SYMMETRIES OF STOCHASTIC COLORED VERTEX MODELS

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ABSTRACT. We discover a new property of the stochastic colored six-vertex model called *flip-invariance*. We use it to show that for a given collection of observables of the model, any transformation that preserves the distribution of each individual observable also preserves their joint distribution. This generalizes recent shift-invariance results of Borodin–Gorin–Wheeler. As limiting cases, we obtain similar statements for the Brownian last passage percolation, the Kardar–Parisi–Zhang equation, the Airy sheet, and directed polymers. Our proof relies on an equivalence between the stochastic colored six-vertex model and the Yang–Baxter basis of the Hecke algebra. We conclude by discussing the relationship of the model with Kazhdan–Lusztig polynomials and positroid varieties in the Grassmannian.

CONTENTS

1. INTRODUCTION

We study various symmetries of the *stochastic colored six-vertex model*. It is a special case of a "higher spin" colored vertex model introduced in [KMMO16], related to the stochastic version of the R-matrix for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_{n+1})$ studied earlier in [Baz85, FRT88, Jim86a, Jim86b]. The model admits a very simple description: one fixes a subdomain of the square grid and considers a family of paths that enter it from the bottom left. Once two paths meet at a vertex of the grid, they either cross with probability b or do not cross with probability $1-b$. The value of b depends (in a certain *integrable* way) on the coordinates of the vertex and on the parity of the number of times these two paths have crossed before, see Figure 1 and Section 1.1.

The main motivation for our work comes from the recent *shift-invariance* results of [BGW19]. In fact, the *flip-invariance* property (Theorem 1.1 and Figure 2) was originally formulated as an attempt to give a simple explanation for the shift-invariance of [BGW19]: one can
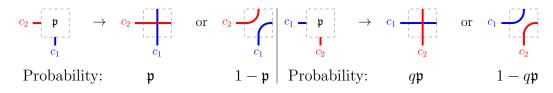
FIGURE 1. When two paths of colors $c_1 < c_2$ enter a cell (i,j), they proceed in the up-right direction according to these probabilities. Here $\mathfrak{p}$ is equal to $\mathfrak{p}_{i,j} = \frac{y_j - x_i}{y_j - qx_i}$, see (1.1).

$$\mathbf{P}^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y}) \; = \; \mathbf{P}^{180°(\mathbb{H}),\mathbb{V}}(\mathbf{x}, \operatorname{rev}(\mathbf{y}))$$
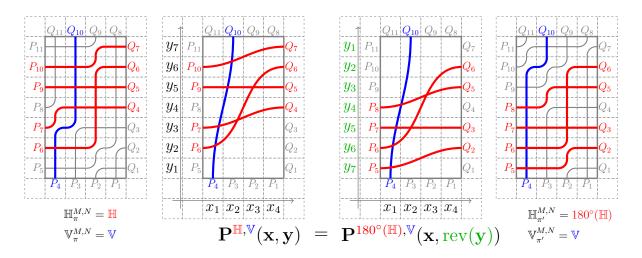
FIGURE 2. The flip theorem states that the two partition functions shown in the middle are equal to each other. The picture on the far left (resp., far right) represents a configuration of the stochastic colored six-vertex model contributing to $\mathbf{P}^{\mathbb{H},\mathbb{V}}$ (resp., to $\mathbf{P}^{180°(\mathbb{H}),\mathbb{V}}$). See Example 1.2.

realize their shift as a composition of two flips, see Figure 5 and Remark 4.10. We show in Theorem 1.6 that this happens more generally: whenever a transformation preserves the one-dimensional distributions of a given vector of *height functions*, we prove that it can essentially be obtained as a composition of several flips, and therefore it preserves the joint distribution of this vector of height functions. Thus, in a sense, one can view flip-invariance as the *fundamental* symmetry of the model inside the class of distribution-preserving transformations that we consider.

Our proofs rely on a connection between the stochastic colored six-vertex model and the *Hecke algebra* $\mathcal{H}_q(S_n; \mathbf{z})$ of the symmetric group. Specifically, we make a simple observation (Proposition 2.3) showing that the probabilities induced by the model coincide with the coefficients in the expansion of the *Yang–Baxter basis* [LLT97] of $\mathcal{H}_q(S_n; \mathbf{z})$ in the standard basis. This observation allows us to greatly simplify the proofs of our results. We also use it to give a simple proof (discovered independently by Bufetov [Buf20]) of the *color-position symmetry* [BB19] for the Asymmetric Simple Exclusion Process (ASEP).

The stochastic colored six-vertex model specializes to many other probabilistic objects of interest. Following [BGW19], we obtain versions of our results for the Brownian last passage percolation and directed $(1+1)$d polymers in random media, as well as two universal objects, the Kardar–Parisi–Zhang (KPZ) equation and the Airy sheet, see Section 1.4. We refer the

reader to [BGW19, Figure 2] for the full chart of objects that can be obtained as limits of the stochastic colored six-vertex model (which includes many objects not considered here, for example, the colored q-PushTASEP, TASEP, ASEP, and the Bernoulli-Exponential last passage percolation).

