{y}_j = \tilde{v}$ for some $1 \leq i, j \leq p$. Let $(g_{ij}(\ell_0), g_{ij}(\ell_0 + 1), \dots)$ be a sequence of polynomials in \mathbf{x} defined by

$$a_{ij}(t) = \sum_{\ell \geq \ell_0} g_{ij}(\ell) t^\ell.$$

Then clearly we have $f(\ell) = g_{ij}(\ell)$ for $\ell \geq 0$. Since $a_{ij}(t)$ is a rational function with denominator $Q_L(t)$, it follows by Propositions 3.1 and 3.4 that the sequence f satisfies a linear recurrence with characteristic polynomial $Q_N(t)$. \square

4. LINEAR RECURRENCES FOR TUPLES OF PATHS

In the previous section, we have established Theorem 2.1 that gave a linear recurrence relation for single paths in \tilde{N} . We now want to give a proof to Theorem 2.3 for r -paths in \tilde{N} .

4.1. Background on plethysms. Consider a monic polynomial

$$Q(t) \in K[t]; \quad Q(t) = t^d - \alpha_1 t^{d-1} + \cdots + (-1)^d \alpha_d.$$

It factors as a product of linear terms in the algebraic closure \overline{K} of K : $Q(t) = \prod_{j=1}^d (t - \gamma_j)$, where $\gamma_1, \dots, \gamma_d \in \overline{K}$. The coefficient α_k of $Q(t)$ equals

$$\alpha_k = e_k(\gamma_1, \dots, \gamma_d) := \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq d} \gamma_{i_1} \cdots \gamma_{i_k}.$$

Here e_k is the k -th *elementary symmetric polynomial*, see [24, Section 7.4]. Recall also that the *complete homogeneous symmetric polynomial* $h_k(\gamma_1, \dots, \gamma_d)$ is given by

$$h_k(\gamma_1, \dots, \gamma_d) := \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq d} \gamma_{i_1} \cdots \gamma_{i_k}.$$

Let $r \geq 1$ be an integer and consider the polynomials

$$(4.1) \quad Q^{[h_r]}(t) := \prod_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq d} (t - \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_k});$$

$$(4.2) \quad Q^{[e_r]}(t) := \prod_{1 \leq j_1 < j_2 < \cdots < j_k \leq d} (t - \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_k}).$$

The coefficient of t^{D-k} (where D is the degree of the corresponding polynomial) in $Q^{[h_r]}(t)$ equals $(-1)^k e_k[h_r](\gamma_1, \dots, \gamma_d)$, where the polynomial $e_k[h_r]$ denotes the *plethysm* of e_k with h_r . Similarly, the coefficient of t^{D-k} in $Q^{[e_r]}(t)$ equals $(-1)^k e_k[e_r](\gamma_1, \dots, \gamma_d)$. We refer the reader to [24, Definition A2.6] for the definition of a plethysm. Loosely speaking, given two symmetric functions f and g , the *plethysm* $f[g]$ of f with g is a symmetric function obtained from f by substituting the monomials of g into the variables of f .

Since $e_k[h_r]$ and $e_k[e_r]$ are again symmetric functions, they can be expressed as polynomials in the elementary symmetric functions e_i by the fundamental theorem of symmetric functions [24, Theorem 7.4.4]. Since $e_i(\gamma_1, \dots, \gamma_d) = \alpha_i$ is the coefficient of t^{d-i} in $Q(t)$, we get that the coefficients of $Q^{[h_r]}(t)$ and $Q^{[e_r]}(t)$ are polynomial expressions in the coefficients of $Q(t)$ and thus $Q^{[h_r]}(t), Q^{[e_r]}(t) \in K[t]$ rather than $\overline{K}[t]$.

Example 4.1. Let $Q(t) = t^3 - \alpha_1 t^2 + \alpha_2 t - \alpha_3$. Then

$$(4.3) \quad Q^{[e_2]}(t) = t^3 - \alpha_2 t^2 + \alpha_3 \alpha_1 t - \alpha_3^3.$$

For example, we get $\alpha_3 \alpha_1 t$ because we have the identity $e_2[e_2] = e_3 e_1 - e_4$ for symmetric functions and since $e_4(\gamma_1, \gamma_2, \gamma_3, 0, 0, \dots) = 0$, we have

$$e_2[e_2](\gamma_1, \gamma_2, \gamma_3) = (e_3 e_1)(\gamma_1, \gamma_2, \gamma_3).$$

Indeed,

$$\begin{aligned} e_2[e_2](\gamma_1, \gamma_2, \gamma_3) &= e_2(\gamma_1 \gamma_2, \gamma_1 \gamma_3, \gamma_2 \gamma_3) = \gamma_1 \gamma_2 \gamma_1 \gamma_3 + \gamma_1 \gamma_2 \gamma_2 \gamma_3 + \gamma_1 \gamma_3 \gamma_2 \gamma_3 \\ &= \gamma_1 \gamma_2 \gamma_3 (\gamma_1 + \gamma_2 + \gamma_3) = e_3(\gamma_1, \gamma_2, \gamma_3) e_1(\gamma_1, \gamma_2, \gamma_3). \end{aligned}$$

Similarly,

$$\begin{aligned} Q^{[h_2]}(t) &= t^6 - (\alpha_1^2 - \alpha_2) t^5 + (\alpha_2 \alpha_1^2 - \alpha_2^2 - \alpha_3 \alpha_1) t^4 \\ &\quad - (\alpha_2^2 + \alpha_3 \alpha_1^3 - 4 \alpha_3 \alpha_2 \alpha_1 + 2 \alpha_3^2) t^3 + (\alpha_3 \alpha_2^2 \alpha_1 - \alpha_3^2 \alpha_1^2 - \alpha_3^2 \alpha_2) t^2 \\ &\quad - (\alpha_3^2 \alpha_2^2 - \alpha_3^3 \alpha_1) t + \alpha_3^4. \end{aligned}$$

One easily observes that the degree of $Q^{[h_r]}(t)$ is $\binom{d}{r} = \binom{d+r-1}{r}$ while the degree of $Q^{[e_r]}(t)$ is $\binom{d}{r}$. For example, $Q^{[h_d]}(t)$ has degree $\binom{2d-1}{d}$ but $Q^{[e_d]}(t) = t - \alpha_d$ has degree 1. It is obvious from the definitions that $Q^{[h_r]}(t)$ is always divisible by $Q^{[e_r]}(t)$ in $\overline{K}[t]$, and by an argument similar to the one above, their ratio even belongs to $K[t]$.

The following proposition follows from [25, Propositions 4.2.2 and 4.2.5].

Proposition 4.2. *Suppose that the sequences $f_1, f_2, \dots, f_r : \mathbb{N} \rightarrow K$ each satisfy a linear recurrence with characteristic polynomial $Q(t)$. Then the sequence $f = (f_1 f_2 \cdots f_r) : \mathbb{N} \rightarrow K$ defined by*

$$f(\ell) = f_1(\ell) f_2(\ell) \cdots f_r(\ell)$$

satisfies a linear recurrence with characteristic polynomial $Q^{[h_r]}(t)$.

Let now A be an $n \times n$ matrix over K . We view A as a linear map $A : W \rightarrow W$ where $W = K^n$. Let w_1, \dots, w_n be the basis of W . The r -th exterior power of W is the linear space $\Lambda^r(W)$ with basis

$$\{w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_r} \mid 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}.$$

The r -th exterior power of A is the linear map $\Lambda^r(A) : \Lambda^r(W) \rightarrow \Lambda^r(W)$ defined on every basis element by

$$(4.4) \quad \Lambda^r(A)(w_{i_1} \wedge w_{i_2} \wedge \cdots \wedge w_{i_r}) = (Aw_{i_1}) \wedge (Aw_{i_2}) \wedge \cdots \wedge (Aw_{i_r}).$$

Here we use the multilinearity and antisymmetry of \wedge to expand the right hand side of (4.4) in the basis of $\Lambda^r(W)$.

Equivalently, $\Lambda^r(A)$ is an $\binom{n}{r} \times \binom{n}{r}$ matrix whose rows and columns are indexed by r -element subsets of $[n] := \{1, 2, \dots, n\}$ and for two such subsets I, J , the corresponding entry of $\Lambda^r(A)$ equals the value of the minor of A with rows I and columns J .

The following fact is an application of the general theory of λ -rings, see e.g. [22, Section 4]:

Proposition 4.3. *Let $Q(t) \in K[t]$ be the characteristic polynomial of A . Then the characteristic polynomial of $\Lambda^r(A)$ is $Q^{[er]}(t)$.*

Example 4.4. Let

$$A = \begin{pmatrix} a & 0 & d \\ 0 & b & e \\ 0 & f & c \end{pmatrix}.$$

Then the characteristic polynomial of A is

$$Q(t) = \det(t \text{Id} - A) = t^3 - (a+b+c)t^2 + (ab+bc+ac-ef)t - (abc-ae f).$$

Order the two-element subsets of $\{1, 2, 3\}$ as $(\{1, 2\}, \{1, 3\}, \{2, 3\})$. Then for this ordering we can write

$$\Lambda^2(A) = \begin{pmatrix} ab & ae & -bd \\ af & ac & -df \\ 0 & 0 & bc - ef \end{pmatrix}.$$

Thus the characteristic polynomial of $\Lambda^2(A)$ is

$$(t - (bc - ef)) ((t - ab)(t - ac) - a^2ef).$$

We encourage the reader to check that this polynomial equals to $Q^{[e2]}(t)$, i.e. the polynomial given by (4.3) for

$$\alpha_1 = a + b + c; \quad \alpha_2 = ab + bc + ac - ef; \quad \alpha_3 = abc - ae f.$$

Another well-known property of the exterior power is its multiplicativity: for two $n \times n$ matrices A, B , we have

$$(4.5) \quad \Lambda^r(AB) = (\Lambda^r(A))(\Lambda^r(B)).$$

This is an obvious consequence of (4.4).

