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Positroids, knots, and q,t-Catalan numbers

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Abstract. We relate the mixed Hodge structure on the cohomology of open positroid varieties (in particular, their Betti numbers over $\mathbb{C}$ and point counts over $\mathbb{F}_q$) to Khovanov–Rozansky homology of the associated links. We deduce that the mixed Hodge polynomials of top-dimensional open positroid varieties are given by rational q,t-Catalan numbers. Via the curious Lefschetz property, this implies the q,t-symmetry and unimodality properties of rational q,t-Catalan numbers. We show that the q,t-symmetry phenomenon is a manifestation of Koszul duality for category $\mathcal{O}$, and discuss relations with equivariant derived categories of flag varieties, and open Richardson varieties.

Keywords: Positroid varieties, q,t-Catalan numbers, HOMFLY polynomial, Khovanov–Rozansky homology, mixed Hodge structure, equivariant cohomology, Koszul duality.

1 Introduction

The Poincaré polynomial of the complex Grassmannian $\mathrm{Gr}(k,n)$ is well known to be given by the Gaussian polynomial $\left[{n \atop k}\right]_q$. The number of points of $\mathrm{Gr}(k,n)$ over a finite field $\mathbb{F}_q$ is given by the same polynomial. The reason these two polynomials coincide is that the mixed Hodge structure on the cohomology of $\mathrm{Gr}(k,n)$ is pure. The situation is different when one considers the top-dimensional positroid variety $\Pi^\circ_{k,n} \subset \mathrm{Gr}(k,n)$, introduced in [15] building on the results of [21]. The space $\Pi^\circ_{k,n}$ consists of row spans of full rank $k \times n$ matrices whose cyclically consecutive maximal minors are all nonzero. It turns out that the Poincaré polynomial and the point count of $\Pi^\circ_{k,n}$ are given by two different q-analogs of (rational) Catalan numbers: one of our main results is that when $\gcd(k,n) = 1$, the Poincaré polynomial of $\Pi^\circ_{k,n}$ is given by $\sum_{P \in \mathrm{Dyck}_{k,n-k}} q^{\mathrm{area}(P)}$ while the number of points of $\Pi^\circ_{k,n}$ over $\mathbb{F}_q$ equals $\frac{1}{[n]_q}\left[{n \atop k}\right]_q$, up to a simple factor. The mixed Hodge structure on $H^\bullet(\Pi^\circ_{k,n})$ is non-pure, and we show that its bigraded Poincaré polynomial $\mathcal{P}(\Pi^\circ_{k,n}; q,t)$ coincides with the rational q,t-Catalan number $C_{k,n-k}(q,t)$ introduced

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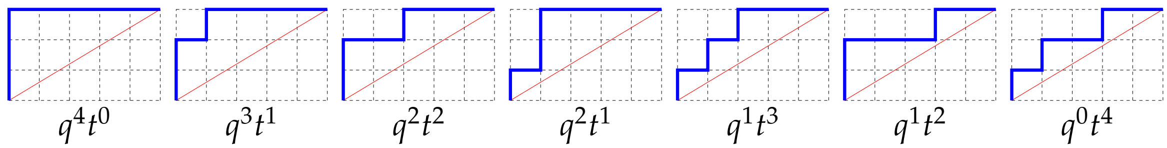


Figure 1: Computing the rational q,t-Catalan number $C_{3,5}(q,t)$.

in [7, 18]. Our proof proceeds via relating both sides to Khovanov–Rozansky knot homology [14]. Our results apply more generally to arbitrary positroid and Richardson varieties.

2 Positroid varieties and Catalan numbers

Rational q,t-Catalan numbers. Let a and b be coprime positive integers. The *rational q,t-Catalan number* $C_{a,b}(q,t) \in \mathbb{N}[q,t]$ was introduced by Loehr–Warrington [18], generalizing the work of Garsia–Haiman [7]. It is defined as follows:

$$C_{a,b}(q,t) := \sum_{P \in \mathrm{Dyck}_{a,b}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)}, \tag{2.1}$$

where $\mathrm{Dyck}_{a,b}$ is the set of lattice paths P inside a rectangle of height a and width b that stay above the diagonal, $\mathrm{area}(P)$ is the number of unit squares fully contained between P and the diagonal, and $\mathrm{dinv}(P)$ is the number of pairs (h,v) satisfying the following conditions: h is a horizontal step of P, v is a vertical step of P that appears to the right of h, and there exists a line of slope a/b (parallel to the diagonal) intersecting both h and v. For example, we have

$$C_{3,5}(q,t) = q^4 + q^3t + q^2t^2 + q^2t + qt^3 + qt^2 + t^4, \tag{2.2}$$

as shown in Figure 1.

