Séminaire Lotharingien de Combinatoire XX (2021)
Article #YY, 12 pp.

Proceedings of the 33rd Conference on Formal Power Series and Algebraic Combinatorics (Ramat Gan)

Critical varieties in the Grassmannian

Pavel Galashin*

Department of Mathematics, University of California, Los Angeles

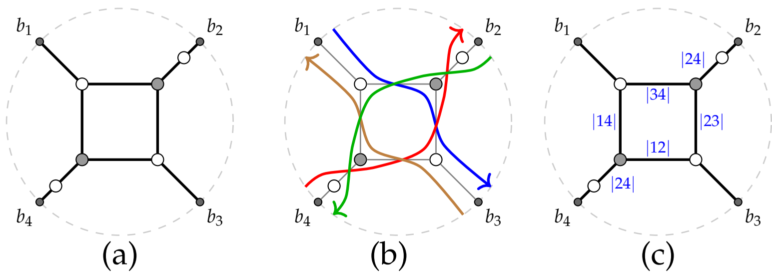
Abstract. We introduce a family of spaces called *critical varieties*. The positive part of each critical variety is a subset of one of Postnikov's positroid cells inside the totally nonnegative Grassmannian. The combinatorics of positroid cells is captured by the dimer model on a planar bipartite graph G, and the critical variety is obtained by restricting to Kenyon's critical dimer model associated to a family of isoradial embeddings of G. This model is invariant under square/spider moves on G, and we give an explicit boundary measurement formula for critical varieties which does not depend on the choice of G. Special cases include critical electrical networks and Baxter's critical Z-invariant Ising model associated to rhombus tilings of polygons in the plane. In the case of regular polygons, our formula yields new simple expressions for response matrices of electrical networks and for correlation matrices of the Ising model.

1 Dimer model

Let G be a planar bipartite graph embedded in a disk. We assume that G has n black boundary vertices, each of degree 1, labeled $b_1, b_2, \dots, b_n$ in clockwise order. A *strand* (or a *zig-zag path*) in G is a path that makes a sharp right turn at each black vertex and a sharp left turn at each white vertex. Thus G gives rise to a *strand permutation* $f_G \in S_n$: for each $1 \le p \le n$, the strand that starts at b_p terminates at $b_{f_G(p)}$. See Figure 1(b). We say that G is *reduced* [19] if it has the minimal number of faces among all graphs with the same strand permutation. It is known that a reduced graph contains no closed strands, thus each strand starts and ends at the boundary of G. For simplicity, we always assume that f_G has no fixed points ($f_G(p) = p$), although the theory extends to that case in a straightforward fashion.

Given a reduced planar bipartite graph G, one can consider the dimer model on it. Let us assign a positive real weight $\mathrm{wt}(e)$ to each edge e of G. An *almost perfect matching* $\mathcal{A}$ of G is a collection of edges of G that uses each interior vertex exactly once, and uses some subset of boundary vertices. We denote by $\{b_p\}_{p \in I_{\mathcal{A}}}$ for $I_{\mathcal{A}} \subset [n] := \{1, 2, \dots, n\}$ the set of boundary vertices used by $\mathcal{A}$. It is easy to check that there exists an integer

*galashin@math.ucla.edu. P.G. was supported by the National Science Foundation under Grant No. DMS-1954121.



$$\begin{aligned}
\Delta_{12} &= |23|\cdot|24| \\
\Delta_{23} &= |34|\cdot|24| \\
\Delta_{34} &= |14|\cdot|24| \\
\Delta_{14} &= |12|\cdot|24| \\
\Delta_{13} &= |24|\cdot|24| \\
\Delta_{24} &= |14|\cdot|23| + |12|\cdot|34| \overset{\text{by Ptolemy's theorem}}{=} |13|\cdot|24|
\end{aligned}$$

(d)

Figure 1: (a) A (reduced) planar bipartite graph G; (b) strands in G; (c) edge weights $\mathrm{wt}_{\mathbf{u}}$, where the unmarked edges have weight 1 and we abbreviate $|pq| := |u_q - u_p|$; (d) the boundary measurements $\Delta_I(G, \mathrm{wt}_{\mathbf{u}})$.

$0 \le k \le n$ such that any almost perfect matching of G satisfies $|I_{\mathcal{A}}| = k$. The number k depends only on f_G: we have

$$k = \#\{1 \le p \le n \mid f_G(p) < p\}. \tag{1.1}$$

Denote the set of k-element subsets of $[n]$ by $\binom{[n]}{k}$, and for $I \in \binom{[n]}{k}$, define

$$\Delta_I(G, \mathrm{wt}) := \sum_{\mathcal{A}:\, I_{\mathcal{A}} = I} \mathrm{wt}(\mathcal{A}), \quad \text{where} \quad \mathrm{wt}(\mathcal{A}) := \prod_{e \in \mathcal{A}} \mathrm{wt}(e),$$

where the summation runs over almost perfect matchings of G. As we explain in Section 4, the *boundary measurements* $\mathrm{Meas}(G, \mathrm{wt}) := (\Delta_I(G, \mathrm{wt}))_{I \in \binom{[n]}{k}}$ give rise to a point in the *totally nonnegative Grassmannian* [16, 19]. We consider the tuple $(\Delta_I(G, \mathrm{wt}))_{I \in \binom{[n]}{k}}$ to be defined up to multiplication by a common scalar.