4.2. Lindström-Gessel-Viennot theorem. We give a short background on the Lindström-Gessel-Viennot method introduced in [16, 21] adapted to the case of cylindrical networks.

Let \tilde{N} be a cylindrical network and consider two r -vertices $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_r)$ and $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, \dots, \tilde{v}_r)$ in \tilde{N} . Let $A(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = (a_{ij})$ be the $r \times r$ matrix whose entries are given by

$$a_{ij} = \tilde{N}(\tilde{u}_i, \tilde{v}_j).$$

The Lindström-Gessel-Viennot theorem gives a combinatorial interpretation to the determinant of $A(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$.

Theorem 4.5 ([16]). *We have*

$$(4.6) \quad \det A(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \sum_{\sigma \in \mathfrak{S}_r} \text{sgn}(\sigma) \tilde{N}(\tilde{\mathbf{u}}, \sigma \tilde{\mathbf{v}}).$$

This theorem can be proven using a simple path-cancelling argument. We refer the reader to [16] for the details.

Example 4.6. For the network \tilde{N} in Figure 1, consider $\tilde{\mathbf{u}} = (\tilde{u}, \tilde{v} - \tilde{g})$ and $\tilde{\mathbf{v}} = (\tilde{u} + \tilde{g}, \tilde{v})$. Then we have

$$A(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \begin{pmatrix} a + cd & b + ce + ac + c^2d \\ d & e + cd \end{pmatrix}.$$

Therefore

$$\det A(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = (a + cd)(e + cd) - d(b + ce + ac + c^2d) = ae - bd.$$

This agrees with (4.6) since there is just one 2-path in $\tilde{\mathcal{P}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}})$ with weight ae and one 2-path in $\tilde{\mathcal{P}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}')$ with weight bd , where $\tilde{\mathbf{v}}' = (\tilde{v}, \tilde{u} + \tilde{g}) = \sigma \tilde{\mathbf{v}}$ for $\sigma = (12)$ being the unique transposition in \mathfrak{S}_2 .

4.3. A recurrence for tuples of paths.

Proof of Theorem 2.3, part (1). Let $a_{ij}^{(\ell)}$ be defined so that $A(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}_\ell) = (a_{ij}^{(\ell)})_{i,j=1}^r$. Then by Theorem 2.1, for all $1 \leq i, j \leq r$, the sequence $(a_{ij}^{(\ell)})_{\ell=0}^\infty$ satisfies a linear recurrence with characteristic polynomial $Q_N(t)$. Since by Theorem 4.5, $f(\ell)$ is a linear combination of r -term products of $a_{ij}^{(\ell)}$'s, the result follows by Propositions 4.2 and 3.2. \square

Recall that the polynomial $Q_N^{[er]}(t)$ in general divides the polynomial $Q_N^{[hr]}(t)$ and has a much smaller degree for large r . We now would like to prove the second part of Theorem 2.3.

Definition 4.7. We say that a cylindrical network \tilde{N} is *local* if one can choose a lifting L of V to \tilde{N} so that the entries of the matrix $B(t)$ given by (3.1) would be *linear polynomials* in t , i.e. $B(t) = C + tD$ for some matrices C, D whose entries do not depend on t . In other words, \tilde{N} is *local* if there exists a lifting L of V such that every edge that starts at L ends either at L or at $L + \tilde{g}$. In this case, we say that L is a *local lifting* for \tilde{N} .

We will later see in Proposition 4.13 that every planar cylindrical network \tilde{N} is local. However, this property is more general:

Example 4.8. The cylindrical network \tilde{N} in Figure 1 is local. Indeed, if we choose a different lift $L = \{\tilde{u} + \tilde{g}, \tilde{v}\}$ then the matrix $B(t)$ becomes

$$B(t) = \begin{pmatrix} at & c + bt \\ dt & et \end{pmatrix}.$$

Thus we have

$$B(t) = C + tD, \quad \text{where } C = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} a & b \\ d & e \end{pmatrix}.$$

Theorem 4.9. *Suppose that \tilde{N} is a local network and let L be a local lifting for \tilde{N} . Let $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ be two r -vertices in \tilde{N} . Then the sequence $f(\ell)$ from Theorem 2.3 satisfies a linear recurrence with characteristic polynomial $Q_N^{[e_r]}(t)$.*

Proof. Let $B(t) = C + tD$. Then since \tilde{N} is acyclic, the matrix C has to be nilpotent, and therefore $(\text{Id} - C)^{-1} = \text{Id} + C + \dots + C^l$ for some $l \geq 0$. We are interested in the matrix $A(t) = (\text{Id} - B(t))^{-1} = (\text{Id} - C - tD)^{-1}$. Let

$$(4.7) \quad S = (\text{Id} - C)^{-1}D,$$

then we have

$$(\text{Id} - C - tD) = (\text{Id} - C)(\text{Id} - t(\text{Id} - C)^{-1}D) = (\text{Id} - C)(\text{Id} - tS),$$

and thus

$$(4.8) \quad A(t) = (\text{Id} - tS)^{-1}(\text{Id} - C)^{-1}.$$

In particular,

$$(4.9) \quad Q_L(t) = \det(\text{Id} - B(t)) = \det(\text{Id} - tS),$$

since $\det(\text{Id} - C) = 1$ for any nilpotent matrix C . Thus $Q_N(t) = t^d Q_L(1/t)$ is the characteristic polynomial $\det(t\text{Id} - S)$ of S , possibly multiplied by a power of t .

Let us denote

$$A(t) = \sum_{\ell \geq \ell_0} A^{(\ell)} t^\ell.$$

Thus the entry $a_{ij}^{(\ell)}$ of $A^{(\ell)}$ counts the paths from \tilde{y}_i to $\tilde{y}_j + \ell\tilde{g}$ in \tilde{N} . Using (4.8), we get

$$A^{(\ell)} = S^\ell (\text{Id} - C)^{-1}.$$

We first consider the case when all the vertices of $\tilde{\mathbf{u}}$ and of $\tilde{\mathbf{v}}$ belong to L . Consider the sequence $f : \mathbb{N} \rightarrow K$ from Theorem 2.3. By Theorem 4.5, $f(\ell)$ is a certain $r \times r$ minor of the matrix $A^{(\ell)}$, or

equivalently, it is a certain entry of the matrix $\Lambda^r(A^{(\ell)})$. Using the multiplicativity (4.5) of the exterior power, we get

$$\Lambda^r(A^{(\ell)}) = (\Lambda^r(S))^\ell (\Lambda^r((\text{Id} - C)^{-1})).$$

Define the matrix

$$A^{(\wedge^r)}(t) = \sum_{\ell \geq \ell_0} t^\ell (\Lambda^r(A^{(\ell)})) = (\text{Id} - t(\Lambda^r(S)))^{-1} (\Lambda^r((\text{Id} - C)^{-1})).$$

The generating function $\sum_{\ell} t^\ell f(\ell)$ for f appears as an entry in $A^{(\wedge^r)}(t)$ and therefore is a rational function with denominator $\det(\text{Id} - t(\Lambda^r(S)))$ which is just the palindromic image of the characteristic polynomial of $\Lambda^r(S)$. By Proposition 4.3, this characteristic polynomial equals $Q_N^{[e_r]}(t)$ since $Q_N(t)$ is the characteristic polynomial of S by (4.9).