Positroid varieties in the Grassmannian. The *Grassmannian* $\mathrm{Gr}(k,n)$ is the space of linear k-dimensional subspaces of $\mathbb{C}^n$. Alternatively, it can be identified with the space of full rank $k \times n$ matrices modulo row operations. Building on Postnikov's cell decomposition [21] of its totally nonnegative part, Knutson–Lam–Speyer [15] constructed a stratification $\mathrm{Gr}(k,n) = \bigsqcup_{f \in S_{k,n}} \Pi^\circ_f$ of the Grassmannian into *(open) positroid varieties*. Roughly speaking,[1] each positroid variety Π°_f corresponds to a permutation $f \in S_n$ such

[1]More precisely, positroid varieties are in bijection with *decorated* permutations, where a decoration of f is an arbitrary coloring of fixed points of f into black and white colors. The actual set $S_{k,n}$ consists of decorated permutations f satisfying $\#\{1 \le i \le n \mid f(i) < i\} + \#\{\text{black fixed points of } f\} = k$. The most interesting special case for us occurs when f is a single cycle, where the decoration is trivial.

that $\#\{1 \leq i \leq n \mid f(i) < i\} = k$; the set of such permutations is denoted by $S_{k,n}$. For each $f \in S_{k,n}$, the space Π_f° is a smooth algebraic variety. Two basic questions one can ask about such a space are: what is the number of points in $\Pi_f^\circ(\mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ with q elements, and what is the cohomology of Π_f° considered as a variety over $\mathbb{C}$?

These two questions turn out to be closely related to each other through the work of Deligne [3] on *mixed Hodge structures*, explored in the case of cluster varieties in [16]. Since the work of Scott [23], positroid varieties have been expected to admit a natural *cluster algebra* structure arising from Postnikov diagrams. We recently proved this conjecture building on the results of [17, 20, 24].

Theorem 2.1 ([6]). *The coordinate ring of each positroid variety Π_f° is isomorphic to the associated cluster algebra.*

This result allows one to study Π_f° as a *cluster variety*, in which case Deligne's mixed Hodge structure can be explored using the machinery developed by Lam–Speyer [16]. The mixed Hodge structure endows the cohomology $H^\bullet(\Pi_f^\circ)$ of Π_f° with a second grading, and the suitably renormalized Poincaré polynomial $\mathcal{P}(\Pi_f^\circ; q, t)$ of this bigraded vector space answers both of the above questions simultaneously:

Theorem 2.2 ([16, 6]). *For each $f \in S_{k,n}$, the* bigraded Poincaré polynomial $\mathcal{P}(\Pi_f^\circ; q, t) \in \mathbb{N}[q^{\frac{1}{2}}, t^{\frac{1}{2}}]$ *satisfies the following properties:*

(i) *q,t-symmetry: $\mathcal{P}(\Pi_f^\circ; q, t) = \mathcal{P}(\Pi_f^\circ; t, q)$;*

(ii) *q,t-unimodality: for each d, the coefficients of $\mathcal{P}(\Pi_f^\circ; q, t)$ at $q^d t^0, q^{d-1} t^1, \dots, q^0 t^d$ form a unimodal sequence;*

(iii) *$\mathcal{P}(\Pi_f^\circ; q^2, 1)$ equals the Poincaré polynomial of Π_f° (considered as a variety over $\mathbb{C}$);*

(iv) *$q^{\frac{1}{2}\dim \Pi_f^\circ} \cdot \mathcal{P}(\Pi_f^\circ; q, t)\big|_{t^{\frac{1}{2}} = -q^{-\frac{1}{2}}}$ equals the point count $\#\Pi_f^\circ(\mathbb{F}_q)$.*

The positroid stratification contains a unique open stratum, the *top-dimensional positroid variety* $\Pi_{k,n}^\circ := \Pi_{f_{k,n}}^\circ$ corresponding to the permutation $f_{k,n} \in S_{k,n}$ sending $i \mapsto i + k$ modulo n for all $1 \leq i \leq n$. It is given by

$$\Pi_{k,n}^\circ := \{\mathrm{RowSpan}(A) \mid A \in \mathrm{Mat}(k, n; \mathbb{C}) : \Delta_{1,2,\dots,k}(A), \Delta_{2,3,\dots,k+1}(A), \dots, \Delta_{n,1,\dots,k-1}(A) \neq 0\}.$$

Here $\Delta_I(A)$ is the maximal minor of A with column set $I \subset \{1, 2, \dots, n\}$, $|I| = k$. We are ready to state (the most important special case of) our main result.