One can apply certain *moves* to (G, wt) which preserve the boundary measurements. The most interesting transformation is known as the *square move*: given a bipartite square face F of G, one can uncontract some edges so that all vertices of F become trivalent, then swap the colors of the vertices of F, and then contract the unicolored edges to obtain a new planar bipartite graph G'; see Figure 2(a) for an example. The weights of the edges are changed appropriately, and the resulting weighted graph (G', wt') satisfies $\mathrm{Meas}(G, \mathrm{wt}) = \mathrm{Meas}(G', \mathrm{wt}')$ and $f_G = f_{G'}$. Conversely, any two reduced bipartite graphs with the same strand permutation can be related by a sequence of square moves.

2 Critical dimer model

Let G be a reduced bipartite graph with strand permutation f. Choose n points $\mathbf{u} := (u_1, u_2, \ldots, u_n)$ on the unit circle. For simplicity, assume that they are listed in counterclockwise order (more generally, see Definition 5.1). The main idea of *critical varieties*

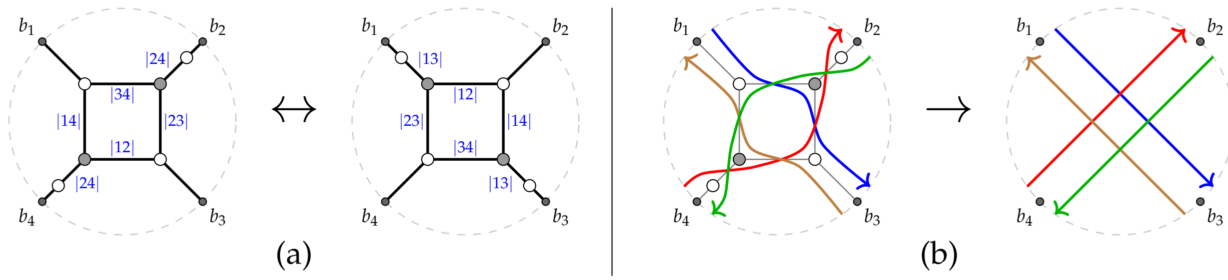


Figure 2: (a) A square move and its effect on $\mathrm{wt}_{\mathbf{u}}$; (b) converting a plabic graph G into a circular diagram of f_G from Section 5.

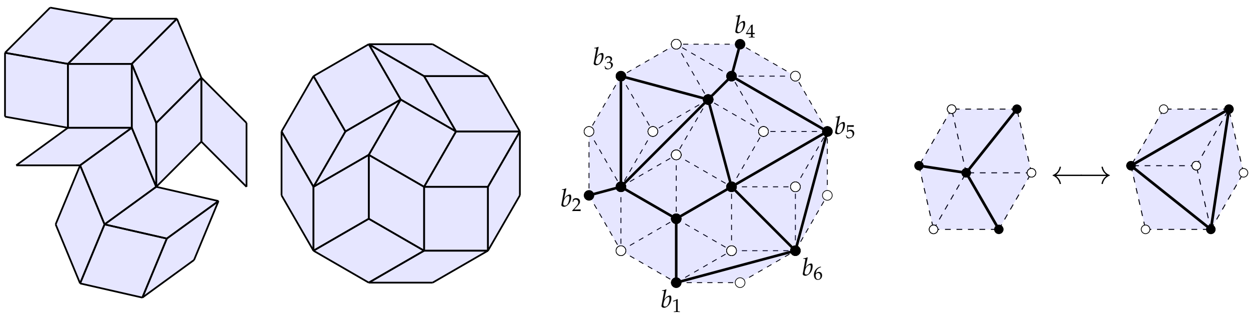
defined below is to consider not just an arbitrary weight function $\mathrm{wt} : E(G) \to \mathbb{R}_{>0}$, but a particular weight function $\mathrm{wt}_{\mathbf{u}}$ defined as follows. Observe that every edge e of G belongs to exactly two strands. Suppose that one strand terminates at b_p and the other strand terminates at b_q for some $1 \le p, q \le n$. If e is not adjacent to a boundary vertex, we let

$$\mathrm{wt}_{\mathbf{u}}(e) := |u_q - u_p| \tag{2.1}$$

be the distance between u_p and u_q. If e is adjacent to the boundary, we set $\mathrm{wt}_{\mathbf{u}}(e) := 1$.