We are done with the case when all the vertices of $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ belong to L . We are going to deduce the general case as a simple consequence. Consider any r -path $\tilde{\mathbf{P}} = (\tilde{P}_1, \dots, \tilde{P}_r)$ from $\tilde{\mathbf{u}}$ to $\sigma\tilde{\mathbf{v}}_\ell$. We claim that it can be decomposed as a concatenation of three r -paths $\tilde{\mathbf{P}}^{(1)}, \tilde{\mathbf{P}}^{(2)}, \tilde{\mathbf{P}}^{(3)}$ in such a way that the endpoints $\tilde{\mathbf{u}}'$ and $\tilde{\mathbf{v}}'$ of $\tilde{\mathbf{P}}^{(2)}$ satisfy the following: all vertices of $\tilde{\mathbf{u}}'$ belong to $L + \ell_1\tilde{g}$ and all vertices of $\tilde{\mathbf{v}}'$ belong to $L + (\ell + \ell_2)\tilde{g}$, for some constants ℓ_1 and ℓ_2 . Indeed, consider a path \tilde{P}_i of $\tilde{\mathbf{P}}$ from \tilde{u}_i to $\tilde{v}_{\sigma(i)} + \ell\tilde{g}$. Let $(\tilde{p}_1, \tilde{p}_2, \dots, \tilde{p}_M)$ be the vertices of \tilde{P}_i , and define $z(\tilde{p}_i)$ to be the unique integer z so that $\tilde{p}_i \in L + z\tilde{g}$. Then the sequence $(z(\tilde{p}_i))_{i=1}^M$ is weakly increasing and *Lipschitz*, i.e.

$$z(\tilde{p}_i) \leq z(\tilde{p}_{i+1}) \leq z(\tilde{p}_i) + 1.$$

Let

$$z_0 = \min\{z(\tilde{u}_i) \mid 1 \leq i \leq r\}, \quad z'_0 = \max\{z(\tilde{u}_i) \mid 1 \leq i \leq r\};$$

$$z_1 = \min\{z(\tilde{v}_i) \mid 1 \leq i \leq r\}, \quad z'_1 = \max\{z(\tilde{v}_i) \mid 1 \leq i \leq r\}.$$

Then we can define the decomposition of \tilde{P}_i into three parts as follows:

- $\tilde{P}_i^{(1)}$ consists of vertices \tilde{p}_i for which $z_0 \leq z(\tilde{p}_i) < z'_0$;
- $\tilde{P}_i^{(2)}$ consists of vertices \tilde{p}_i for which $z'_0 \leq z(\tilde{p}_i) \leq z_1 + \ell$;
- $\tilde{P}_i^{(3)}$ consists of vertices \tilde{p}_i for which $z_1 + \ell < z(\tilde{p}_i) \leq z'_1 + \ell$.

Indeed, if we set $\ell_1 = z'_0$ and $\ell_2 = z_1$ then $\tilde{P}_i^{(2)}$ starts at a vertex from $L + \ell_1\tilde{g}$ and ends at a vertex from $L + (\ell + \ell_2)\tilde{g}$. It is clear that the total number of choices for $\tilde{\mathbf{P}}^{(1)}$ and $\tilde{\mathbf{P}}^{(3)}$ is finite, and for each such choice the corresponding sequence f' for the points $\tilde{\mathbf{u}}'$ and $\tilde{\mathbf{v}}'$ satisfies a linear recurrence with characteristic polynomial $Q_N^{[e_r]}$. Thus we are done by Proposition 3.2. \square

Even though we can prove Theorem 4.9 for only local cylindrical networks, we suspect that it holds for *any* cylindrical network:

Conjecture 4.10. If \tilde{N} is a cylindrical network and $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ are two r -vertices then the sequence $f(\ell)$ from Theorem 2.3 satisfies a linear recurrence with characteristic polynomial $Q_N^{[e_r]}(t)$ for all but finitely many ℓ .

4.4. Planar cylindrical networks. Even though Theorem 4.5 holds for all networks, in the majority of the situations it is applied to *planar* networks.

Definition 4.11. We say that a cylindrical network \tilde{N} is *planar* if for each edge $\tilde{e} = (\tilde{u}, \tilde{v}) \in \tilde{E}$ there is an embedding $h_{\tilde{e}} : [0, 1] \rightarrow \mathcal{S}$ satisfying the following properties:

- $h_{\tilde{e}}(0) = \tilde{u}$ and $h_{\tilde{e}}(1) = \tilde{v}$;
- the interior $h_{\tilde{e}}((0, 1))$ of \tilde{e} does not intersect the image of any other $h_{\tilde{e}'}$ for $\tilde{e}' \in \tilde{E}$ and does not contain any vertex of \tilde{N} ;
- shift-invariance: $h_{\tilde{e}+\tilde{g}}(t) = h_{\tilde{e}}(t) + \tilde{g}$ for all $\tilde{e} \in \tilde{E}$ and $t \in [0, 1]$.

First, we prove a simpler formula (2.2) for $Q_N(t)$ in the planar case:

Proposition 4.12. *If \tilde{N} is a planar cylindrical network then $Q_N(t)$ is given by (2.2), that is,*

$$Q_N(t) = \sum_{r \geq 0} (-t)^{d-r} \sum_{\mathbf{C} \in \mathcal{C}^r(N)} \text{wt}(\mathbf{C}).$$

Proof. As we have already mentioned in Section 2, it suffices to show every simple cycle in a planar cylindrical network has winding number 1. Indeed, every simple cycle in N represents a non-self-intersecting loop in the cylinder \mathcal{O} with a positive winding number which therefore has to be equal to 1. \square

By Theorem 4.9, in order to prove the second part of Theorem 2.3 it suffices to show the following:

Proposition 4.13. *Every planar cylindrical network \tilde{N} is local.*

Proof. Showing that \tilde{N} is local amounts to constructing a function $z : \tilde{V} \rightarrow \mathbb{Z}$ such that $z(\tilde{v} + \tilde{g}) = z(\tilde{v}) + 1$ and such that for every edge $\tilde{e} = (\tilde{u}, \tilde{v})$ of \tilde{N} we have

$$(4.10) \quad z(\tilde{u}) \leq z(\tilde{v}) \leq z(\tilde{u}) + 1.$$

Given such a function, it is clear that the set $L := z^{-1}(0)$ is a local lift for \tilde{N} . We prove the existence of z by induction on the number of vertices in N . Throughout, we assume that \tilde{N} is connected since

it suffices to prove the result for any connected component of \tilde{N} . The base case is when there is just one vertex v in N , so let \tilde{v}_0 be any of its lifts and define $\tilde{v}_\ell = \tilde{v}_0 + \ell\tilde{g}, \ell \in \mathbb{Z}$ to be the remaining lifts of v . We claim that all the edges coming out of \tilde{v}_0 end at \tilde{v}_1 . Indeed, suppose that $\tilde{e} = (\tilde{v}_0, \tilde{v}_\ell)$ is an edge that ends at some \tilde{v}_ℓ . Then its projection is a simple loop in the cylinder \mathcal{O} with winding number ℓ . As we have noted earlier, this can only happen when $\ell = 1$. Thus we get $\ell = 1$ for every edge $\tilde{e} = (\tilde{v}_0, \tilde{v}_\ell)$, and it suffices to set $z(\tilde{v}_k) = k$ for all $k \in \mathbb{Z}$ to complete the base case.

To do the induction step, consider a general connected planar cylindrical network \tilde{N} . An edge $\tilde{e} = (\tilde{u}, \tilde{v})$ is called a *cover relation* in \tilde{N} if $\text{proj}(\tilde{u}) \neq \text{proj}(\tilde{v})$ and there is no path in \tilde{N} with at least two edges that starts at \tilde{u} and ends at \tilde{v} . It is clear that if N is connected and has at least two vertices then such a cover relation exists in \tilde{N} by part (4) of Definition 3.3. So let $\tilde{e} = (\tilde{u}, \tilde{v})$ be a cover relation in \tilde{N} . We can contract this edge \tilde{e} and all of its shifts, and this operation produces a smaller connected planar cylindrical network \tilde{N}' for which we already have a function z' satisfying (4.10). Let us put $z(\tilde{w}) = z'(\tilde{w}')$ for any vertex $\tilde{w} \in \tilde{V}$, where \tilde{w}' is the vertex in \tilde{N}' corresponding to \tilde{w} . It is clear that this defines a function $z : \tilde{V} \rightarrow \mathbb{Z}$ satisfying (4.10). We are done with the proof. \square

5. APPLICATIONS

5.1. A recurrence for Schur polynomials. Since the second part of Theorem 2.3 holds for any planar cylindrical network, we first demonstrate how it yields new results in one especially well-studied case of *Schur polynomials*.

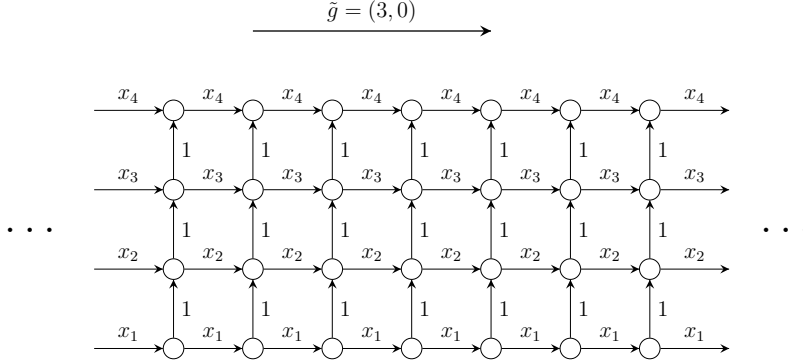
Fix two integers $n, m \geq 1$ and consider the following planar cylindrical network $\tilde{N}_{n,m}$. Its set of vertices \tilde{V} is identified with $\mathbb{Z} \times [n]$. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a family of indeterminates. A vertex $\tilde{v} = (i, j)$ with $j \in [n] = \{1, 2, \dots, n\}$ is connected to $(i+1, j)$ by an edge of weight x_j and, assuming $j < n$, it is connected to $(i, j+1)$ by an edge of weight 1. We set $\tilde{g} := (m, 0)$. This defines the network $\tilde{N}_{n,m}$ whose projection $N_{n,m}$ is a cylinder $[n] \times \mathbb{Z}_m$ of height n and width m . See Figure 2.