Theorem 2.3. *Assume that* $\gcd(k, n) = 1$. *Then*

$$\mathcal{P}(\Pi_{k,n}^\circ; q, t) = \left(q^{\frac{1}{2}} + t^{\frac{1}{2}}\right)^{n-1} C_{k,n-k}(q, t). \tag{2.3}$$

The equality (2.3) arises as a conjecture from the works [26, 25] and we thank Vivek Shende for drawing our attention to the conjecture. We generalize Theorem 2.3 to all positroid varieties in Theorem 4.2 below.

Our proof of Theorem 2.3 involves a number of ingredients, including Khovanov–Rozansky knot homology and equivariant perverse sheaves. The point count specialization ($t^{\frac12} = -q^{-\frac12}$) turns out to require less advanced machinery. Namely, let us denote $[n]_q := 1 + q + \cdots + q^{n-1}$, $[n]_q! := [1]_q[2]_q \cdots [n]_q$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$. We give an elementary proof of the following special case of Theorem 2.3.

Proposition 2.4. *Assume that* $\gcd(k,n) = 1$. *Then* $\#\Pi^\circ_{k,n}(\mathbb{F}_q) = (q-1)^{n-1} \cdot \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q$. *In other words, the probability that a uniformly random k-dimensional subspace of* $(\mathbb{F}_q)^n$ *belongs to* $\Pi^\circ_{k,n}(\mathbb{F}_q)$ *equals* $\frac{(q-1)^n}{q^n-1}$.

Remark 2.5. Surprisingly, this probability $\frac{(q-1)^n}{q^n-1}$ does not depend on k. We do not have a combinatorial explanation for this phenomenon.

Our proof proceeds by associating a *link* $\hat\beta_f$ to each positroid variety Π°_f (Section 3) and then comparing the point count $\#\Pi^\circ_f(\mathbb{F}_q)$ to the HOMFLY polynomial of $\hat\beta_f$. The HOMFLY polynomial is categorified by Khovanov–Rozansky knot homology, and our proof of Theorem 2.3 may be considered a "categorification" of Proposition 2.4.

Remark 2.6. After discovering the proof of Proposition 2.4 via the HOMFLY polynomial, we found that it can also be deduced from the results of [26, 25]. Our proof is new and yields a generalization (Theorem 3.4) of Proposition 2.4 to arbitrary open positroid varieties.

Torus action. The appearance of the extra factor $\left(q^{\frac12} + t^{\frac12}\right)^{n-1}$ in (2.3), as well as the condition $\gcd(k,n) = 1$ are neatly explained by the *torus action* on $\mathrm{Gr}(k,n)$. Let $T \cong (\mathbb{C}^*)^{n-1}$ be the quotient of the group of diagonal $n \times n$ matrices by the group of scalar matrices. The group T acts on $\mathrm{Gr}(k,n)$ by rescaling the columns of $k \times n$ matrices. This action leaves each positroid variety Π°_f invariant. We say that T acts *freely* on Π°_f if all non-identity elements of T act on Π°_f without fixed points. It is straightforward to check that the action of T on Π°_f is free if and only if the permutation f is a single cycle. Note that $f_{k,n}$ is a single cycle if and only if $\gcd(k,n) = 1$. We will see later in Definition 3.2 that f is a single cycle precisely when the associated link $\hat\beta_f$ has a single component, i.e., is a knot.

Let $\mathrm{ncyc}(f)$ denote the number of cycles of $f \in S_n$, and let $S^{\mathrm{ncyc}=1}_{k,n} := \{f \in S_{k,n} \mid \mathrm{ncyc}(f) = 1\}$. For $f \in S^{\mathrm{ncyc}=1}_{k,n}$, the quotient Π°_f/T is again a smooth cluster variety, and Theorem 2.2 applies to it. The associated bigraded Poincaré polynomials are related as

H^k	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$k-p=0$	1	0	1	0	1	0	1	0	1
$k-p=1$					1	0	1		

Table 1: The mixed Hodge table recording the dimensions of $H^{k,(p,p)}(\Pi^\circ_{3,8}/T)$ for the cluster algebra of type E_8; see [16, Table 5]. The dimensions agree with the coefficients of $C_{3,5}(q,t)$; see Example 2.9.