Another consequence of the Hecke algebra approach is a family of unexpected connections between the stochastic colored six-vertex model and well studied algebraic objects such as *Kazhdan–Lusztig polynomials* [KL79, KL80] and *positroid varieties* [Pos06, BGY06, KLS13] reviewed in Section 7. In particular, we give an interpretation of the flip-invariance property in the language of Grassmannians in Proposition 7.5. We hope to further explore the connections between the above objects in future work.

1.1. Stochastic colored six-vertex model.
An *up-left path* P is a lattice path in the positive quadrant with unit steps that go either up or left. A *skew domain* is a pair (P, Q) of up-left paths with common start and end points such that P is weakly below Q. We denote by $n := |P| = |Q|$ the number of steps in P and Q, and we label their steps by $P_1, P_2, \ldots, P_n$ and $Q_1, Q_2, \ldots, Q_n$, respectively. The area between P and Q is subdivided into unit squares called *cells* whose Cartesian coordinates are pairs of positive integers, see Figure 4(left).

Fix a skew domain (P, Q), two sets of formal variables $\mathbf{x} = (x_1, x_2, \ldots)$ and $\mathbf{y} = (y_1, y_2, \ldots)$, called, respectively, *column* and *row rapidities*, and a parameter $0 < q < 1$. To describe the model, we consider colored paths, where a *color* is just an integer assigned to each path that determines its "priority" in the model dynamics. Suppose that for each $c = 1, 2, \ldots, n$, a path of color c enters the domain through the edge P_c. The n paths then propagate in the up-right direction according to the following rule: once two paths enter a cell (i, j) from the left and from the bottom, they proceed in the up-right direction according to the probabilities given in Figure 1. The probabilities depend on a parameter $\mathfrak{p}$, which for a given cell (i, j) equals[1]

$$(1.1) \qquad \qquad \mathfrak{p}_{i,j} := \frac{y_j - x_i}{y_j - q x_i}.$$

We apply this sampling procedure to all cells between P and Q, proceeding in the up-right direction, see Figure 4(right) for an example. This procedure gives rise to a random *color permutation* $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$, where for $c = 1, 2, \ldots, n$, the path of color c exits the domain through the edge $Q_{\pi(c)}$. We let S_n be the group of permutations of $\{1, 2, \ldots, n\}$, and for each $\pi \in S_n$, we denote by $\mathbf{P}_\pi^{P,Q}(\mathbf{x}, \mathbf{y})$ the total probability[2] of observing π as the color permutation.

1.2. Flips in rectangular domains.
We formulate our first result which we call the *flip theorem*. The proof of the main result (Theorem 1.6) will essentially consist of repeated applications of this fundamental hidden symmetry of the model.

For positive integers M, N, an $M \times N$-*rectangular domain* is a skew domain (P, Q) such that P goes M steps left and then N steps up while Q goes N steps up and then M steps left. The lengths of P and Q are given by $n := M + N$. We let $\mathbf{x} = (x_1, \ldots, x_M)$ and $\mathbf{y} = (y_1, \ldots, y_N)$.

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[1]This choice of parameters ensures that the model is *integrable*, i.e., satisfies the *Yang–Baxter equation*.

[2]This is a probability distribution in the sense that $\mathbf{P}_\pi^{P,Q}(\mathbf{x}, \mathbf{y}) \geq 0$ and $\sum_{\pi \in S_n} \mathbf{P}_\pi^{P,Q}(\mathbf{x}, \mathbf{y}) = 1$ when $0 < q < 1$ and $\mathbf{x}, \mathbf{y}$ are specialized so that $0 < x_i < y_j$ for all i, j.
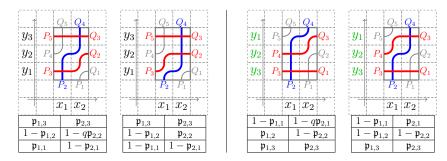
FIGURE 3.   Applying the flip theorem to a 2×3 rectangle leads to a non-trivial identity, see Example 1.3. For each of the four configurations, its probability equals the product of the entries in the corresponding table shown below it.

We will be interested in the probability of observing a color permutation satisfying given *horizontal and vertical boundary conditions*. For a permutation $\pi \in S_n$, we let

$$\mathbb{H}_\pi^{M,N} := \{(i, \pi(i)) \mid i > M \text{ and } \pi(i) \leq N\}, \quad \mathbb{V}_\pi^{M,N} := \{(i, \pi(i)) \mid i \leq M \text{ and } \pi(i) > N\}.$$

For two sets $\mathbb{H} = \{(l_1, r_1), \ldots, (l_h, r_h)\}$ and $\mathbb{V} = \{(d_1, u_1), \ldots, (d_v, u_v)\}$ of pairs, we denote

$$\mathbf{P}^{\mathbb{H}, \mathbb{V}}(\mathbf{x}, \mathbf{y}) := \sum_{\pi \in S_n: \ \mathbb{H}_\pi^{M,N} = \mathbb{H} \text{ and } \mathbb{V}_\pi^{M,N} = \mathbb{V}} \mathbf{P}_\pi^{P,Q}(\mathbf{x}, \mathbf{y}).$$

Thus $\mathbf{P}^{\mathbb{H}, \mathbb{V}}(\mathbf{x}, \mathbf{y})$ is the probability of observing a color permutation with specified endpoints of *each* path that connects the opposite boundaries of the rectangle.