It is clear from (2.2) that the polynomial $Q_{N_{n,m}}(t)$ is equal to

$$Q_{N_{n,m}}(t) = (t - x_1^m)(t - x_2^m) \cdots (t - x_n^m).$$

The polynomial $Q_{N_{n,m}}(t)$ has degree $d = n$ and the coefficient of t^{d-r} in it equals $(-1)^r e_r(x_1^m, \dots, x_n^m)$.

Let now $1 \leq r \leq n$ be an integer. A *partition with at most r parts* is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq 0)$ of

FIGURE 2. The network $\tilde{N}_{n,m}$ for $n = 4$ and $m = 3$.

r nonnegative integers. To each such partition one can associate a certain symmetric polynomial $s_\lambda(x_1, x_2, \dots, x_n)$ called the *Schur polynomial* of λ , see [24, Section 7.10] for the definition. Given a partition λ with at most r parts, we introduce an r -vertex $\tilde{\mathbf{v}}(\lambda) = ((\lambda_r + 1, n), (\lambda_{r-1} + 2, n), \dots, (\lambda_1 + r, n))$ in $\tilde{N}_{n,m}$. We also fix an r -vertex $\tilde{\mathbf{u}} = ((1, 1), (2, 1), \dots, (r, 1))$. In terms of our network $\tilde{N}_{n,m}$, the Schur polynomial $s_\lambda(x_1, \dots, x_n)$ equals

$$s_\lambda(x_1, x_2, \dots, x_n) = \tilde{N}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}(\lambda)).$$

We define the sum of two partitions $\lambda + \mu$ componentwise, that is, $(\lambda + \mu)_i = \lambda_i + \mu_i$, similarly, $\ell\mu$ denotes a partition with $(\ell\mu)_i = \ell\mu_i$. We fix a partition $\mu = m^r := (m, m, \dots, m)$ with r parts, in other words, the *Young diagram* of μ is an $m \times r$ rectangle.

Theorem 5.1. *The sequence $f(\ell) = s_{\lambda + \ell\mu}(x_1, \dots, x_n)$ satisfies a linear recurrence for all but finitely many ℓ with characteristic polynomial $Q_{N_{n,m}}^{[e_r]}(t)$.*

Proof. By Proposition 4.13, the planar cylindrical network \tilde{N} is local. It is clear that the only permutation σ that gives a non-zero contribution in (4.6) is the identity (in this case the r -vertices $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are called *non-permutable*, see [16]). Thus the sequence $f(\ell)$ is exactly a sequence satisfying the assumptions of Theorem 4.9 and the result follows. \square

The polynomial $Q_{N_{n,m}}^{[e_r]}(t)$ has $\binom{n}{r}$ roots of the form $(x_{i_1}x_{i_2} \cdots x_{i_r})^m$ for all $1 \leq i_1 < i_2 < \cdots < i_r \leq n$.

Remark 5.2. Theorem 5.1 extends trivially to the sequence $f(\ell) = s_{(\lambda + \ell\mu)/\kappa}(x_1, \dots, x_n)$ of *skew Schur polynomials* for any partitions $\lambda, \mu, \nu, \kappa$ with at most r parts. More generally, given four partitions $\lambda, \mu, \nu, \kappa$, one can consider a sequence $f(\ell) = s_{(\lambda + \ell\mu)/(\kappa + \ell\nu)}(x_1, \dots, x_n)$. It was

shown by Alexandersson [1] that this sequence satisfies (for all but finitely many ℓ under a mild condition on $\lambda, \mu, \nu, \kappa$) a linear recurrence with characteristic polynomial

$$Q_{\mu/\nu}(t) = \prod_T (t - \mathbf{x}^{\text{wt}(T)}),$$

where T runs over all semistandard Young tableaux of shape μ/ν with entries in $[n]$. In [1, Conjecture 24], he conjectured that f satisfies a shorter linear recurrence with characteristic polynomial

$$P_{\mu/\nu}(t) = \prod_{\mathbf{w} \in W} (t - \mathbf{x}^{\mathbf{w}}),$$

where W is a certain subset of \mathbb{N}^n defined explicitly. In the case $\mu = m^r, \nu = 0^r = \emptyset$, it is easy to see that the polynomial $Q_{N_{n,m}^{[e_r]}}(t)$ equals the polynomial $P_{\mu/\nu}(t)$ and they both have degree $\binom{n}{r}$. This number is much smaller³ than the degree of $Q_{\mu/\nu}(t)$ which is the number of semistandard Young tableaux of rectangular shape m^r with entries in $[n]$. Thus Theorem 5.1 gives in this case a new, shorter linear recurrence which is conjectured in [1, Conjecture 24] to be the minimal recurrence for the sequence $f(\ell) = s_{(\lambda+\ell\mu)/\kappa}(x_1, \dots, x_n)$.

5.2. Lozenge tilings, plane partitions, and Carlitz q -Fibonacci polynomials. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, define its *Young diagram* $\mathcal{Y}(\lambda)$ to be the set of 1×1 “boxes” in the plane centered at (i, j) for every pair (i, j) satisfying $1 \leq i \leq m$ and $1 \leq j \leq \lambda_i$. We use the *English notation* and draw the boxes of $\mathcal{Y}(\lambda)$ using matrix coordinates, see e.g. Figure 3 (a). Consider two partitions λ, μ such that $\mathcal{Y}(\mu) \subset \mathcal{Y}(\lambda)$. Then we define the *skew shape* $\mathcal{Y}(\lambda/\mu)$ to be the set-theoretic difference $\mathcal{Y}(\lambda) \setminus \mathcal{Y}(\mu)$.

Fix four integers $a, b, c, d \geq 1$ such that $a + b = c + d$. Then for each $\ell \geq 0$, define $\mathcal{Y}(a, b, c, d, \ell)$ to be the *skew shape* depicted in Figure 3 (a). Explicitly, we define $\mathcal{Y}(a, b, c, d, \ell)$ to be the difference of two Young diagrams $\mathcal{Y} = \mathcal{Y}(\lambda/\mu)$ where

$$\lambda = ((\ell + a)^b, \ell + a - 1, \ell + a - 2, \dots, c), \quad \mu = (\ell, \ell - 1, \dots, 1).$$

Here $(\ell + a)^b$ denotes the number $\ell + a$ repeated b times.

Fix some integer $r \geq 1$.

Definition 5.3. Given a skew shape λ/μ , a (*weak*) *reverse plane partition of shape* λ/μ is a filling π of the boxes of $\mathcal{Y}(\lambda/\mu)$ with integers

³Indeed, by the Weyl dimension formula [24, Corollary 7.21.4], $Q_{\mu/\nu}(t)$ has degree $O(n^{mr})$ as $n \rightarrow \infty$. On the other hand, $Q_{N_{n,m}^{[e_r]}}(t)$ has degree $\binom{n}{r} = O(n^r)$ as $n \rightarrow \infty$.

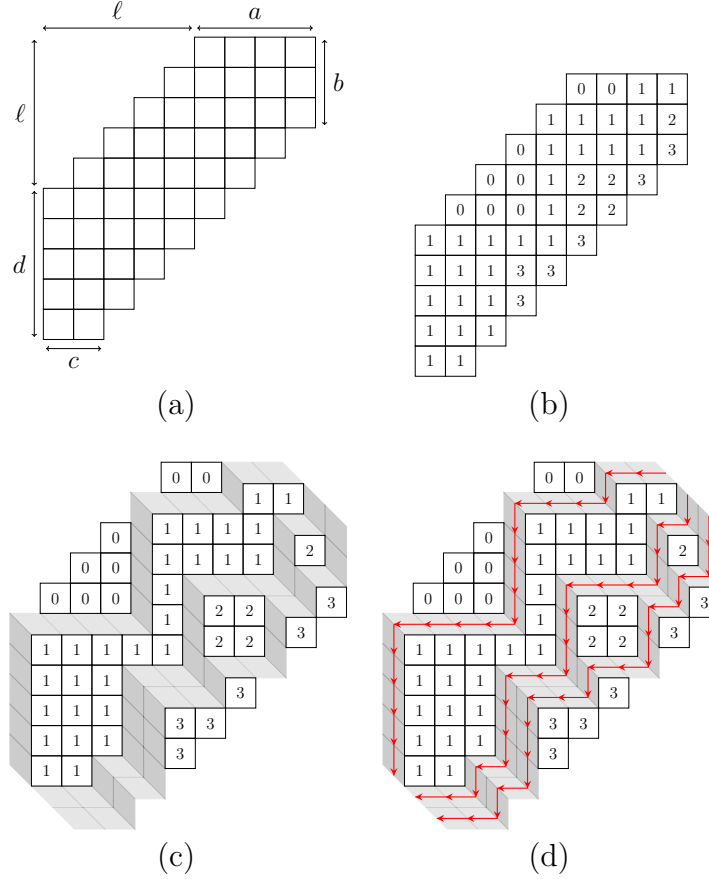


FIGURE 3. (a) The skew shape $\mathcal{Y} = \mathcal{Y}(a, b, c, d, \ell)$ for $a = 4, b = 3, c = 2, d = 5$, and $\ell = 5$. (b) A weak reverse plane partition π of shape \mathcal{Y} with $r = 3$. (c) A lozenge tiling corresponding to π . (d) An r -path $\tilde{\mathbf{P}}(\pi)$ in \tilde{N}_m corresponding to π .

from 0 to r such that the numbers increase weakly along every row and column of λ/μ . Define $|\pi|$ to be the sum of values of π .