$\mathcal{P}(\Pi^\circ_f;q,t) = \left(q^{\frac{1}{2}}+t^{\frac{1}{2}}\right)^{n-1} \cdot \mathcal{P}(\Pi^\circ_f/T;q,t)$. In particular, in the setting of Theorem 2.3, we find

$$\mathcal{P}(\Pi^\circ_{k,n}/T;q,t) = C_{k,n-k}(q,t). \tag{2.4}$$

Combining this with Theorem 2.2 has consequences for q,t-Catalan numbers and positroid varieties which can be stated in an elementary way. Let us denote $d_{k,n} := (k-1)(n-k-1) = \dim(\Pi^\circ_{k,n}/T)$.

Corollary 2.7. *Assume that* $\gcd(k,n)=1$. *We have:*

(i) *q,t-symmetry: $C_{k,n-k}(q,t) = C_{k,n-k}(t,q)$;*
(ii) *q,t-unimodality: for each d, the coefficients of $C_{k,n-k}(q,t)$ at $q^dt^0, q^{d-1}t^1, \dots, q^0t^d$ form a unimodal sequence;*
(iii) *the Poincaré polynomial of $\Pi^\circ_{k,n}/T$ is given by*

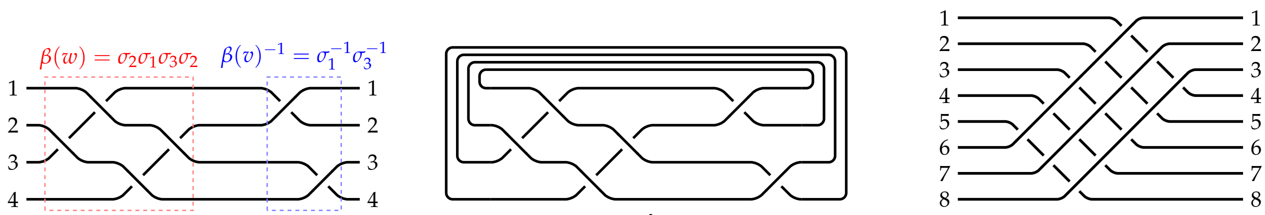
$$\sum_d q^{\frac{d}{2}} \dim H^{d_{k,n}-d}(\Pi^\circ_{k,n}/T) = C_{k,n-k}(q,1) = \sum_{P\in \mathrm{Dyck}_{k,n-k}} q^{\mathrm{area}(P)}; \tag{2.5}$$

(iv) *the number of $\mathbb{F}_q$-points of $\Pi^\circ_{k,n}/T$ is given by*

$$\#(\Pi^\circ_{k,n}/T)(\mathbb{F}_q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{\frac{1}{2}d_{k,n}} \cdot C_{k,n-k}(q,1/q). \tag{2.6}$$

Remark 2.8. When $a=n$ and $b=n+1$, $C_{a,b}(q,t)$ recovers the famous *q,t-Catalan numbers* $C_n(q,t)$ of Garsia and Haiman [7]. The fact that $C_n(q,t)$ is q,t-symmetric and q,t-unimodal follows from the results of Haiman [10, 9]. For arbitrary a,b, the q,t-symmetry property follows from the celebrated recent proof of the rational shuffle conjecture [19]. To our knowledge, q,t-unimodality of $C_{k,n-k}(q,t)$ is a new result.

Example 2.9. For $k=3$, $n=8$, the coordinate ring of $\Pi^\circ_{k,n}/T$ is a cluster algebra of type E_8 (with no frozen variables). The associated *mixed Hodge table* is given in Table 1; see [16, Table 5]. The grading conventions are chosen so that the first row contributes q^4+q^3t+



braid $\beta_f = \beta(w) \cdot \beta(v)^{-1}$ closure $\hat{\beta}_f$ of β_f $\beta_{f_{k,n}}$ for $k = 3$, $n = 8$

Figure 2: Braids and links associated to positroid varieties.

$q^2t^2 + qt^3 + t^4$ while the second row contributes $q^2t + qt^2$ to $\mathcal{P}(\Pi^\circ_{k,n}; q, t)$. Comparing the result with (2.2), we find $\mathcal{P}(\Pi^\circ_{k,n}/T; q, t) = C_{k,n-k}(q, t)$.