A crucial property of such a choice is that the result is invariant under square moves: for any two reduced graphs G, G' with the same strand permutation f, we have $\mathrm{Meas}(G, \mathrm{wt}_{\mathbf{u}}) = \mathrm{Meas}(G', \mathrm{wt}'_{\mathbf{u}})$. For instance, the two graphs in Figure 2(a) produce the same boundary measurements (up to a common scalar). Thus $\mathrm{Meas}(G, \mathrm{wt}_{\mathbf{u}})$ depends only on f and on $\mathbf{u}$, therefore it makes sense to denote $\mathrm{Meas}(f, \mathbf{u}) := \mathrm{Meas}(G, \mathrm{wt}_{\mathbf{u}})$. Our main result is an explicit simple formula for $\mathrm{Meas}(f, \mathbf{u})$ which depends only on f and $\mathbf{u}$, and does not involve choosing a reduced graph G.

Remark 2.1. The origin of (2.1) comes from the notion of a *critical dimer model* [10] associated to *isoradial graphs* [17]. An embedded planar graph G is called *isoradial* if each interior face of G is inscribed in a circle of radius one. To a given reduced bipartite graph G and a collection $\mathbf{u}$ of points on a circle one can associate a planar embedding of G (as well as of its planar dual) known under the name *plabic tiling* [18]. This embedding is easily seen to be isoradial. The assignment of weights [10] to the edges of G simplifies considerably in the case of plabic tilings, giving rise to the expression (2.1) in terms of strands. While the critical dimer model and its connections to electrical networks and the Ising model are well known, the specialization plabic tilings and the totally nonnegative Grassmannian appears to not have been studied before.



(a) arbitrary polygon R (b) regular polygon R_n (c) isoradial graph $G_{\mathbb{T}}$ (d) star-triangle move

Figure 3: (a) A rhombus tiling of an arbitrary polygon R; (b) a rhombus tiling of a regular polygon R_N for $N = 6$; (c) the associated isoradial graph $G_{\mathbb{T}}$ consists of black vertices and black solid edges; (d) a flip of a rhombus tiling resulting in a star-triangle move on $G_{\mathbb{T}}$.

3 Electrical networks and the Ising model

We shall come back to the dimer model and explain our boundary measurement formula for critical varieties in the next sections. First, as a warm up, we review two particularly neat applications of the formula in the "cyclically symmetric" case.

Both critical electrical networks and the critical Ising model are defined on isoradial graphs. To obtain an isoradial graph, take a rhombus tiling $\mathbb{T}$ of a polygonal region R, such as the one in Figure 3(a), color its vertices black and white in a bipartite way, and let $G_{\mathbb{T}}$ be the graph consisting of all diagonals of rhombi that connect their black vertices; see Figure 3(c). The graph $G_{\mathbb{T}}$ is isoradial, with white vertices of $\mathbb{T}$ being the centers of the corresponding unit circles. One then associates a weight to each edge e of $G_{\mathbb{T}}$ that depends on the geometry of the rhombus containing e, and treats this weight as either the resistance (in the case of electrical networks) or as the interaction constant (in the case of the Ising model) associated to e, as explained below.

Denote by $b_1, b_2, \dots, b_N$ the vertices of $G_{\mathbb{T}}$ that belong to the boundary of R, listed in clockwise order. We now give some background on electrical networks and the Ising model, and explain how in each case one can apply natural boundary measurements to pairs (b_p, b_q) of boundary vertices.

Electrical networks. We consider $G_{\mathbb{T}}$ as an electrical resistor network, replacing every edge by a resistor. Given a rhombus $ABCD$ of $\mathbb{T}$ with black vertices A and C, the edge AC of $G_{\mathbb{T}}$ is treated as a resistor whose resistance equals the ratio of the diagonals $\frac{|AC|}{|BD|}$.

Let us apply the voltage of 1 to b_p and the voltage of 0 to all other boundary vertices.

Then the voltages at all interior vertices, as well as the currents through all edges, can be computed from the Ohm's and Kirchhoff's laws. Denote by $\Lambda^{\mathbb{T}}_{p,q}$ the current that flows out of b_q.[1] The matrix $\Lambda^{\mathbb{T}}_{\mathrm{elec}} = (\Lambda^{\mathbb{T}}_{p,q})$ is known as the *response matrix* of the electrical network associated with $G_{\mathbb{T}}$.

Ising model. Denote by $V := V(G_{\mathbb{T}})$ and $E := E(G_{\mathbb{T}})$ the vertex and edge sets of $G_{\mathbb{T}}$. A *spin configuration* is an assignment $\sigma = (\sigma_v)_{v \in V} \in \{\pm 1\}^V$ of spins to the vertices of $G_{\mathbb{T}}$, where we have $\sigma_v = \pm 1$ for each $v \in V$. For an edge $e \in E$, we denote $x_e := \cot(\theta_e/2)$, where $2\theta_e \in (0, \pi)$ is the angle at a white vertex of the rhombus of $\mathbb{T}$ containing e. The *critical Z-invariant Ising model* [1, 2] is a probability distribution on the set $\{\pm 1\}^V$ of all spin configurations: the probability of a given spin configuration σ equals

$$\mathrm{Prob}_{\mathbb{T}}(\sigma) := \frac{1}{Z_{\mathbb{T}}} \prod_{\{u,v\} \in E:\, \sigma_u = \sigma_v} x_{\{u,v\}}, \quad \text{where} \quad Z_{\mathbb{T}} := \sum_{\sigma \in \{\pm 1\}^V} \prod_{\{u,v\} \in E:\, \sigma_u = \sigma_v} x_{\{u,v\}}$$

is the *partition function*. Given two boundary vertices b_p, b_q, we define their *correlation* as

$$\langle \sigma_p \sigma_q \rangle_{\mathbb{T}} := \mathrm{Prob}_{\mathbb{T}}(\sigma_p = \sigma_q) - \mathrm{Prob}_{\mathbb{T}}(\sigma_p \neq \sigma_q).$$