An example of a weak reverse plane partition π of shape $\mathcal{Y}(a, b, c, d, \ell)$ is shown in Figure 3 (b). We have

$$|\pi| = 8 \times 0 + 28 \times 1 + 5 \times 2 + 6 \times 3 = 56.$$

Let q be an indeterminate. For each $\ell \geq 0$, we define

$$(5.1) \quad f(\ell) = \sum_{\pi} q^{|\pi|},$$

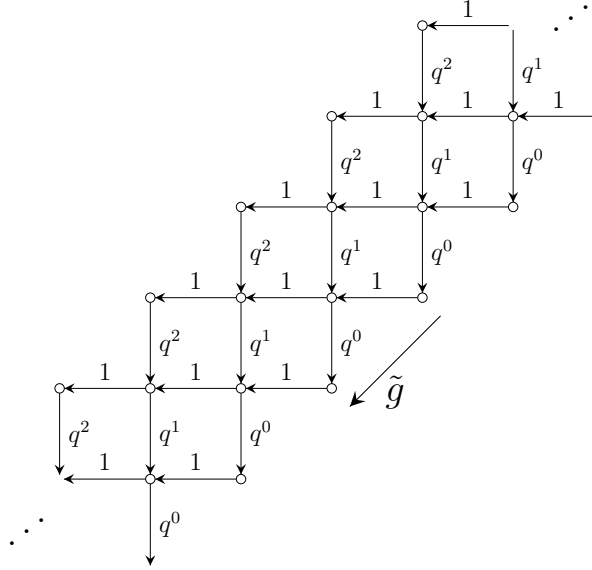


FIGURE 4. The cylindrical network \tilde{N}_m for $m = 4$.

where the sum is taken over all weak reverse plane partitions of shape $\mathcal{Y}(a, b, c, d, \ell)$ with values from 0 to r .

Let us now put $m = a + b + r$. For example, in the case of Figure 3, we have $m = 4 + 3 + 3 = 10$. For each $m \geq 2$, we introduce a planar cylindrical network \tilde{N}_m as follows. We put $\tilde{g} = (-1, 1)$, which is not a horizontal vector but we can rotate the whole picture by a 45 degree angle. Next, the vertex set \tilde{V} of \tilde{N}_m consists of all points $(i, j) \in \mathbb{R}^2$ with integer coordinates satisfying $0 \leq j - i \leq m - 1$. If two vertices (i, j) and $(i + 1, j)$ both belong to \tilde{V} then \tilde{N}_m contains an edge $(i + 1, j) \rightarrow (i, j)$ with weight 1. If two vertices (i, j) and $(i, j + 1)$ both belong to \tilde{V} then \tilde{N}_m contains an edge $(i, j + 1) \rightarrow (i, j)$ with weight q^{j-i} . Thus the weights of edges in \tilde{N}_m range from $q^0 = 1$ to q^{m-2} . An example of the network \tilde{N}_m for $m = 4$ is shown in Figure 4.

Proposition 5.4. *For each a, b, c, d, r , there exist two r -vertices $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ in \tilde{N}_m such that for any $\ell \geq 1$, there is a bijection $\pi \rightarrow \tilde{\mathbf{P}}(\pi)$ from the set of all weak reverse plane partitions of shape $\mathcal{Y}(a, b, c, d, \ell)$ with values from 0 to r to the set $\tilde{\mathcal{P}}(\tilde{\mathbf{u}}, \tilde{\mathbf{v}} + \ell\tilde{g})$ in \tilde{N}_m . Moreover, there exist two integers α, β depending on a, b, c, d, r such that for all ℓ we have*

$$(5.2) \quad q^{|\pi|} = \text{wt}(\tilde{\mathbf{P}}(\pi))q^\alpha q^{\ell\beta}$$

for any weak reverse plane partition π of shape $\mathcal{Y}(a, b, c, d, \ell)$.

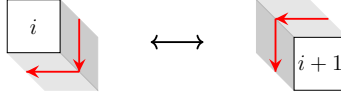


FIGURE 5. A local move that connects any two lozenge tilings with each other.

Proof. As Figure 3 (c) demonstrates, each weak reverse plane partition (with parts bounded by r) corresponds to a lozenge tiling of a certain planar region. We refer the reader to [10, Figure 1] for an explicit description of this bijection. As one can see from Figure 3 (d), each such lozenge tiling corresponds to a unique r -path in \tilde{N}_m for $m = a + b + r$. The endpoints of this path are precisely $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}} + \ell\tilde{\mathbf{g}}$ for some $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ that do not depend on ℓ . Thus the only claim we need to prove is (5.2). It is clear that all lozenge tilings are connected by the local move shown in Figure 5. Moreover, this local move increases the power of q by exactly 1 in both π and $\tilde{\mathbf{P}}(\pi)$. It suffices to analyze the image of the reverse plane partition π_0 whose all parts are equal to zero, and it is straightforward to check that the weight of $\tilde{\mathbf{P}}(\pi_0)$ is of the form $q^\alpha q^{\ell\beta}$ for some $\alpha, \beta \in \mathbb{Z}$. This finishes the proof. \square

Thus we can now apply Theorem 2.3 to the network \tilde{N}_m in order to obtain the main result of this section:

Theorem 5.5. *The sequence f given by (5.1) satisfies a linear recurrence with characteristic polynomial $Q_{N_m}^{[e_r]}(tq^\beta)$.*

Proof. Follows immediately from the previous proposition combined with the second part of Theorem 2.3. \square

Let us now analyze the polynomial $Q_{N_m}(t)$. There are $m - 1$ cycles C_0, C_1, \dots, C_{m-2} in N with respective weights q^0, q^1, \dots, q^{m-2} . Two cycles C_i and C_j are vertex disjoint if and only if $|i - j| > 1$. Thus by (2.2), we get that the degree d of $Q_{N_m}(t)$ is equal to $\lfloor m/2 \rfloor$ and

$$Q_{N_m}(t) = \sum_{r=0}^d (-t)^{d-r} H_r,$$

where

$$(5.3) \quad H_r = \sum_{(i_1, i_2, \dots, i_r)} q^{i_1 + i_2 + \dots + i_r}$$

and the sum is taken over all r -tuples $0 \leq i_1 < i_2 < \dots < i_r \leq m - 2$ of integers satisfying $i_k + 1 < i_{k+1}$ for all $k = 1, 2, \dots, r - 1$. For example,

for the network \tilde{N}_4 in Figure 4, we get

$$Q_{N_4}(t) = t^2 - (1 + q + q^2)t + q^{0+2}.$$

It turns out that these polynomials have already been extensively studied under the name *Carlitz q -Fibonacci polynomials*.

Definition 5.6 (see [7] or [9]). For $n \geq 0$, define the *Carlitz q -Fibonacci polynomial* $F_n(t)$ by

$$F_0(t) = 0, \quad F_1(t) = 1, \quad F_n(t) = F_{n-1}(t) + q^{n-3}tF_{n-2}(t).$$

The first few values of $F_n(t)$ are therefore

$$F_2(t) = 1, \quad F_3(t) = t+1, \quad F_4(t) = (1+q)t+1, \quad F_5(t) = q^2t^2+(1+q+q^2)t+1.$$

Proposition 5.7. For each $m \geq 2$, we have

$$Q_{N_m}(t) = (-t)^d F_{m+1}(-1/t),$$

where $d = \lfloor m/2 \rfloor$ is the degree of both polynomials.

Proof. It follows from (5.3) that both sides satisfy the same recurrence relation with the same initial conditions. \square

5.3. Domino tilings and the octahedron recurrence. In this section, we reprove (and slightly generalize) the results of [14, Section 3] using our machinery. Let $m \geq 2$ be an integer, and consider the strip $\mathcal{S}_m = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq y \leq m\}$. We are going to be interested in domino tilings of regions inside \mathcal{S}_m .

Let $n \geq 1$ be another integer and define the vector $\tilde{g} = (2n, 0)$. To each lattice point $(i, j) \in \mathbb{Z}^2, 0 \leq j \leq m$ of \mathcal{S}_m we assign a *weight* $w_{i,j}$ that satisfies $w_{i,j} = w_{i+2n,j}$ for all (i, j) and $w_{0,j} = w_{m,j} = 1$. For each lattice point $(i, j) \in \mathbb{Z}^2$ such that $0 \leq i < 2n$ and $0 < j < m$, we put $w_{i,j} = x_{ij}$ to be an indeterminate and let \mathbf{x} be the set of all these $2n(m-1)$ indeterminates. This defines $w_{i,j}$ for all lattice points $(i, j) \in \mathcal{S}_m$.

Given integers i, j , a *square* S with center $(i + 1/2, j + 1/2)$ is the convex hull of four lattice points with coordinates $(i + 1/2, j + 1/2) + (\pm 1/2, \pm 1/2)$. We say that S is *white* (resp., *black*) if $i + j$ is even (resp., odd).