For a set of pairs $\mathbb{H} = \{(l_1, r_1), \ldots, (l_h, r_h)\}$, we define its *180-degree rotation* by

$$180^\circ(\mathbb{H}) := \{(n + 1 - r_1, n + 1 - l_1), \ldots, (n + 1 - r_h, n + 1 - l_h)\}.$$

Additionally, denote $\mathrm{rev}(\mathbf{y}) := (y_N, y_{N-1}, \ldots, y_1)$. We are ready to state the flip theorem, which is a special case of Theorem 1.6 below.

Theorem 1.1. *For an $M \times N$-rectangular domain (P, Q) and any $\mathbb{H}$ and $\mathbb{V}$, we have*

$$(1.2) \qquad\qquad \mathbf{P}^{\mathbb{H}, \mathbb{V}}(\mathbf{x}, \mathbf{y}) = \mathbf{P}^{180^\circ(\mathbb{H}), \mathbb{V}}(\mathbf{x}, \mathrm{rev}(\mathbf{y})).$$

Example 1.2. Figure 2 illustrates Theorem 1.1 in the case $M = 4$, $N = 7$, and

$$\mathbb{H} = \{(6, 6), (7, 4), (9, 5), (10, 7)\}, \quad 180^\circ(\mathbb{H}) = \{(5, 2), (6, 6), (7, 3), (8, 5)\}, \quad \mathbb{V} = \{(4, 10)\}.$$

Example 1.3. Let $M = 2$, $N = 3$, $\mathbb{H} = \{(3, 2), (5, 3)\}$, $\mathbb{V} = \{(2, 4)\}$. The left (resp., right) hand side of (1.2) is the sum of probabilities of the two configurations shown in Figure 3 on the left (resp., right). Thus we get

$$\mathbf{P}^{\mathbb{H}, \mathbb{V}}(\mathbf{x}, \mathbf{y}) = \mathfrak{p}_{1,3}\mathfrak{p}_{2,3}(1 - \mathfrak{p}_{1,2})(1 - \mathfrak{p}_{2,1})\left(\mathfrak{p}_{1,1}(1 - q\mathfrak{p}_{2,2}) + \mathfrak{p}_{2,2}(1 - \mathfrak{p}_{1,1})\right)$$
$$\mathbf{P}^{180^\circ(\mathbb{H}), \mathbb{V}}(\mathbf{x}, \mathrm{rev}(\mathbf{y})) = \mathfrak{p}_{1,3}\mathfrak{p}_{2,3}(1 - \mathfrak{p}_{1,1})(1 - \mathfrak{p}_{2,2})\left(\mathfrak{p}_{1,2}(1 - q\mathfrak{p}_{2,1}) + \mathfrak{p}_{2,1}(1 - \mathfrak{p}_{1,2})\right).$$

A quick calculation verifies that the right hand sides coincide as rational functions in $(q, \mathbf{x}, \mathbf{y})$.

Remark 1.4. In the above example, the *number* of configurations contributing to each side of (1.2) was the same (equal to 2). This is true in general, see Section 7.4.
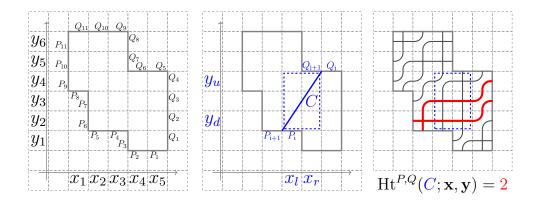
FIGURE 4. Left: a skew domain (P, Q) shown together with row/column rapidities. Middle: a (P, Q)-cut $C = (l, d, u, r)$. Right: a configuration of the model for which the height function $\mathrm{Ht}^{P,Q}(C; \mathbf{x}, \mathbf{y})$ is equal to 2 since there are two paths that connect the left and right boundaries of the dashed rectangle.

Remark 1.5. Theorem 1.1 is stated in the case of *ordered* incoming colors: for each c, the color of the path entering through the edge P_c equals to c. As we explain in Remark 3.4, Theorem 1.1 holds more generally when the colors entering from the left are ordered and are larger than the colors entering from the bottom (which may be ordered arbitrarily).

Our proof of Theorem 1.1 is given in Section 3.2. It is quite short and makes repeated use of the Yang–Baxter equation (2.7) combined with the machinery of Hecke algebras that we develop in Section 2. In contrast, deducing Theorem 1.6 from Theorem 1.1 requires a long technical argument which is presented in Sections 4 and 5.

1.3. **Main result: height functions.** Following [BGW19], we study *height functions* defined as follows. Given a pair of integers $1 \leq i, j \leq n$ and a permutation $\pi \in S_n$, the value of the associated height function is simply given by $\mathrm{Ht}_\pi(i, j) := \#\{c > i \mid \pi(c) \leq j\}$. Let us reformulate this slightly, taking the geometry of the domain into account.