The matrix $M^{\mathbb{T}}_{\mathrm{Ising}} = (\langle \sigma_p \sigma_q \rangle_{\mathbb{T}})$ is known as the *boundary correlation matrix* of the Ising model on $G_{\mathbb{T}}$.

Star-triangle moves. It is known [11] that any two rhombus tilings of the same region can be related by a sequence of *flips* as in Figure 3(d). Applying a flip to a rhombus tiling results in applying a *star-triangle move* to the weighted graph $G_{\mathbb{T}}$. A very well known property of the electrical response matrix $\Lambda^{\mathbb{T}}_{\mathrm{elec}}$ is that it is preserved by such moves. The same property also holds for the critical Z-invariant Ising model: the boundary correlation matrix $M^{\mathbb{T}}_{\mathrm{Ising}}$ is preserved when $\mathbb{T}$ changes by a flip. Therefore both $\Lambda^{\mathbb{T}}_{\mathrm{elec}}$ and $M^{\mathbb{T}}_{\mathrm{Ising}}$ depend only on the region R itself, and not on the particular choice of a rhombus tiling $\mathbb{T}$. It is thus natural to denote $\Lambda^{R}_{\mathrm{elec}} := \Lambda^{\mathbb{T}}_{\mathrm{elec}}$ and $M^{R}_{\mathrm{Ising}} := M^{\mathbb{T}}_{\mathrm{Ising}}$. A consequence of our main results is a formula for $\Lambda^{R}_{\mathrm{elec}}$ and M^{R}_{Ising} that depends manifestly only on the region R.

Regular polygons. For arbitrary regions R, our formula (Corollary 7.2) involves computing the inverse of an $N \times N$ matrix. However, in the most symmetric case when the region is a regular $2N$-gon, the matrix can be inverted explicitly, which gives rise to the following results. Let us denote the regular $2N$-gon by R_N.

[1] By linearity of the Ohm's and Kirchhoff's laws, knowing $\Lambda^{\mathbb{T}}_{p,q}$ for all $1 \leq p, q \leq N$ allows one to solve the more general problem: for any known voltages that are applied to the boundary vertices, one finds the resulting currents flowing through each boundary vertex.

Theorem 3.1. *For $1 \leq p, q \leq N$ and $d := |p-q|$, we have*

$$\Lambda_{p,q}^{R_N} = \frac{\sin(\pi/N)}{N \cdot \sin((2d-1)\pi/2N) \cdot \sin((2d+1)\pi/2N)}. \tag{3.1}$$

Example 3.2. Consider the star electrical network as in Figure 3(d) inside a regular hexagon R_3. Then the resistance of each edge equals $\frac{1}{\sqrt{3}}$. Applying the voltage of 1 to b_1 and the voltage of 0 to b_2 and b_3, we calculate that the resulting voltage at the unique interior vertex is $\frac{1}{3}$, and thus the currents through b_2 and b_3 are both equal to $\frac{1}{\sqrt{3}}$. This agrees with (3.1) for $N = 3$ and $d = 1, 2$. For $d = 0$, we also obtain the correct value $-\frac{2}{\sqrt{3}}$ for the current through b_1, the negative sign representing the fact that the current flows *into* the network.

Theorem 3.3. *For $1 \leq p, q \leq N$ and $d := |p-q|$, we have*

$$\langle \sigma_p \sigma_q \rangle_{R_N} = \frac{2}{N} \left(\frac{1}{\sin((2d-1)\pi/2N)} - \frac{1}{\sin((2d-3)\pi/2N)} + \cdots \pm \frac{1}{\sin(\pi/2N)} \right) \mp 1.$$

As explained in [4, Remark 1.15], the formula in Theorem 3.3 describes the unique $N \times N$ boundary correlation matrix of the Ising model that is invariant under the Kramers–Wannier duality [14].

Example 3.4. For the square R_2, the graph $G_{\mathbb{T}}$ consists of a single edge e with $x_e = \cot(\pi/8) = \sqrt{2}+1$. By definition, we find $\langle \sigma_1 \sigma_2 \rangle_{R_2} = \sqrt{2}-1$, which agrees with Theorem 3.3 for $N = 2$ and $d = 1$.

Remark 3.5. Despite the simplicity of these two results, both of them are apparently new. They also lead to new asymptotic consequences (including a convergence result for the Ising model in a disk to a conformally invariant limit [4]) which fall outside the scope of the present extended abstract.