The polynomial $C_{3,5}(q, t)$ given in (2.2) is indeed q, t-symmetric and q, t-unimodal: fixing the total degree of q and t, it splits into polynomials $q^4 + q^3t + q^2t^2 + qt^3 + t^4$ and $q^2t + qt^2$. We also have $C_{3,5}(q, 1) = q^4 + q^3 + 2q^2 + 2q + 1$; the coefficient of $q^{d/2}$ is equal to $\dim H^{d_{k,n}-d}(\Pi^\circ_{k,n}/T)$ for each d.

3 Links associated to positroid varieties

In order to explain how knot theory comes into play, we need a way to represent $f \in S_{k,n}$ in a slightly different form. Let us say that a permutation $w \in S_n$ is *k-Grassmannian* if $w^{-1}(1) < w^{-1}(2) < \cdots < w^{-1}(k)$ and $w^{-1}(k+1) < \cdots < w^{-1}(n)$. We denote by $\leq$ the (strong) Bruhat order on S_n. The following result is well known.

Proposition 3.1 ([15]). *For every $f \in S_{k,n}$, there exists a unique*[2] *pair of permutations $v, w \in S_n$ such that $v \leq w$, w is k-Grassmannian, and $f = wv^{-1}$.*

For example, when $f = f_{k,n}$, we have $w = f$ and $v = \mathrm{id}$. The dimension of Π°_f equals $\ell_{v,w} := \ell(w) - \ell(v)$, where $\ell(w)$ is the number of inversions of w.

The group S_n is generated by simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. Similarly, let $\mathcal{B}_n$ be the *braid group* on n strands, generated by $\sigma_1, \dots, \sigma_{n-1}$ with relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| > 1$. Connecting the corresponding endpoints of a braid β gives rise to a *link* called the *closure* $\hat{\beta}$ of β; see Figure 2.

For each element $u \in S_n$, let $\beta(u)$ denote the corresponding braid, obtained by choosing a reduced word $u = s_{i_1}s_{i_2}\cdots s_{i_{\ell(u)}}$ for u and then replacing each s_i with σ_i.

Definition 3.2. For $f \in S_{k,n}$ and $v \leq w \in S_n$ as in Proposition 3.1, let $\beta_f := \beta(w) \cdot \beta(v)^{-1}$. We refer to the closure $\hat{\beta}_f$ as *the link associated to* f. See Figure 2 for an example.

[2]If f has fixed points, the pair (v, w) must be compatible with the decoration of f.

Observe that $\hat{\beta}_f$ is a *knot* (i.e., has one connected component) if and only if $f \in S_{k,n}^{\mathrm{ncyc}=1}$. We note that two other (more complicated) ways of assigning a Legendrian or a transverse link to a positroid variety have appeared recently in [25, 5].

The *HOMFLY polynomial* $P(L) = P(L; a, z)$ of an (oriented) link L is defined by a skein relation $aP(L_+) - a^{-1}P(L_-) = zP(L_0)$ and $P(\bigcirc) = 1$, where $\bigcirc$ denotes the unknot and L_+, L_-, L_0 are three links whose planar diagrams locally differ as follows:

L_+ L_- L_0

Example 3.3. For $n = 2$, we may take L_+ to be the closure of σ_1, in which case L_- is the closure of σ_1^{-1} and $L_0 = \bigcirc\bigcirc$ is the 2-component unlink. Applying the skein relation, we find $P(L_0) = \frac{a-a^{-1}}{z}$.

Surprisingly, the HOMFLY polynomial computes the number of $\mathbb{F}_q$-points of *any* positroid variety.

Theorem 3.4. *For all $f \in S_{k,n}$, let $P_f^{\mathrm{top}}(q)$ be obtained from the top a-degree term of $P(\hat{\beta}_f; a, z)$ by substituting $a := q^{-\frac{1}{2}}$ and $z := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$. Then $\#\Pi_f^\circ(\mathbb{F}_q) = (q-1)^{n-1} \cdot P_f^{\mathrm{top}}(q)$.*

Remark 3.5. When $\gcd(k, n) = 1$, we have $f_{k,n} \in S_{k,n}^{\mathrm{ncyc}=1}$, and the associated knot $\hat{\beta}_{f_{k,n}}$ is the $(k, n-k)$-*torus knot*; see Figure 2(right). The value of $P(\hat{\beta}_{f_{k,n}}; a, z)$ was computed in [11], and its relationship with Catalan numbers was clarified in [8]. Thus Proposition 2.4 follows from Theorem 3.4 as a direct corollary.