Let $\mathbb{Z}_{\geq 1} := \{1, 2, \dots\}$ and denote by $(P, Q)_{\mathbb{Z}} \subseteq \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ the set of cells inside the domain (P, Q). A (P, Q)-*cut* is a quadruple $C = (l, d, u, r)$ of positive integers satisfying $l \leq r$, $d \leq u$, and such that $(l, d), (r, u) \in (P, Q)_{\mathbb{Z}}$ while $(l - 1, d - 1), (r + 1, u + 1) \notin (P, Q)_{\mathbb{Z}}$. Define $1 \leq i, j \leq n$ so that the bottom left corner of the cell (l, d) belongs to both P_i and P_{i+1} while the top right corner of the cell (r, u) belongs to both Q_j and Q_{j+1}, see Figure 4(middle). Given a permutation $\pi \in S_n$, we set

$$(1.3) \qquad \mathrm{Ht}_\pi^{P,Q}(C) := \mathrm{Ht}_\pi(i, j) = \#\{c > i \mid \pi(c) \leq j\}.$$

In other words, $\mathrm{Ht}_\pi^{P,Q}(C)$ counts the number of colored paths that connect the left and right boundaries of the sub-rectangle $\{l, l + 1, \dots, r\} \times \{d, d + 1, \dots, u\}$ of $(P, Q)_{\mathbb{Z}}$, see Figure 4(right). We let $\mathrm{Ht}^{P,Q}(C; \mathbf{x}, \mathbf{y})$ denote the associated random variable.

Let $\mathrm{supp}_H(C; \mathbf{x}) := \{x_l, x_{l+1}, \dots, x_r\}$ and $\mathrm{supp}_V(C; \mathbf{y}) := \{y_d, y_{d+1}, \dots, y_u\}$ denote the *unordered* sets of column and row rapidities covered by C. Let us say that $\mathbf{x}' = (x_1', x_2', \dots)$ is a *permutation of the variables in* $\mathbf{x}$ if there exists a bijection $\phi : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ such that $x_i' = x_{\phi(i)}$ for all $i \in \mathbb{Z}_{\geq 1}$. Our main result is the following equality between joint distributions of two vectors of height functions, conjectured by Borodin–Gorin–Wheeler.
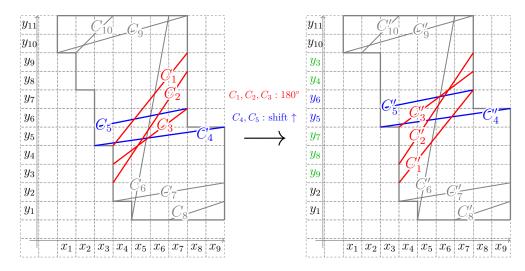
FIGURE 5.   An application of Theorem 1.6: the joint distributions of the two vectors of height functions are the same. This transformation is called a *double* $\mathbb{H}$-*flip*, see Lemma 4.8.

Theorem 1.6. *Suppose that we are given the following data:*

- *two skew domains (P, Q) and (P', Q');*
- *a permutation $\mathbf{x}'$ of the variables in $\mathbf{x}$ and a permutation $\mathbf{y}'$ of the variables in $\mathbf{y}$;*
- *a tuple $(C_1, C_2, \ldots, C_m)$ of (P, Q)-cuts and a tuple $(C'_1, C'_2, \ldots, C'_m)$ of (P', Q')-cuts.*

Assume that for each $i = 1, 2, \ldots, m$, we have

$$(1.4) \qquad \operatorname{supp}_H(C_i; \mathbf{x}) = \operatorname{supp}_H(C'_i; \mathbf{x}') \quad and \quad \operatorname{supp}_V(C_i; \mathbf{y}) = \operatorname{supp}_V(C'_i; \mathbf{y}').$$

Then the distributions of the following two vectors of height functions agree:

$$\left(\operatorname{Ht}^{P,Q}(C_1; \mathbf{x}, \mathbf{y}), \ldots, \operatorname{Ht}^{P,Q}(C_m; \mathbf{x}, \mathbf{y}) \right) \overset{d}{=} \left(\operatorname{Ht}^{P',Q'}(C'_1; \mathbf{x}', \mathbf{y}'), \ldots, \operatorname{Ht}^{P',Q'}(C'_m; \mathbf{x}', \mathbf{y}') \right).$$

An illustration of Theorem 1.6 is shown in Figure 5. For example, we check that $\operatorname{supp}_V(C_3; \mathbf{y}) = \{y_4, y_5, y_6\} = \operatorname{supp}_V(C'_3; \mathbf{y}')$, in agreement with (1.4).

Remark 1.7. The distribution of an individual height function $\operatorname{Ht}^{P,Q}(C_i; \mathbf{x}, \mathbf{y})$ depends[3] on the variables in $\operatorname{supp}_H(C_i; \mathbf{x})$ and $\operatorname{supp}_V(C_i; \mathbf{y})$, and it is straightforward to check that the condition (1.4) is necessary in order for a distributional identity

$$\operatorname{Ht}^{P,Q}(C_i; \mathbf{x}, \mathbf{y}) \overset{d}{=} \operatorname{Ht}^{P',Q'}(C'_i; \mathbf{x}', \mathbf{y}')$$

to hold. The content of Theorem 1.6 is that this condition is also sufficient even when one considers joint distributions of multiple height functions.

Examples of various transformations satisfying the assumptions of Theorem 1.6 are given in Figure 5 and Section 4.2 (cf. Figures 12 and 13). These examples include the shift-invariance of [BGW19, Theorems 1.2 and 4.13], see Remark 4.10.