4 The totally nonnegative Grassmannian

The *Grassmannian* $\mathrm{Gr}(k,n)$ is the space of k-dimensional linear subspaces of $\mathbb{R}^n$. It can also be identified with the set of full rank $k \times n$ real matrices modulo row operations, via the bijection sending a matrix M to $\mathrm{RowSpan}(M)$. Given such a matrix M and a k-element set $I \in \binom{[n]}{k}$ of columns, let $\Delta_I(M)$ denote the maximal minor of M with column set I. The collection $(\Delta_I(M))_{I \in \binom{[n]}{k}}$ of *Plücker coordinates* of M uniquely determines the point $\mathrm{RowSpan}(M) \in \mathrm{Gr}(k,n)$, and is defined up to a multiplication by a common scalar. The *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geq 0}(k,n)$ is the subset of $\mathrm{Gr}(k,n)$ where all nonzero Plücker coordinates have the same sign. Loosely speaking, we can write

$$\mathrm{Gr}_{\geq 0}(k,n) := \left\{ X \in \mathrm{Gr}(k,n) \,\middle|\, \Delta_I(X) \geq 0 \text{ for all } I \in \binom{[n]}{k} \right\}.$$

The space $\mathrm{Gr}_{\geq 0}(k,n)$ is naturally stratified into *positroid cells* by looking at which Plücker coordinates are zero and which are strictly positive.

Given a reduced bipartite planar graph G with weight function wt, there exists a (necessarily unique) point $X \in \mathrm{Gr}_{\geq 0}(k,n)$ such that $\Delta_I(X) = \Delta_I(G,\mathrm{wt})$ for all $I \in \binom{[n]}{k}$. By an abuse of notation, we also denote this point by $\mathrm{Meas}(G,\mathrm{wt})$. Fixing G and letting $\mathrm{wt} \in \mathbb{R}_{>0}^{E(G)}$ vary over all possible assignments of positive real edge weights, we obtain the *positroid cell* $\Pi_G^{>0} \subset \mathrm{Gr}_{\geq 0}(k,n)$ defined by

$$\Pi_G^{>0} := \{\mathrm{Meas}(G,\mathrm{wt}) \mid \mathrm{wt} \in \mathbb{R}_{>0}^{E(G)}\}.$$

The set $\Pi_G^{>0}$ depends only on f_G and is denoted $\Pi_{f_G}^{>0}$. The positroid cells give a cell decomposition[2] of $\mathrm{Gr}_{\geq 0}(k,n)$.

Thus in particular we may take $\mathrm{wt} := \mathrm{wt}_{\mathbf{u}}$ to come from the critical dimer model, in which case we get a point $\mathrm{Meas}(f_G,\mathbf{u}) := \mathrm{Meas}(G,\mathrm{wt}_{\mathbf{u}}) \in \Pi_{f_G}^{>0}$.

Remark 4.1. Taking f to be the permutation sending $p \mapsto p+k$ modulo n for all $p \in [n]$ (corresponding to the *top-dimensional positroid cell*), and taking the points in $\mathbf{u}$ to be equally spaced on the circle, $\mathrm{Meas}(f,\mathbf{u})$ becomes the unique cyclically symmetric point of $\mathrm{Gr}_{\geq 0}(k,n)$ studied in [5, 8].

5 Critical cells

We have defined positroid cells above. Their Zariski closures in the complex Grassmannian are called *positroid varieties* [13]. Similarly, we will first introduce *critical cells*; critical varieties will be their Zariski closures, and critical cells will be the totally positive parts of critical varieties.

Let $f \in S_n$ be a permutation (without fixed points), and let $k \in [n]$ be defined by (1.1). Place the points $b_1, b_2, \dots, b_n$ on a circle, and for each $p \in [n]$, let p_- (resp., p_+) be the point slightly before (resp., after) b_p in clockwise order. The *circular diagram* of f is obtained by drawing a straight arrow $a_+ \to p_-$ whenever $f(a) = p$; see Figure 2(b). We say that $p \neq q \in [n]$ *form a crossing* if the arrows $a_+ \to p_-$ and $b_+ \to q_-$ cross, where $a := f^{-1}(p)$ and $b := f^{-1}(q)$. For instance, in Figure 2(b), 1 and 2 form a crossing but 1 and 3 do not.

The critical cell $\mathrm{Crit}_f^{>0}$ is parametrized by the following set.

Definition 5.1. A tuple $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n)$ is called *f-admissible* if for all $1 \leq p < q \leq n$ such that p and q form a crossing, we have

$$\theta_p < \theta_q < \theta_p + \pi.$$

[2] To obtain the whole $\mathrm{Gr}_{\geq 0}(k,n)$, one needs to consider *decorated permutations* with fixed points [19].

Thus in the case of Figure 2(b), $\boldsymbol{\theta}$ is f-admissible if and only if $\theta_1 < \theta_2 < \theta_3 < \theta_4 < \theta_1 + \pi$.