Example 3.6. For $k = 3$, $n = 8$, one calculates (for instance, using Sage[3]) that the top a-degree term of $P(\hat{\beta}_{f_{k,n}}; a, z)$ equals $\frac{z^8+8z^6+21z^4+21z^2+7}{a^8}$. Substituting $a := q^{-\frac{1}{2}}$ and $z := q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, we get

$$P_f^{\mathrm{top}}(q) = q^8 + q^6 + q^5 + q^4 + q^3 + q^2 + 1 = q^4 \cdot C_{3,5}(q, 1/q).$$

This agrees with (2.6) and Theorem 3.4.

Links associated to Richardson varieties. By Proposition 3.1, positroid varieties correspond to pairs $v \le w$ of permutations such that w is k-Grassmannian. There is a more general class of *(open) Richardson varieties* $R_{v,w}^\circ$, indexed by all pairs $v \le w \in S_n$, and the majority of the above story generalizes to this setting. The varieties $R_{v,w}^\circ$ form a stratification of the *complete flag variety* $\mathrm{Fl}_n(\mathbb{C})$. For a permutation $f \in S_{k,n}$ corresponding to a pair $v \le w \in S_n$ via Proposition 3.1, the projection map $\mathrm{Fl}_n(\mathbb{C}) \to \mathrm{Gr}(k, n)$ restricts

[3] https://doc.sagemath.org/html/en/reference/knots/sage/knots/link.html

to an isomorphism $\Pi_f^\circ \cong R_{v,w}^\circ$. Thus positroid varieties are special cases of Richardson varieties.

Now, let G be a complex semisimple algebraic group of adjoint type, and choose a pair $B, B_- \subset G$ of opposite Borel subgroups. Let $T := B \cap B_-$ be the maximal torus and $W := N_G(T)/T$ the associated Weyl group. For the case $G = \mathrm{SL}_n(\mathbb{C})$, we have $W = S_n$, the subgroups $B, B_- \subset G$ consist of upper and lower triangular matrices, and $T \cong (\mathbb{C}^*)^{n-1}$ is the group of diagonal matrices modulo scalar matrices.[4] We have Bruhat decompositions $G = \bigsqcup_{w \in W} BwB = \bigsqcup_{v \in W} B_- vB$, and the intersection $BwB \cap B_- vB$ is nonempty if and only if $v \leq w$ in the Bruhat order on W. For $v \leq w$, we denote by $R_{v,w}^\circ := (BwB \cap B_- vB)/B$ an *open Richardson variety* inside the *complete flag variety* G/B. For $G = \mathrm{SL}_n(\mathbb{C})$, $R_{v,w}^\circ$ is the subset of $G/B = \mathrm{Fl}_n(\mathbb{C})$ obtained by specifying the dimensions of the intersections of a given flag with a given coordinate flag and its opposite coordinate flag.

In the case $G = \mathrm{SL}_n(\mathbb{C})$, one can similarly associate a braid $\beta_{v,w} := \beta(w) \cdot \beta(v)^{-1}$ to any pair $v \leq w$ and consider its closure $\hat{\beta}_{v,w}$. We refer to the links of the form $\hat{\beta}_{v,w}$ as *Richardson links*. The point count $\#R_{v,w}^\circ(\mathbb{F}_q)$ is given by the *Kazhdan–Lusztig R-polynomial* [12, 13], and both the statement and the proof of Theorem 3.4 generalize to this setting.

4 Main results

All of the above results are actually special cases of a single statement which applies to arbitrary Richardson varieties. This includes all positroid varieties Π_f° for $f \in S_{k,n}$, where $\mathrm{ncyc}(f)$ can be arbitrary. As a warm up, we start with the non-equivariant version.

Ordinary cohomology. Let $\mathfrak{h} := \mathrm{Lie}(T)$ be the Cartan subalgebra of $\mathrm{Lie}(G)$ corresponding to T, and denote $R := \mathbb{C}[\mathfrak{h}^*]$. For $G = \mathrm{SL}_n(\mathbb{C})$, $R = \mathbb{C}[y_1, \dots, y_{n-1}]$ is the polynomial ring. Since W is a Coxeter group, we can consider the category $\mathbb{S}\mathrm{Bim}$ of *Soergel bimodules*. Each element $B \in \mathbb{S}\mathrm{Bim}$ is a graded R-bimodule, and we will be interested in its R-invariants, which by definition form the *zeroth Hochschild cohomology* $HH^0(B)$ *of* B. Denote $HH^0_{\mathbb{C}}(B) := HH^0(B) \otimes_R \mathbb{C}$, where $\mathbb{C} = R/(\mathfrak{h}^*)$ is an R-module on which $\mathfrak{h}^*$ acts by 0. While the functor HH^0 involves Soergel bimodules, the functor $HH^0_{\mathbb{C}}$ involves *Soergel modules* instead.