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[3]In fact, it follows as a simple consequence of the Yang–Baxter equation that the distribution of $\operatorname{Ht}^{P,Q}(C_i; \mathbf{x}, \mathbf{y})$ is symmetric in the variables $\operatorname{supp}_H(C_i; \mathbf{x})$ and (separately) $\operatorname{supp}_V(C_i; \mathbf{y})$, see Section 3.2.
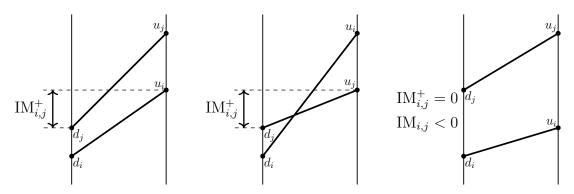
FIGURE 6. The geometric meaning of the intersection matrix $\mathrm{IM}^+(\mathbf{d}, \mathbf{u})$.

1.4. **Applications.** Following [BGW19], we describe some probabilistic models and universal objects which can be obtained as limiting cases of the stochastic colored six-vertex model and state analogs of Theorem 1.6 for them. In particular, the generalized shift-invariance property [BGW19, Conjecture 1.5] of the KPZ equation is given below in Theorem 1.10. We closely follow the notation and exposition of [BGW19, Section 1].

For simplicity, we restrict to the case where the skew domain is a vertical strip of some fixed width M. We start by introducing the notion of an *intersection matrix* which, in view of our results below, plays a role similar to the covariance matrix of a multivariate Gaussian distribution.

Definition 1.8. Given vectors $\mathbf{d} = (d_1, d_2, \ldots, d_m)$ and $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ in $\mathbb{R}^m$, we introduce two $m \times m$ symmetric matrices $\mathrm{IM}(\mathbf{d}, \mathbf{u}), \mathrm{IM}^+(\mathbf{d}, \mathbf{u})$ whose entries are given by

$$(1.5) \quad \mathrm{IM}_{i,j} := \min(u_i, u_j) - \max(d_i, d_j), \quad \mathrm{IM}_{i,j}^+ = \max(\mathrm{IM}_{i,j}, 0) \quad \text{for all } i, j = 1, 2, \ldots, m.$$

We refer to $\mathrm{IM}^+(\mathbf{d}, \mathbf{u})$ as the *intersection matrix of* $(\mathbf{d}, \mathbf{u})$ since its (i, j)-th entry equals the length of the intersection of line segments $[d_i, u_i] \cap [d_j, u_j]$, see Figure 6.

Brownian last passage percolation. Fix a collection $\{B_n(t)\}_{n \in \mathbb{Z}}$ of independent standard Brownian motions on the real line. Given $l, r \in \mathbb{Z}$ and $d, u \in \mathbb{R}$ satisfying $l \le r$ and $d \le u$, the *last passage time* $\mathfrak{Z}_{(l,d) \to (r,u)}$ is defined by

$$(1.6) \qquad \mathfrak{Z}_{(l,d) \to (r,u)} := \max_{d = t_l < t_{l+1} < \cdots < t_{r+1} = u} \left[\sum_{i=l}^{r} (B_i(t_{i+1}) - B_i(t_i)) \right].$$

The following result is a special case of Theorem 6.10.

Theorem 1.9. *Fix $M \in \mathbb{Z}_{\ge 1}$ and consider vectors $\mathbf{d}, \mathbf{d}', \mathbf{u}, \mathbf{u}' \in \mathbb{R}^m$ satisfying $d_i \le u_i$ and $d_i' \le u_i'$ for all $i = 1, \ldots, m$. Suppose that the intersection matrices of $(\mathbf{d}, \mathbf{u})$ and $(\mathbf{d}', \mathbf{u}')$ coincide: $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$. Then we have a distributional identity*

$$(1.7) \qquad \left(\mathfrak{Z}_{(0,d_1) \to (M,u_1)}, \ldots, \mathfrak{Z}_{(0,d_m) \to (M,u_m)} \right) \overset{d}{=} \left(\mathfrak{Z}_{(0,d_1') \to (M,u_1')}, \ldots, \mathfrak{Z}_{(0,d_m') \to (M,u_m')} \right).$$

Similarly to Remark 1.7, we expect that the condition $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$ is not only sufficient but also necessary in order for (1.7) to hold.

KPZ equation. Consider a two-dimensional white Gaussian noise $\eta(x,t)$. For $y \in \mathbb{R}$, define a random function $\mathcal{Z}^{(y)}(t,x)$ as a solution to the following *stochastic heat equation with multiplicative white noise*:

$$\mathcal{Z}_t^{(y)} = \frac{1}{2}\mathcal{Z}_{xx}^{(y)} + \eta\mathcal{Z}^{(y)}, \quad t \in \mathbb{R}_{\geq 0}, x \in \mathbb{R}; \quad \mathcal{Z}^{(y)}(0,x) = \delta(x-y),$$

where the initial condition is given by the delta function at y. We will consider the random variables $\mathcal{Z}^{(y)}(t,x)$ for fixed $t \in \mathbb{R}_{\geq 0}$ and different pairs (x,y), and we assume that the white noise η is the same for different values of y. The formal logarithm $\mathcal{H} := -\ln(\mathcal{Z}^{(y)})$ satisfies the celebrated Kardar–Parisi–Zhang (KPZ) equation [KPZ86]:

$$\mathcal{H}_t = \frac{1}{2}\mathcal{H}_{xx} - \frac{1}{2}(\mathcal{H}_x)^2 - \eta.$$

The KPZ universality class has been a subject of intense interest throughout the past two decades, see [Cor12, QS15] for reviews.