Let now G be a reduced bipartite planar graph with strand permutation f and let $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n$ be a tuple of angles. We define a weight function $\mathrm{wt}_{\boldsymbol{\theta}}$ on the edges of G as follows. When $e \in E(G)$ is adjacent to a boundary vertex, we set $\mathrm{wt}_{\boldsymbol{\theta}}(e) := 1$, and otherwise we set

$$\mathrm{wt}_{\boldsymbol{\theta}}(e) := \sin(\theta_q - \theta_p), \tag{5.1}$$

where $1 \leq p < q \leq n$ are the indices such that the two strands passing through e terminate at b_p and b_q. It is not hard to check that all edge weights are positive if and only if $\boldsymbol{\theta}$ is f-admissible.

Remark 5.2. In Section 2, we worked with a tuple of n points $\mathbf{u} := (u_1, u_2, \dots, u_n)$ on the unit circle. Let us regard these points as complex numbers of modulus 1. Then the connection between (2.1) and (5.1) is obtained by setting $u_q := \exp(2i\theta_q)$ for all $q \in [n]$. Then we have $|u_q - u_p| = \sin(\theta_q - \theta_p)$.

Recall from Section 2 that $\mathrm{Meas}(G, \mathrm{wt}_{\boldsymbol{\theta}})$ depends only on f_G and $\boldsymbol{\theta}$, and is denoted $\mathrm{Meas}(f_G, \boldsymbol{\theta})$. The following is the main definition of this paper.

Definition 5.3. For a permutation f, the *critical cell* $\mathrm{Crit}_f^{>0} \subset \Pi_f^{>0}$ is given by

$$\mathrm{Crit}_f^{>0} := \{\mathrm{Meas}(f, \boldsymbol{\theta}) \mid \boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n) \text{ is an } f\text{-admissible tuple}\}.$$

The *critical variety* $\mathrm{Crit}_f^{\mathbb{C}}$ is the Zariski closure of $\mathrm{Crit}_f^{>0}$ inside the complex Grassmannian.

6 Boundary measurement formula

Currently, in order to compute $\mathrm{Meas}(f, \boldsymbol{\theta})$, one needs to choose a reduce planar bipartite graph G. And then the result does not depend on this choice. It is therefore natural to look for an expression for $\mathrm{Meas}(f, \boldsymbol{\theta})$ purely in terms f and $\boldsymbol{\theta}$. Our main result gives a solution to this problem.

Take the circular diagram of f as in Figure 2(b). For each $r \in [n]$, introduce the set

$$J_r := \{q \in [n] \mid b_r \text{ is to the left of the arrow } p_+ \to q_-\}.$$

Here we assume $p = f^{-1}(q)$. Observe that we always have $r \notin J_r$. The collection $(J_r \sqcup \{r\})_{r \in [n]}$ is known as the *Grassmann necklace* [19] of f.

For an index $r \in [n]$, let $\epsilon_r \in \{\pm 1\}$ be given by $\epsilon_r := (-1)^{\#\{p \in [n] \mid f(p) < p < r\}}$.

Definition 6.1. Let $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n)$ be a tuple of angles. Consider a curve $\gamma_{f,\boldsymbol{\theta}} : \mathbb{R} \to \mathbb{R}^n$ with coordinates $\gamma_{f,\boldsymbol{\theta}}(t) = (\gamma_1(t), \gamma_2(t), \dots, \gamma_n(t))$ given by

$$\gamma_r(t) := \epsilon_r \prod_{p \in J_r} \sin(t - \theta_p) \qquad \text{for } r \in [n].$$

It is possible to give a formula for an arbitrary f-admissible tuple $\boldsymbol{\theta}$, but for simplicity, we restrict to the case when $\boldsymbol{\theta}$ is *generic*, that is, when all angles in $\boldsymbol{\theta}$ are pairwise non-congruent modulo π. We are ready to state our main result.

Theorem 6.2. *Suppose that $\boldsymbol{\theta}$ is a generic f-admissible tuple. Then the linear span* $\mathrm{Span}(\gamma_{f,\boldsymbol{\theta}}) \subset \mathbb{R}^n$ *has dimension k and we have*

$$\mathrm{Meas}(f,\boldsymbol{\theta}) = \mathrm{Span}(\gamma_{f,\boldsymbol{\theta}}) \quad \textit{inside } \mathrm{Gr}(k,n).$$

Example 6.3. Let $k=2$, $n=4$, and f be the permutation sending $p \mapsto p+2$ modulo 4. The boundary measurement map $\mathrm{Meas}(f,\boldsymbol{\theta})$ was computed in Figure 1(d). Since the Plücker coordinates are defined up to a common scalar, the term $|24|$ cancels out. The sets J_r are given by $J_1 = \{2\}$, $J_2 = \{3\}$, $J_3 = \{4\}$, and $J_4 = \{1\}$, so $\gamma_{f,\boldsymbol{\theta}}$ has coordinates

$$\gamma_{f,\boldsymbol{\theta}}(t) = (\sin(t-\theta_2), \sin(t-\theta_3), \sin(t-\theta_4), -\sin(t-\theta_1)).$$

We can choose a basis consisting of e.g. $\gamma_{f,\boldsymbol{\theta}}(0)$ and $\gamma_{f,\boldsymbol{\theta}}(\pi/2)$, which we can write in the rows of the following matrix:

$$A = \begin{pmatrix} -\sin(\theta_2) & -\sin(\theta_3) & -\sin(\theta_4) & \sin(\theta_1) \\ \cos(\theta_2) & \cos(\theta_3) & \cos(\theta_4) & -\cos(\theta_1) \end{pmatrix}.$$

We see that the maximal minors of A coincide with the values computed in Figure 1(d).