To any element $w \in W$, Rouquier [22] associates a cochain complex $F^\bullet(w)$ of Soergel bimodules. He also associates another complex $F^\bullet(w)^{-1}$ such that their tensor product $F^\bullet(w) \otimes_R F^\bullet(w)^{-1}$ is homotopic to the identity. For a braid $\beta_{v,w} = \beta(w) \cdot \beta(v)^{-1}$, we set $F^\bullet_{v,w} := F^\bullet(w) \otimes_R F^\bullet(v)^{-1}$. Applying the functor $HH^0_{\mathbb{C}}$ to each term of this complex yields

[4] Note that G is assumed to be of *adjoint type*, thus in type A we should have $G = \mathrm{PGL}_n(\mathbb{C})$. However, we choose to work with $G = \mathrm{SL}_n(\mathbb{C})$ for simplicity.

a complex $HH^0_{\mathbb{C}}(F^\bullet_{v,w})$ of graded R-modules. Taking its cohomology $HHH^0_{\mathbb{C}}(F^\bullet_{v,w}) := H^\bullet(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$, we get a bigraded R-module. We denote by $H^{k,(p)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$ the polynomial degree $2p$ part of $H^k(HH^0_{\mathbb{C}}(F^\bullet_{v,w}))$. By convention, the elements of $\mathfrak{h}^* \subset R$ are assumed to have polynomial degree 2. On the other hand, let us denote by $H^{k,(p,p)}(R^\circ_{v,w})$ the (p,p) part of the mixed Hodge structure on $H^k(R^\circ_{v,w})$. See Table 1 for an example.

Theorem 4.1. *For all $v \le w \in W$ and $k,p \in \mathbb{Z}$, we have*

$$\dim H^{k,(p,p)}(R^\circ_{v,w}) = \dim H^{-k,(p)}(HH^0_{\mathbb{C}}(F^\bullet_{v,w})). \tag{4.1}$$

Equivariant cohomology. The spaces $HHH^0(F^\bullet_{v,w})$ and $HHH^0_{\mathbb{C}}(F^\bullet_{v,w})$ are closely related. By Theorem 4.1, $HHH^0_{\mathbb{C}}(F^\bullet_{v,w})$ yields the cohomology of $R^\circ_{v,w}$. It turns out that $HHH^0(F^\bullet_{v,w})$ yields the *torus-equivariant* cohomology of $R^\circ_{v,w}$.

The algebraic torus T acts on each Richardson variety $R^\circ_{v,w}$, and thus we can consider its *T-equivariant cohomology with compact support*, denoted $H^\bullet_{T,c}(R^\circ_{v,w})$. It is equipped with an action of the ring $H^\bullet_{T,c}(\mathrm{pt}) \cong R$. Similarly to the positroid case, $H^\bullet_{T,c}(R^\circ_{v,w})$ admits a second grading via the mixed Hodge structure and is therefore a bigraded R-module.

Theorem 4.2. *For all $v \le w \in W$, we have an isomorphism of bigraded R-modules*

$$H^\bullet_{T,c}(R^\circ_{v,w}) \cong HHH^0(F^\bullet_{v,w}). \tag{4.2}$$

It restricts to a vector space isomorphism $H^{\ell_{v,w}+2p+k,(p,p)}_{T,c}(R^\circ_{v,w}) \cong H^{k,(p)}(HH^0(F^\bullet_{v,w}))$ for each $k,p \in \mathbb{Z}$, where $\ell_{v,w} = \ell(w) - \ell(v) = \dim R^\circ_{v,w}$.

Koszul duality and q,t-symmetry. One can encode the dimensions of bigraded components of $HHH^0(F^\bullet(\beta))$, resp., $HHH^0_{\mathbb{C}}(F^\bullet(\beta))$ in a two-variable polynomial $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\beta;q,t)$, resp., $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta;q,t)$.[5] For any $f \in S^{\mathrm{ncyc}=1}_{k,n}$, the positroid variety Π°_f/T is a cluster variety [6], so the polynomial $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_f;q,t)$ satisfies the properties (i)–(iv) listed in Theorem 2.2 by the results of [16]. In particular, it is q,t-symmetric and q,t-unimodal.