For the following result, we consider vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ that do not necessarily satisfy $x_i \leq y_i$. In fact, we will swap $\mathbf{x}$ and $\mathbf{y}$ and use the matrix $\mathrm{IM}(\mathbf{y}, \mathbf{x})$ from Definition 1.8 rather than the intersection matrix $\mathrm{IM}^+(\mathbf{y}, \mathbf{x})$. The geometric meaning of the entries of $\mathrm{IM}(\mathbf{y}, \mathbf{x})$ is that for all large enough L, the length of the intersection $[y_i, x_i + L] \cap [y_j, x_j + L]$ is given by $\mathrm{IM}_{i,j} + L$.

Theorem 1.10. *Let* $t \in \mathbb{R}_{\geq 0}$ *and* $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathbb{R}^m$. *If* $\mathrm{IM}(\mathbf{y}, \mathbf{x}) = \mathrm{IM}(\mathbf{y}', \mathbf{x}')$ *then*

$$\left(\mathcal{Z}^{(y_1)}(t, x_1), \dots, \mathcal{Z}^{(y_m)}(t, x_m)\right) \overset{d}{=} \left(\mathcal{Z}^{(y_1')}(t, x_1'), \dots, \mathcal{Z}^{(y_m')}(t, x_m')\right).$$

In particular, this result implies [BGW19, Conjecture 1.5].

Airy sheet. It is believed [CQR15] that the large time limit of $\mathcal{Z}^{(y)}(t, x)$ (as well as the universal limit of various directed polymers and last-passage percolation models) is described by the *Airy sheet* $\mathcal{A}(x, y)$. We define it in (6.3) building on recent results of [DOV18]. For $(x, y) \in \mathbb{R}^2$, the random variable $\mathcal{A}(x, y)$ has the Tracy–Widom distribution [TW93, TW94].

Theorem 1.11. *Let* $t \in \mathbb{R}_{\geq 0}$ *and* $\mathbf{x}, \mathbf{x}', \mathbf{y}, \mathbf{y}' \in \mathbb{R}^m$. *If* $\mathrm{IM}(\mathbf{x}, \mathbf{y}) = \mathrm{IM}(\mathbf{x}', \mathbf{y}')$ *then*

$$(\mathcal{A}(x_1, y_1), \dots, \mathcal{A}(x_m, y_m)) \overset{d}{=} (\mathcal{A}(x_1', y_1'), \dots, \mathcal{A}(x_m', y_m')).$$

Directed polymers. The limit transition from the stochastic colored six-vertex model to the objects described above is obtained through a sequence of intermediate steps. First, one passes to the *fused* vertex model [BW18, Kua18, KMMO16] which then gets degenerated further to the *Beta polymer* [BC17], the *Gamma polymer* [CSS15, OO15], and the *O'Connell–Yor polymer* [OY01]. We will define these objects in Section 6.

Note that our proof of Theorem 1.6 has the following advantage. The general Beta polymer depends on a choice of parameters $(\sigma_i)_{i \in \mathbb{Z}_{\geq 0}}$ and $(\rho_j)_{j \in \mathbb{Z}_{\geq 1}}$. The shift-invariance of [BGW19] is stated for the Beta polymer that is *homogeneous* in the vertical direction, i.e., when all parameters ρ_j are equal to each other. The result is expected [BGW19, Remark 7.4] to hold more generally for the Beta polymer that is inhomogeneous in both directions. Our proof allows to confirm this: we show that the analog of Theorem 1.6 holds for Beta polymers with arbitrary parameters $(\sigma_i)_{i \in \mathbb{Z}_{\geq 0}}$ and $(\rho_j)_{j \in \mathbb{Z}_{\geq 1}}$, see Theorem 6.6.

1.5. **Related work.** In a very recent preprint [Dau20], the author relies on the *geometric RSK correspondence* [Kir01, NY04] to describe several transformations similar to the ones we construct. The models considered in [Dau20] are mostly disjoint from the models studied here: we focus on the stochastic six-vertex model and Beta polymers, while [Dau20] works with the last passage percolation with geometric or exponential weights and log-gamma polymers. There is a certain overlap among the limiting Gaussian objects, such as the Brownian last passage percolation or the KPZ equation. For example, [BGW19, Conjecture 1.5] that we obtain as a consequence of Theorem 1.10 can also be deduced from [Dau20, Theorem 1.1].

The central transformation of this paper, Theorem 1.1, shares some similarities with the notion of a *decoupled polymer model* [Dau20, Definition 2.2] that captures the class of models to which the RSK correspondence applies, see [Dau20, Theorem 2.3]. It would be interesting to understand whether our flip theorem leads to an analog of the geometric RSK correspondence for the stochastic colored six-vertex model. We thank Vadim Gorin for bringing the paper [Dau20] to our attention.