A few remarks are in order.

- For a generic $\boldsymbol{\theta}$, an explicit basis of $\mathrm{Span}(\gamma_{f,\boldsymbol{\theta}})$ can be chosen by taking any k distinct points on the curve $\gamma_{f,\boldsymbol{\theta}}$.
- A more canonical (and computationally robust) way to produce a basis of $\mathrm{Span}(\gamma_{f,\boldsymbol{\theta}})$ is to observe that each coordinate $\gamma_r(t)$ is a trigonometric polynomial of degree $k-1$. Therefore it has precisely k non-trivial Fourier coefficients. The rows of the resulting $k \times n$ matrix of Fourier coefficients form a basis of $\mathrm{Span}(\gamma_{f,\boldsymbol{\theta}})$ which does not depend on anything besides f and the angles $\theta_1, \theta_2, \dots, \theta_n$.
- After a change of variables $T_p := \exp(i\theta_p)$, $T := \exp(it)$, our boundary measurement formula becomes an algebraic expression in $T_1, T_2, \dots, T_n$. We have $\sin(\theta_q - \theta_p) = \frac{1}{2i}\left(T_q/T_p - T_p/T_q\right)$, thus the edge weights of G are also algebraic. Allowing T_p to run over complex numbers, we obtain a Zariski open subset of $\mathrm{Crit}_f^{\mathbb{C}}$.
- Let $\overline{\mathrm{Crit}}_f^{\geq 0}$ be the closure of $\mathrm{Crit}_f^{>0}$ inside $\mathrm{Gr}_{\geq 0}(k,n)$. An immediate next direction is to determine the boundary cell structure of $\overline{\mathrm{Crit}}_f^{\geq 0}$ in the spirit of the positroid cell decomposition of $\mathrm{Gr}_{\geq 0}(k,n)$. This will be explored in a forthcoming paper.

7 Back to electrical networks and the Ising model

We promised in Section 3 to give a formula for the electrical response matrix $\Lambda^R_{\mathrm{elec}}$ and the boundary correlation matrix M^R_{Ising} associated to a rhombus tiling $\mathbb{T}$ of a polygonal region R. Recall that N is the number of boundary vertices of $G_{\mathbb{T}}$, and let $n := 2N$ be the number of boundary edges of R. Suppose that the boundary vectors of R are $u_1, u_2, \dots, u_{2N}$, listed and directed in clockwise order (and treated as complex numbers of modulus one). The tiling $\mathbb{T}$ gives rise to a fixed-point-free involution $\tau_R : [2N] \to [2N]$, as follows. For each $p \in [2N]$, the boundary vector u_p is contained in a unique rhombus. The opposite side of that rhombus is contained in a unique other rhombus. We continue in this fashion until we arrive at another boundary vector u_q (which necessarily satisfies $u_q = -u_p$), and we set $\tau_R(p) := q$ and $\tau_R(q) := p$. Clearly, τ_R depends only on R and not on $\mathbb{T}$.

Let $f^{\mathrm{elec}}_R, f^{\mathrm{Ising}}_R : [2N] \to [2N]$ be defined by $f^{\mathrm{elec}}_R(p) := \tau_R(p) + 1$ (modulo $2N$) and $f^{\mathrm{Ising}}_R(p) := \tau_R(p)$. One can uniquely extract the square roots to obtain the tuple $\boldsymbol{\theta}_R = (\theta_1, \theta_2, \dots, \theta_{2N})$ that is both f^{elec}_R- and f^{Ising}_R-admissible, such that $u_p = \exp(2i\theta_p)$ for all $p \in [2N]$, and such that $\theta_{\tau_R(p)} = \theta_p + \pi/2$ whenever $p < \tau_R(p)$.

Lam [15] has constructed an embedding ϕ^{elec} of the space of $N \times N$ electrical response matrices into $\mathrm{Gr}_{\geq 0}(N+1, 2N)$.[3] Together with Pylyavskyy [6], we constructed an embedding ϕ^{Ising} of the space of $N \times N$ Ising boundary correlation matrices into $\mathrm{Gr}_{\geq 0}(N, 2N)$.

Theorem 7.1. *For any region R, we have*

$$\phi^{\mathrm{elec}}(\Lambda^R_{\mathrm{elec}}) = \mathrm{Meas}(f^{\mathrm{elec}}_R, \boldsymbol{\theta}_R) \quad \textit{and} \quad \phi^{\mathrm{Ising}}(M^R_{\mathrm{Ising}}) = \mathrm{Meas}(f^{\mathrm{Ising}}_R, \boldsymbol{\theta}_R)$$

The matrices $\Lambda^R_{\mathrm{elec}}$ and M^R_{Ising} can be easily recovered from their respective images inside the Grassmannian. Let us explain the Ising model case in detail, the case of electrical networks being similar. Introduce a $2N \times N$ matrix K_N defined as

$$K_N = \frac{1}{2}\begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

[3] More precisely, Lam's embedding lands in $\mathrm{Gr}_{\geq 0}(N-1, 2N)$. To get an element of $\mathrm{Gr}_{\geq 0}(N+1, 2N)$, one needs to take the orthogonal complement and then change the sign of every second column.