For integers i, j, ℓ , we define the *Aztec diamond* $\mathcal{A}(i, j, \ell)$ to be the union of all squares S that are fully contained in the region

$$\{(x, y) \in \mathbb{R}^2 : |x - i| + |y - j| \leq \ell + 1\}.$$

Thus for example $\mathcal{A}(i, j, 1)$ is the union of four squares. Define the *truncated Aztec diamond* $\mathcal{A}_m(i, j, \ell)$ to be the intersection of $\mathcal{A}(i, j, \ell)$ with \mathcal{S}_m . A *domino tiling* of $\mathcal{A}_m(i, j, \ell)$ is a covering of $\mathcal{A}_m(i, j, \ell)$ by

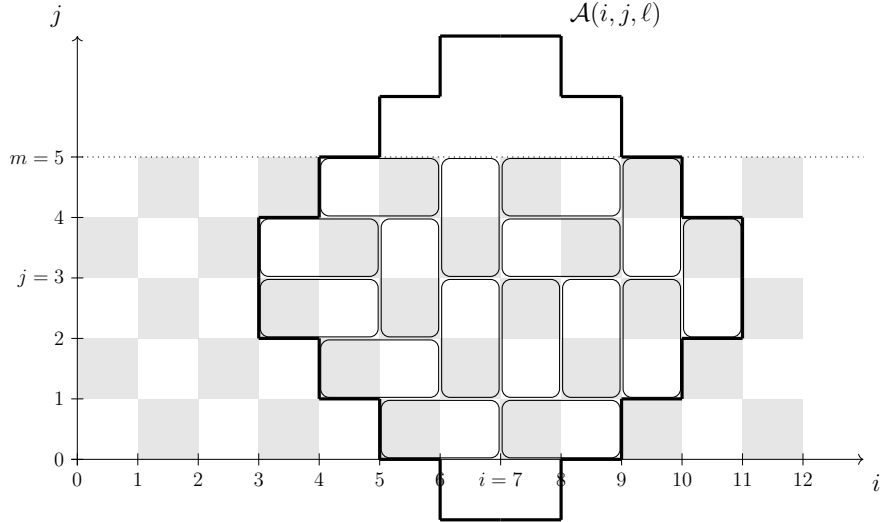


FIGURE 6. A domino tiling of $\mathcal{A}_m(i, j, \ell)$ for $i = 7, j = 3, \ell = 4$, and $m = 5$.

2×1 rectangles such that their vertices belong to \mathbb{Z}^2 and their interiors do not intersect each other. An example of a domino tiling of $\mathcal{A}_m(i, j, \ell)$ is given in Figure 6.

Given a domino tiling T of $\mathcal{A}_m(i, j, \ell)$, the *weight* $\text{wt}(T)$ of T is the product over all lattice points (i', j') in the interior of $\mathcal{A}_m(i, j, \ell)$ of $w_{i', j'}^\epsilon$, where $\epsilon \in \{-1, 0, +1\}$ is defined as follows:

$$\epsilon = \begin{cases} +1, & \text{if } (i', j') \text{ is adjacent to 4 dominoes of } T; \\ 0, & \text{if } (i', j') \text{ is adjacent to 3 dominoes of } T; \\ -1, & \text{if } (i', j') \text{ is adjacent to 2 dominoes of } T. \end{cases}$$

This assignment of weights was introduced by Speyer [23] to give a formula for the values of the octahedron recurrence. There is a more complicated rule that assigns the weights to vertices on the boundary of $\mathcal{A}_m(i, j, \ell)$, however, we will just omit them from $\text{wt}(T)$ for simplicity.

Define $Z_m(i, j, \ell)$ to be the sum of weights of all domino tilings of $\mathcal{A}_m(i, j, \ell)$.

Let us now introduce the following planar cylindrical network $\tilde{N}_{n, m}$. Its vertex set \tilde{V} is the set of all centers of black squares S that lie fully inside \mathcal{S}_m . Suppose that S' is a black square with center $(i' + 1/2, j' + 1/2)$ and S is a black square with center $(i + 1/2, j + 1/2)$ and that both of them belong to \mathcal{S}_m . Then $\tilde{N}_{n, m}$ contains an edge $e = (i' + 1/2, j' + 1/2) \rightarrow (i + 1/2, j + 1/2)$ if and only if either $i = i' + 1, j = j' + 1$ or $i = i' + 2, j = j'$ or $i = i' + 1, j = j' - 1$. For each

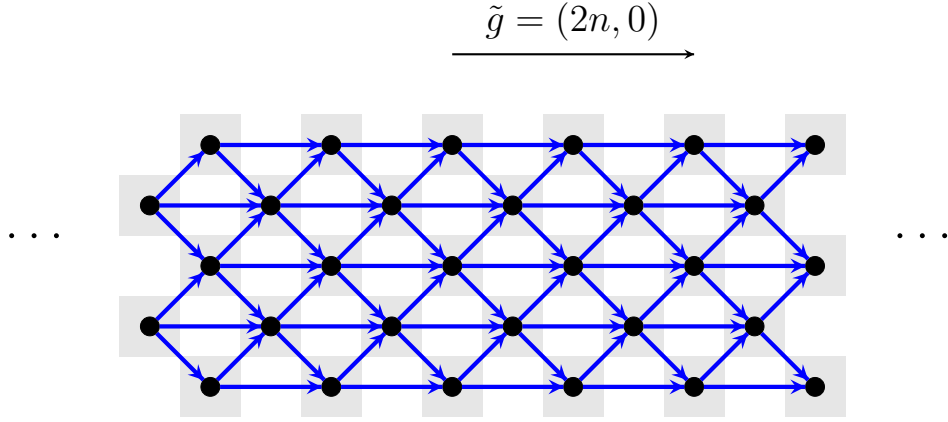


FIGURE 7. The network $\tilde{N}_{n,m}$ for $m = 5$ and $n = 2$.

of these three cases, we assign the *weight* $\text{wt}(e)$ to e as follows:

$$\text{wt}(e) = \begin{cases} \frac{w_{i+1,j+1}}{w_{i,j}}, & \text{if } i = i' + 1, j = j' + 1; \\ \frac{w_{i+1,j+1}w_{i+1,j}}{w_{i,j}w_{i,j+1}}, & \text{if } i = i' + 2, j = j'; \\ \frac{w_{i+1,j}}{w_{i,j+1}}, & \text{if } i = i' + 1, j = j' - 1. \end{cases}$$

This defines the network $\tilde{N}_{n,m}$. An example of $\tilde{N}_{n,m}$ is given in Figure 7.

Now given a lattice point $(i, j) \in \mathcal{S}_m$ and an integer ℓ , there is a simple bijection that associates to each domino tiling T of $\mathcal{A}_m(i, j, \ell)$ an r -path $\tilde{\mathbf{P}}(T)$ between two r -vertices $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$. Here $r, \tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ only depend on i, j, ℓ and not on T .

We construct $\tilde{\mathbf{P}}(T)$ using a local rule in Figure 8. Consider a black square S with center $(i' + 1/2, j' + 1/2)$ such that the white square S' with center $(i' + 3/2, j' + 1/2)$ to the right of S lies inside $\mathcal{A}_m(i, j, \ell)$. There is exactly one domino D of T that contains S' , and there are four possibilities for the black square S'' contained in D . Let $(i'' + 1/2, j'' + 1/2)$ be the center of S'' . Our local rule reads as follows:

- (1) If $i'' = i', j'' = j'$ then there is no edge in $\tilde{\mathbf{P}}(T)$ coming out of $(i' + 1/2, j' + 1/2)$;
- (2) otherwise there an edge $(i' + 1/2, j' + 1/2) \rightarrow (i'' + 1/2, j'' + 1/2)$ in $\tilde{\mathbf{P}}(T)$.

See Figure 8 for an illustration.

It is easy to see that in the second case, the edge $(i' + 1/2, j' + 1/2) \rightarrow (i'' + 1/2, j'' + 1/2)$ always is an edge of $\tilde{N}_{n,m}$. Thus our local rule defines $\tilde{\mathbf{P}}(T)$ as a collection of edges of $\tilde{N}_{n,m}$ and it follows that every vertex $(i' + 1/2, j' + 1/2)$ in the interior of $\mathcal{A}_m(i, j, \ell - 1)$ either is isolated

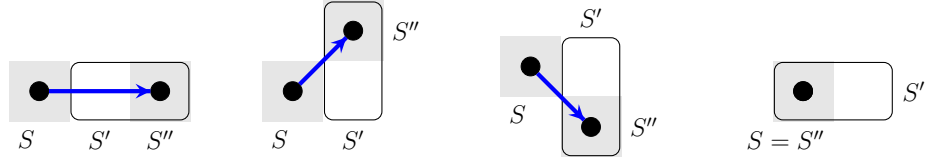


FIGURE 8. Local rules for constructing an r -path $\tilde{\mathbf{P}}(T)$ in $\tilde{N}_{n,m}$ from a given domino tiling T .