1.6. **Outline.** We give background on Hecke algebras in Section 2 and use them to prove Theorem 1.1 in Section 3. The next two sections are devoted to the proof of Theorem 1.6: in Section 4, we describe a family of transformations and use Theorem 1.1 to show that they preserve joint distributions of height functions. In Section 5, we show that any transformation satisfying the assumptions of Theorem 1.6 can be obtained as a composition of the transformations constructed in Section 4. In Section 6, we describe the limiting transitions of [BGW19] and use them to prove the results stated in Section 1.4. Finally, we discuss the relationship of the stochastic colored six-vertex model with Kazhdan–Lusztig polynomials and positroid varieties in Section 7. We include some examples and a conjecture relating Theorem 1.6 to more general wiring diagram domains in Section 7.5.

2. Hecke algebra and the Yang–Baxter basis

One of our main tools is a certain direct relationship between the stochastic colored six-vertex model (which we from now on abbreviate as the *SC6V model*) and the *Yang–Baxter basis* of the Hecke algebra of S_n introduced in [LLT97]. A simple proof of the color-position symmetry of [BB19] is given in Section 2.2. Another non-trivial property of the SC6V model that becomes obvious in the language of Hecke algebras is known as the *non-local relations* of Borodin–Wheeler [BW18], see (2.3) below.

Fix $n \geq 1$. For integers $i \leq j$, we denote $[i, j] := \{i, i + 1, \ldots, j\}$, and for $i > j$, we set $[i, j] := \emptyset$. For $i \in [1, n-1]$, denote by $s_i \in S_n$ the transposition $(i, i+1)$. For indeterminates q and $\mathbf{z} := (z_1, z_2, \ldots, z_n)$, we consider the *Hecke algebra* $\mathcal{H}_q(S_n; \mathbf{z})$, which is an associative algebra over $\mathbb{C}(q; \mathbf{z}) := \mathbb{C}(q, z_1, \ldots, z_n)$ with basis $\{T_w\}_{w \in S_n}$ and relations

(2.1) $T_u T_w = T_{uw}$ if $\ell(uw) = \ell(u) + \ell(w)$, and $(T_i + q)(T_i - 1) = 0$ for $i \in [1, n-1]$,

where $T_i := T_{s_i}$ and $\ell(w)$ denotes *length* of $w \in S_n$, i.e., the number of inversions of w. For the identity permutation $\mathrm{id} \in S_n$, we have $T_{\mathrm{id}} = 1 \in \mathcal{H}_q(S_n; \mathbf{z})$.

For $j \in [1, n]$ and $u, w \in S_n$, we use the convention that $(uw)(j) := w(u(j))$. For $k \in [1, n-1]$ and $\mathfrak{p} \in \mathbb{C}(q; \mathbf{z})$, we write

$$R_k(\mathfrak{p}) := \mathfrak{p} T_k + (1 - \mathfrak{p}) \in \mathcal{H}_q(S_n; \mathbf{z}).$$

Given an arbitrary permutation $\pi \in S_n$ and $k \in [1, n-1]$, the above relations imply the following rules for multiplying T_π by $R_k(\mathfrak{p})$:

$$(2.2) \qquad T_\pi \cdot R_k(\mathfrak{p}) = \begin{cases} \mathfrak{p} T_{\pi s_k} + (1 - \mathfrak{p}) T_\pi, & \text{if } \ell(\pi s_k) = \ell(\pi) + 1, \\ q\mathfrak{p} T_{\pi s_k} + (1 - q\mathfrak{p}) T_\pi, & \text{if } \ell(\pi s_k) = \ell(\pi) - 1; \end{cases}$$

$$(2.3) \qquad R_k(\mathfrak{p}) \cdot T_\pi = \begin{cases} \mathfrak{p} T_{s_k \pi} + (1 - \mathfrak{p}) T_\pi, & \text{if } \ell(s_k \pi) = \ell(\pi) + 1, \\ q\mathfrak{p} T_{s_k \pi} + (1 - q\mathfrak{p}) T_\pi, & \text{if } \ell(s_k \pi) = \ell(\pi) - 1. \end{cases}$$

We note the formal similarity between (2.2) and Figure 1. We will make this precise in Proposition 2.3 by interpreting the SC6V model in terms of products of elements of the form $R_k(\mathfrak{p})$. With this interpretation, (2.3) turns into the *non-local relations* studied in [BW18, Theorem 5.3.1]. In the language of Section 1.3, applying (2.2) (resp., (2.3)) corresponds to adding a single square to the top right (resp., bottom left) boundary of a skew domain (P, Q). Then (2.2) can be proved using a simple bijection on configurations that only changes them locally inside that square. The analogous bijection proving (2.3) is much more complicated and involves changing the configurations globally. See the discussion after [BW18, Theorem 5.3.1] for further details.

2.1. **Wiring diagram domains.** We would like to consider the SC6V model described in Section 1.1 associated to more general domains that we call *wiring diagram domains*.

Let $\mathbf{i} = (i_1, i_2, \ldots, i_r)$ be an arbitrary sequence of elements of $[1, n-1]$, and choose an arbitrary family $\mathbf{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r)$ of elements of $\mathbb{C}(q; \mathbf{z})$. The sequence $\mathbf{i}$ gives rise to a *wiring diagram* as in Example 2.2 and Figure 7: there are n paths called *wires*, and each wire moves horizontally left-to-right. The wires start on the left at heights $1, 2, \ldots, n$. For each $j \in [1, r]$, the wires at heights i_j and $i_j + 1$ *cross* at the point $\left(j, i_j + \frac{1}{2}\right)$ and then proceed to the right at heights $i_j + 1$ and i_j, respectively.