Richardson varieties are not yet known to admit cluster structures (see [17]), in particular, it does not follow from Theorem 2.2 that $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w};q,t)$ is q,t-symmetric for arbitrary $v \le w \in S_n$. We show that the q,t-symmetry phenomenon for such links is a manifestation of *Koszul duality* for mixed perverse sheaves [1, 2].

Theorem 4.3 (Koszul duality)**.** *For any $v \le w \in S_n$, we have*

$$\mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta_{v,w};q,t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR};\mathbb{C}}(\beta_{v,w};t,q).$$

If $\hat{\beta}_{v,w}$ is a knot then it follows that $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w};q,t) = \mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\hat{\beta}_{v,w};t,q)$. This gives a new proof of the q,t-symmetry of $C_{k,n-k}(q,t)$ for $\gcd(k,n) = 1$.

[5] The polynomial $\mathcal{P}^{\mathrm{top}}_{\mathrm{KR}}(\beta;q,t)$ the top a-degree coefficient of *Khovanov–Rozansky homology* [14] of $\hat{\beta}$.

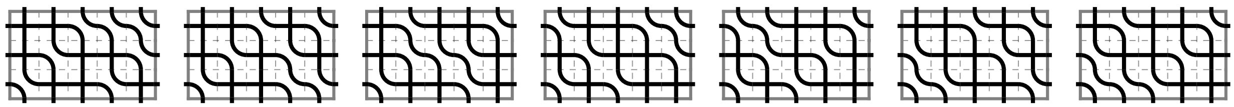


Figure 3: The sets $\mathrm{Deo}^{\max}_{f_{k,n}}$ and $\mathrm{Dyck}_{k,n-k}$ have the same cardinality by (5.1). Compare with Figure 1.

5 Catalan numbers associated to positroid varieties

An important combinatorial consequence of our results is an embedding of rational q,t-Catalan numbers $C_{k,n-k}(q,t)$ into a family of q,t-polynomials $\mathcal{P}(\Pi^\circ_f/T;q,t) \in \mathbb{N}[q^{\frac{1}{2}},t^{\frac{1}{2}}]$ indexed by permutations $f \in S^{\mathrm{ncyc}=1}_{k,n}$ (all of which are q,t-symmetric and q,t-unimodal).

Definition 5.1. For $f \in S^{\mathrm{ncyc}=1}_{k,n}$, define the *$f$-Catalan number $C_f \in \mathbb{Z}$* as the $q=1$ specialization of the point count polynomial $\#(\Pi^\circ_f/T)(\mathbb{F}_q)$.

Let us focus on the case $f = f_{k,n}$ with $\gcd(k,n) = 1$. We will show that $C_{f_{k,n}} = C_{k,n-k}(1,1) = \#\mathrm{Dyck}_{k,(n-k)}$ counts certain pipe dreams inside a $k \times (n-k)$ rectangle. This interpretation extends to arbitrary $f \in S_{k,n}$ in a straightforward fashion.

Definition 5.2. Let $\gcd(k,n) = 1$. A *maximal $f_{k,n}$-Deogram* (short for *Deodhar diagram*) is a way of placing $n-1$ elbows in a $k \times (n-k)$ rectangle and filling the rest with crossings so that (i) the resulting permutation obtained by following the paths is the identity, and (ii) the following *distinguished condition* [4] is satisfied: if any two paths have crossed an odd number of times, they cannot form an elbow. See Figure 3 for an example.

Denote the set of maximal $f_{k,n}$-Deograms by $\mathrm{Deo}^{\max}_{f_{k,n}}$. It follows by combining our results with [4] that $C_{f_{k,n}}$ equals the number of maximal $f_{k,n}$-Deograms:

$$C_{f_{k,n}} = \#\mathrm{Deo}^{\max}_{f_{k,n}}. \tag{5.1}$$

An analogous result holds for arbitrary $f \in S_{k,n}$. It would be interesting to give a bijective proof of (5.1).

Problem 5.3. Find a bijection between $\mathrm{Deo}^{\max}_{f_{k,n}}$ and $\mathrm{Dyck}_{k,(n-k)}$ for the case $\gcd(k,n) = 1$.

For the case $n = 2k+1$ of the standard Catalan numbers, the maximal $f_{k,n}$-Deograms are easily seen (exercise) to be in bijection with *non-crossing alternating trees* on $n+1$ vertices. A recursive proof of (5.1) for the case $n = dk \pm 1$ ($d \geq 2$) was found by David Speyer. We were able to find a recursive proof of (5.1) for arbitrary k,n. The problem of finding a bijective proof remains open.

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