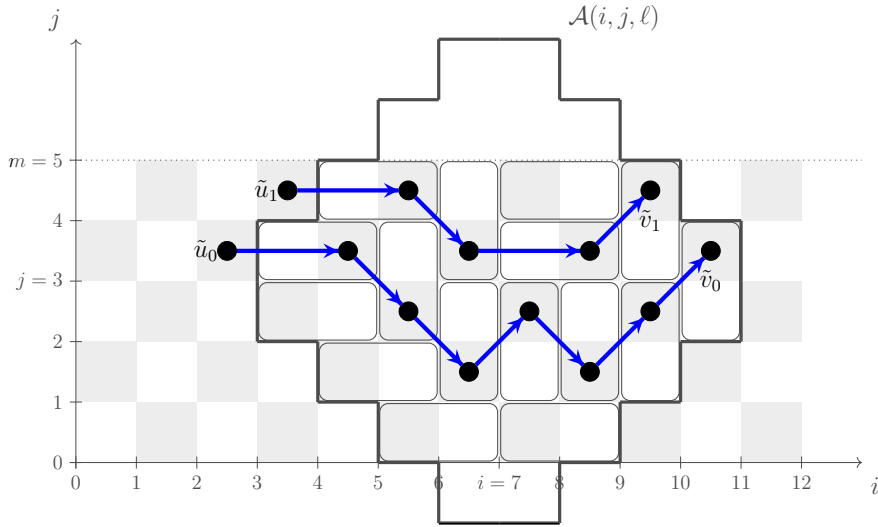


FIGURE 9. The r -path $\tilde{\mathbf{P}}(T)$ where T is the domino tiling from Figure 6.

in $\tilde{\mathbf{P}}(T)$ or has indegree and outdegree 1. Thus $\tilde{\mathbf{P}}(T)$ is an r -path for some $r \geq 0$. Clearly, the start $\tilde{\mathbf{u}}$ and end $\tilde{\mathbf{v}}$ of $\tilde{\mathbf{P}}(T)$ do not actually depend on T and thus the same is true for r . It is also straightforward to check that given any r -path $\tilde{\mathbf{P}}$ in $\tilde{N}_{n,m}$ that starts at $\tilde{\mathbf{u}}$ and ends at $\tilde{\mathbf{v}}$, there is a unique domino tiling T of $\mathcal{A}_m(i, j, \ell)$ such that $\tilde{\mathbf{P}} = \tilde{\mathbf{P}}(T)$. For example, if T is the domino tiling from Figure 6 then $\tilde{\mathbf{P}}(T)$ is shown in Figure 9.

Let us describe the vertices $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ explicitly as functions of i, j, ℓ . By symmetry, we may assume that $i + j - \ell$ is even. Then the square S with center $(i - \ell + 1/2, j + 1/2)$ on the boundary of $\mathcal{A}_m(i, j, \ell)$ is white. Let r be the minimum of $m - j$ and ℓ . Then it is easy to check that the start $\tilde{\mathbf{u}} = (\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{r-1})$ and end $\tilde{\mathbf{v}} = (\tilde{v}_0, \tilde{v}_1, \dots, \tilde{v}_{r-1})$ of $\tilde{\mathbf{P}}(T)$ are given by

$$(5.4) \quad \tilde{u}_k = (i - \ell + k - 1/2, j + k + 1/2), \quad \tilde{v}_k = (i + \ell - k - 1/2, j + k + 1/2)$$

for $k = 0, 1, \dots, r - 1$. In particular, the difference $\tilde{v}_k - \tilde{u}_k$ equals $(2\ell - 2k, 0)$.

Theorem 5.8. *For any lattice point (i, j) of \mathcal{S}_m and any integer ℓ_0 , the sequence $f : \mathbb{N} \rightarrow \mathbb{Z}[\mathbf{x}]$ given by*

$$f(\ell) = Z_m(i + \ell, j, \ell n + \ell_0)$$

satisfies a linear recurrence with characteristic polynomial $Q_{N_{n,m}}^{[e_m-j]}(t)$ for all but finitely many values of ℓ . Here the polynomial $Q_{N_{n,m}}(t)$ is given by (2.2).

Proof. We will deduce the result from the second part of Theorem 2.3. In order to do so, we need to show that

- (i) there exist vertices $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}_0$ such that for any domino tiling T of $\mathcal{A}_m(i + \ell, j, \ell n + \ell_0)$, the r -path $\tilde{\mathbf{P}}(T)$ starts at $\tilde{\mathbf{u}}$ and ends at $\tilde{\mathbf{v}}_\ell := \tilde{\mathbf{v}}_0 + \ell \tilde{\mathbf{g}}$.
- (ii) the only permutation $\sigma \in \mathfrak{S}_r$ such that there exists an r -path from $\tilde{\mathbf{u}}$ to $\sigma \tilde{\mathbf{v}}_\ell$ is the identity;
- (iii) there is a monomial τ in $\mathbf{x}^{\pm 1}$ that depends only on i, j, ℓ_0 such that we have $\text{wt}(T) = \tau \text{wt}(\tilde{\mathbf{P}}(T))$.

The first claim follows from (5.4). The second claim is obvious by inspection. The third claim can be proved as follows. First note that by Thurston's theorem [27], all domino tilings of $\mathcal{A}_m(i + \ell, j, \ell n + \ell_0)$ are connected by flips and that if T' is obtained from T by applying a flip then $\text{wt}(\tilde{\mathbf{P}}(T)) = \text{wt}(\tilde{\mathbf{P}}(T'))$ which is easy to check directly. Thus we can put τ to be the ratio of weights of T_0 and $\tilde{\mathbf{P}}(T_0)$ for some fixed tiling T_0 of $\mathcal{A}_m(i + \ell, j, \ell n + \ell_0)$. Let T_0 be the unique tiling that consists entirely of horizontal dominoes. Then $\text{wt}(T_0) = 1$ while $\text{wt}(\tilde{\mathbf{P}}(T_0))$ only contains the vertices that belong to the boundary of $\mathcal{A}_m(i + \ell, j, \ell n + \ell_0)$ and therefore does not depend on ℓ . Thus τ does not depend on ℓ either and we are done with the proof. \square

It remains to note that one can give a nice combinatorial interpretation to the coefficients of $Q_{N_{n,m}}(t)$ in terms of *domino tilings of the cylinder*. Namely, by (2.2), the coefficient of $(-t)^{d-r}$ is the sum H_r of weights of all r -cycles in $N_{n,m}$. Applying the inverse of the local rules in Figure 8 to any such r -cycle yields a domino tiling of the cylinder with *Thurston height* equal to r . Thus H_r can be interpreted as the sum of weights of all domino tilings of the cylinder with a fixed Thurston height. We refer the reader to [14, Section 3] for the details.

Remark 5.9. One can extend this approach to more general regions inside \mathcal{S}_m . Namely, let R be a region inside \mathcal{S}_m that lies between two

paths P_l and P_r that connect the upper boundary of \mathcal{S}_m with the lower boundary of \mathcal{S}_m and consist of left, right, and down steps. Define the region R_ℓ to be the region between P_l and $P_r + \ell\tilde{g}$ for all $\ell \geq 0$. Suppose in addition that there exists a domino tiling of R_ℓ for each sufficiently large ℓ . Then the sum $f(\ell)$ of weights of domino tilings of R_ℓ satisfies a linear recurrence with characteristic polynomial $Q_{N_{n,m}}^{[e_r]}(t)$ for some $r \geq 0$. The proof is analogous to that of Theorem 5.8.

Remark 5.10. If $n = 1$ then N is an $m \times 2$ cylinder. There are $2m - 1$ cycles $C_1, C_2, \dots, C_{2m-1}$ in N labeled in weakly increasing order (on the cylinder) and just as in the previous section, C_i and C_j are vertex disjoint if and only if $|i - j| > 1$. Thus the total number of r -cycles in N will be the $2m - 1$ 'th Fibonacci number F_{2m-1} . Hence the polynomial $Q_{N_{1,m}}(t)$ has F_{2m-1} terms, however, it is a multivariate polynomial in \mathbf{x} unlike the polynomial $F_n(t)$ in the previous section.

5.4. Other applications. Many other objects correspond to r -paths in various planar cylindrical networks. We briefly list several important examples and refer the reader to specific places where the corresponding bijections are described in the literature.

- (1) *Vicious walkers between two walls*, see [19, Figure 1].
- (2) *Q-Schur functions*, see [26, Figure 4a]
- (3) *Super-Schur functions*, see [6, Figure 1].
- (4) *Cube recurrence in a cylinder*. In our work in progress [12], we give a way to transform a formula from [8] in the language of r -paths in a certain network.
- (5) *States of the six vertex model \leftrightarrow cylindric packed loops*, see [28, Figures 11, 19]. Note that both of these objects are in bijection with domino tilings of the Aztec diamond.

Remark 5.11. In order to get a cylindrical network for domino tilings and related objects, we had to restrict the Aztec diamond to the strip \mathcal{S}_m . Note that however for the remaining four items in the above list, the underlying network is naturally cylindrical and there is no need to impose further restrictions on it. Thus a linear recurrence result for them is an immediate consequence of the second part of Theorem 2.3.

6. CONJECTURES

In this section, we give some additional conjectures for the case when \tilde{N} is a planar cylindrical network. Let us denote

$$H_r = \sum_{\mathbf{C} \in \mathcal{C}^r(N)} \text{wt}(\mathbf{C}),$$

so that $Q_N(t) = \sum_{r=0}^d (-t)^{d-r} H_r$. In particular, we set $H_r = 0$ for $r > d$.