Corollary 7.2. *Let R be a non-alternating region. Choose an $N \times 2N$ matrix A whose row span equals* $\mathrm{Span}(\gamma_R)$*. Let $B = (b_{p,q})$ be the $N \times 2N$ matrix given by*

$$B := (AK_N)^{-1}A.$$

Then, up to a sign, the entries of B are the boundary correlations: we have

$$\langle \sigma_p \sigma_q \rangle_R = \langle \sigma_p \sigma_q \rangle_R = |b_{p,2q-1}| = (-1)^{q-p+1} b_{p,2q-1} \quad \textit{for all } 1 \le p < q \le N.$$

Part of the content of Corollary 7.2 is that the matrix AK_N is always invertible.

Remark 7.3. Our formula involves inverting an $N \times N$ matrix AK_N. There are other celebrated matrix formulas for the dimer and Ising models, e.g., the Kasteleyn matrix or the Kac–Ward matrix [7, 9]. However, these matrices are indexed by the vertices of $G_{\mathbb{T}}$, and therefore have size $O(N^2 \times N^2)$. By contrast, our matrix has size $N \times N$.

Acknowledgements

I am indebted to Pasha Pylyavskyy for his numerous contributions at various stages of the development of [4], where the boundary measurement formula was first discovered in the context of the Ising model. The generalization to the Grassmannian level was inspired by the results of [3, 12], presented by Marianna Russkikh at the "Dimers in Combinatorics and Cluster Algebras" conference at the University of Michigan. I thank Marianna for bringing these results to my attention, and also thank the organizers of the conference (Sebastian Franco, Gregg Musiker, Richard Kenyon, David Speyer, and Lauren Williams) for making such an interaction possible.

References

[1] R. J. Baxter. Solvable eight-vertex model on an arbitrary planar lattice. *Philos. Trans. Roy. Soc. London Ser. A*, 289(1359):315–346, 1978.

[2] R. J. Baxter. Free-fermion, checkerboard and **Z**-invariant lattice models in statistical mechanics. *Proc. Roy. Soc. London Ser. A*, 404(1826):1–33, 1986.

[3] Dmitry Chelkak, Benoît Laslier, and Marianna Russkikh. Dimer model and holomorphic functions on t-embeddings of planar graphs. `arXiv:2001.11871v1`, 2020.

[4] Pavel Galashin. A formula for boundary correlations of the critical Ising model. `arXiv:2010.13345v2`, 2020.

[5] Pavel Galashin, Steven N. Karp, and Thomas Lam. The totally nonnegative Grassmannian is a ball. arXiv:1707.02010, 2017.

[6] Pavel Galashin and Pavlo Pylyavskyy. Ising model and the positive orthogonal Grassmannian. *Duke Math. J.*, 169(10):1877–1942, 2020.

[7] M. Kac and J. C. Ward. A combinatorial solution of the two-dimensional ising model. *Phys. Rev.*, 88:1332–1337, Dec 1952.

[8] Steven N. Karp. Sign variation, the Grassmannian, and total positivity. *J. Combin. Theory Ser. A*, 145:308–339, 2017.

[9] P.W. Kasteleyn. The statistics of dimers on a lattice: I. The number of dimer arrangements on a quadratic lattice. *Physica*, 27(12):1209 – 1225, 1961.

[10] R. Kenyon. The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.*, 150(2):409–439, 2002.

[11] Richard Kenyon. Tiling a polygon with parallelograms. *Algorithmica*, 9(4):382–397, 1993.

[12] Richard Kenyon, Wai Yeung Lam, Sanjay Ramassamy, and Marianna Russkikh. Dimers and Circle patterns. arXiv:1810.05616v2, 2018.

[13] Allen Knutson, Thomas Lam, and David E. Speyer. Positroid varieties: juggling and geometry. *Compos. Math.*, 149(10):1710–1752, 2013.

[14] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet. I. *Phys. Rev. (2)*, 60:252–262, 1941.

[15] Thomas Lam. Electroid varieties and a compactification of the space of electrical networks. *Adv. Math.*, 338:549–600, 2018.

[16] G. Lusztig. Total positivity in partial flag manifolds. *Represent. Theory*, 2:70–78, 1998.

[17] Christian Mercat. Discrete Riemann surfaces and the Ising model. *Comm. Math. Phys.*, 218(1):177–216, 2001.

[18] Suho Oh, Alexander Postnikov, and David E. Speyer. Weak separation and plabic graphs. *Proc. Lond. Math. Soc. (3)*, 110(3):721–754, 2015.

[19] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, http:// math.mit.edu/~apost/papers/tpgrass.pdf, 2006.