We say that a sequence (H_0, H_1, \dots) of polynomials is a *Pólya frequency sequence* if all minors of the following *infinite Toeplitz matrix* $\mathcal{H} = (\mathcal{H}_{ij})_{i,j \geq 1}$ defined by

$$\mathcal{H}_{ij} = \begin{cases} H_{j-i}, & \text{if } j \geq i; \\ 0, & \text{if } j < i; \end{cases} \quad \mathcal{H} = \begin{pmatrix} H_0 & H_1 & H_2 & \dots \\ 0 & H_0 & H_1 & \dots \\ 0 & 0 & H_0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

are polynomials in \mathbf{x} with nonnegative integer coefficients (in other words, the matrix \mathcal{H} is required to be *totally positive*). For example, the fact that all 1×1 minors have nonnegative coefficients means that each H_i has nonnegative coefficients, and the fact that all 2×2 minors have nonnegative coefficients implies that the sequence (H_0, H_1, \dots) is *strongly log-concave* meaning that the polynomial $H_i^2 - H_{i+1}H_{i-1}$ has nonnegative coefficients for each $i > 0$.

Conjecture 6.1. The polynomials H_0, H_1, \dots form a Pólya frequency sequence.

Let Λ be the ring of symmetric functions (see [24, Chapter 7]), and consider the ring homomorphism $\psi : \Lambda \rightarrow \mathbb{Z}[\mathbf{x}]$ defined by

$$\psi(e_r) = H_r, \quad r \geq 1.$$

In other words, $\psi(f)$ is obtained from $f \in \Lambda$ by specializing it to the roots of $Q_N(t)$.

By the dual Jacobi-Trudi identity [24, Corollary 7.16.2], the image $\psi(s_\lambda)$ of a Schur function s_λ is given by a row-solid minor of \mathcal{H} . Arbitrary minors of \mathcal{H} are images $\psi(s_{\lambda/\mu})$ of skew-Schur functions and thus by the Littlewood-Richardson rule [24, Section A1.3] are nonnegative integer combinations of the row-solid minors of \mathcal{H} . Thus Conjecture 6.1 can be equivalently stated as follows:

Conjecture 6.2. The images $\psi(s_\lambda)$ of Schur functions are polynomials in $\mathbb{Z}[\mathbf{x}]$ with nonnegative coefficients.

In particular, Conjecture 6.2 would imply that the coefficient of $(-1)^k t^{D-k}$ in $Q_N^{[e_r]}(t)$ is a nonnegative polynomial in \mathbf{x} , where $D = \binom{d}{r}$ is the degree of $Q_N^{[e_r]}(t)$. This is the case since the plethysm $e_k[e_r]$ of two Schur positive functions e_k and e_r is again Schur positive, but its image $\psi(e_k[e_r])$ is the desired coefficient of $Q_N^{[e_r]}(t)$.

Conjecture 6.3. Fix some $r \geq 1$ and substitute positive real numbers for the variables in \mathbf{x} . After such a substitution, the polynomials $Q_N^{[hr]}$ and $Q_N^{[e_r]}$ have positive real roots.

Of course, by (2.3), it is enough to prove Conjecture 6.3 for $r = 1$. By [5, Theorem 4.5.3], Conjecture 6.3 is a special case of Conjecture 6.1. One way to prove Conjecture 6.3 would be to show the following statement.

Conjecture 6.4. Substitute positive real numbers for the variables in \mathbf{x} . Then there exists a local lift L such that the matrix S defined by (4.7) is *totally positive*, that is, all minors of S are positive.

Indeed, by (4.9) the roots of $Q_N(t)$ are the eigenvalues of S so Conjecture 6.3 follows since totally positive matrices are known to have positive real eigenvalues. We thank Richard Stanley for suggesting this way of proving Conjecture 6.3.

We finish with another conjecture that potentially increases the value of the second part of Theorem 2.3. Let us say that a cylindrical network \tilde{N} is *strongly connected* if for any two vertices \tilde{u}, \tilde{v} of \tilde{N} there exists an integer $\ell \in \mathbb{Z}$ and a directed path in \tilde{N} from \tilde{u} to $\tilde{v} + \ell\tilde{g}$. Equivalently, \tilde{N} is strongly connected if and only if the directed graph N is strongly connected in the usual sense.

Conjecture 6.5. Suppose we are given a strongly connected planar cylindrical network \tilde{N} such that the weights of the edges of N are algebraically independent. Then for any integer $r \geq 1$ and the sequence f from Theorem 2.3 the polynomial $Q_N^{[e_r]}(t)$ is the *minimal* recurrence polynomial for f .

REFERENCES

- [1] Per Alexandersson. Stretched skew Schur polynomials are recurrent. *J. Combin. Theory Ser. A*, 122:1–8, 2014.
- [2] Per Alexandersson. Polynomials defined by tableaux and linear recurrences. *Electron. J. Combin.*, 23(1):Paper 1.47, 24, 2016.
- [3] Roger E. Behrend, Philippe Di Francesco, and Paul Zinn-Justin. On the weighted enumeration of alternating sign matrices and descending plane partitions. *J. Combin. Theory Ser. A*, 119(2):331–363, 2012.
- [4] Frédéric Bosio and Marc A. A. van Leeuwen. A bijection proving the Aztec diamond theorem by combing lattice paths. *Electron. J. Combin.*, 20(4):Paper 24, 30, 2013.
- [5] Francesco Brenti. Unimodal, log-concave and Pólya frequency sequences in combinatorics. *Mem. Amer. Math. Soc.*, 81(413):viii+106, 1989.
- [6] Francesco Brenti. Determinants of super-Schur functions, lattice paths, and dotted plane partitions. *Adv. Math.*, 98(1):27–64, 1993.

- [7] L. Carlitz. Fibonacci notes. IV. q -Fibonacci polynomials. *Fibonacci Quart.*, 13:97–102, 1975.
- [8] Gabriel D. Carroll and David Speyer. The cube recurrence. *Electron. J. Combin.*, 11(1):Research Paper 73, 31 pp. (electronic), 2004.
- [9] Johann Cigler. q -Fibonacci polynomials. *Fibonacci Quart.*, 41(1):31–40, 2003.
- [10] Henry Cohn, Michael Larsen, and James Propp. The shape of a typical boxed plane partition. *New York J. Math.*, 4:137–165, 1998.
- [11] Patrik L. Ferrari and Herbert Spohn. Domino tilings and the six-vertex model at its free-fermion point. *J. Phys. A*, 39(33):10297–10306, 2006.
- [12] Pavel Galashin. Periodicity and integrability for the cube recurrence. *arXiv:1704.05570*, 2017.
- [13] Pavel Galashin and Pavlo Pylyavskyy. The classification of Zamolodchikov periodic quivers. *Amer. J. Math.*, to appear. *arXiv:1603.03942*.
- [14] Pavel Galashin and Pavlo Pylyavskyy. Quivers with subadditive labelings: classification and integrability. *arXiv:1606.04878*, 2016.
- [15] Pavel Galashin and Pavlo Pylyavskyy. Quivers with additive labelings: classification and algebraic entropy. *arXiv:1704.05024*, 2017.
- [16] Ira Gessel and Gérard Viennot. Binomial determinants, paths, and hook length formulae. *Adv. in Math.*, 58(3):300–321, 1985.
- [17] Anthony J. Guttmann, Aleksander L. Owczarek, and Xavier G. Viennot. Vicious walkers and Young tableaux. I. Without walls. *J. Phys. A*, 31(40):8123–8135, 1998.
- [18] Christian Krattenthaler, Anthony J. Guttmann, and Xavier G. Viennot. Vicious walkers, friendly walkers and Young tableaux. II. With a wall. *J. Phys. A*, 33(48):8835–8866, 2000.
- [19] Christian Krattenthaler, Anthony J. Guttmann, and Xavier G. Viennot. Vicious walkers, friendly walkers, and Young tableaux. III. Between two walls. *J. Statist. Phys.*, 110(3-6):1069–1086, 2003. Special issue in honor of Michael E. Fisher’s 70th birthday (Piscataway, NJ, 2001).
- [20] Thomas Lam and Pavlo Pylyavskyy. Crystals and total positivity on orientable surfaces. *Selecta Math. (N.S.)*, 19(1):173–235, 2013.
- [21] Bernt Lindström. On the vector representations of induced matroids. *Bull. London Math. Soc.*, 5:85–90, 1973.
- [22] C. Soulé. *Lectures on Arakelov geometry*, volume 33 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1992. With the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer.
- [23] David E. Speyer. Perfect matchings and the octahedron recurrence. *J. Algebraic Combin.*, 25(3):309–348, 2007.
- [24] Richard P. Stanley. *Enumerative combinatorics. Volume 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [25] Richard P. Stanley. *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.
- [26] John R. Stembridge. Nonintersecting paths, Pfaffians, and plane partitions. *Adv. Math.*, 83(1):96–131, 1990.

- [27] William P. Thurston. Conway's tiling groups. *Amer. Math. Monthly*, 97(8):757–773, 1990.
- [28] P. Zinn-Justin. Six-vertex, loop and tiling models: Integrability and combinatorics. *arXiv:0901.0665*, 2009.
- [29] Paul Zinn-Justin and Philippe Di Francesco. The quantum Knizhnik-Zamolodchikov equation, completely symmetric self-complementary plane partitions, and alternating-sign matrices. *Teoret. Mat. Fiz.*, 154(3):387–408, 2008.

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