Let us also fix a permutation $\sigma \in S_n$ called the *incoming color permutation*.

Definition 2.1. The *SC6V model inside* $(\mathbf{i}, \mathbf{p})$ *with incoming colors* σ is a probability distribution on S_n denoted $(\mathbf{P}_\pi^{\mathbf{i}, \mathbf{p}, \sigma}(\mathbf{z}))_{\pi \in S_n}$, defined as follows. Suppose that for each $c \in [1, n]$, a path of color $\sigma(c)$ enters the wiring diagram of $\mathbf{i}$ on the left at height c. We say that *the incoming colors are ordered* if $\sigma = \mathrm{id}$. For each $j \in [1, r]$, the two paths entering the crossing point $\left(j, i_j + \frac{1}{2}\right)$ from the left proceed to the right according to the probabilities in Figure 1, where the parameter $\mathfrak{p}$ is equal to $\mathfrak{p}_j$. Once all paths reach the right boundary of the wiring diagram of $\mathbf{i}$, they give rise to a random *outgoing color permutation* π defined so that for $c \in [1, n]$, the path of color c exits at height $\pi(c)$. We let $\mathbf{P}_\pi^{\mathbf{i}, \mathbf{p}, \sigma}(\mathbf{z})$ denote the total probability of observing π as the outgoing color permutation.

Example 2.2. Let $n = 5$, $\mathbf{i} = (4, 2, 3, 2, 1, 2, 3)$, and $\mathbf{p} := (\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_7)$. The corresponding wiring diagram domain is shown in Figure 7(left). An example of a configuration of the SC6V model inside $(\mathbf{i}, \mathbf{p})$ with incoming color permutation $\sigma = (3, 1, 2, 5, 4)$ and outgoing color permutation $\pi = (1, 4, 2, 5, 3)$ is shown in Figure 7(right) together with its probability.
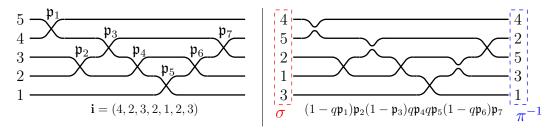
FIGURE 7. Left: a wiring diagram domain from Definition 2.1. Right: a configuration of the SC6V model inside $(\mathbf{i}, \mathfrak{p})$, see Example 2.2.

Our first goal is to interpret the probabilities $\mathbf{P}_\pi^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z})$ in terms of elements of $\mathcal{H}_q(S_n; \mathbf{z})$. Let us write

$$(2.4) \qquad Y^{\mathbf{i}, \mathfrak{p}, \sigma} := T_\sigma R_{i_1}(\mathfrak{p}_1) R_{i_2}(\mathfrak{p}_2) \cdots R_{i_r}(\mathfrak{p}_r) = \sum_{\pi \in S_n} \kappa_\pi^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}) T_\pi.$$

Proposition 2.3. *For all $\mathbf{i}$, $\mathfrak{p}$, σ, and π, we have*

$$(2.5) \qquad \mathbf{P}_\pi^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}) = \kappa_\pi^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}).$$

Proof. We prove this by induction on r. The case $r = 0$ is trivial. Suppose now that (2.5) holds for some $\mathbf{i} = (i_1, i_2, \ldots, i_r)$, $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r)$, $\sigma \in S_n$, and all $\pi \in S_n$. Choose $k \in [1, n-1]$ and $\mathfrak{p} \in \mathbb{C}(q; \mathbf{z})$ and let $\mathbf{i}' := (i_1, i_2, \ldots, i_r, k)$, $\mathfrak{p}' := (\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r, \mathfrak{p})$. Consider a permutation $\pi \in S_n$ such that $\ell(\pi s_k) = \ell(\pi) + 1$.

The following transition formulas are easily deduced respectively from Definition 2.1 and from Equations (2.2), (2.4):

$$\begin{pmatrix} \mathbf{P}_\pi^{\mathbf{i}', \mathfrak{p}', \sigma}(\mathbf{z}) \\ \mathbf{P}_{\pi s_k}^{\mathbf{i}', \mathfrak{p}', \sigma}(\mathbf{z}) \end{pmatrix} = \begin{pmatrix} 1 - \mathfrak{p} & q\mathfrak{p} \\ \mathfrak{p} & 1 - q\mathfrak{p} \end{pmatrix} \begin{pmatrix} \mathbf{P}_\pi^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}) \\ \mathbf{P}_{\pi s_k}^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}) \end{pmatrix}, \qquad \begin{pmatrix} \kappa_\pi^{\mathbf{i}', \mathfrak{p}', \sigma}(\mathbf{z}) \\ \kappa_{\pi s_k}^{\mathbf{i}', \mathfrak{p}', \sigma}(\mathbf{z}) \end{pmatrix} = \begin{pmatrix} 1 - \mathfrak{p} & q\mathfrak{p} \\ \mathfrak{p} & 1 - q\mathfrak{p} \end{pmatrix} \begin{pmatrix} \kappa_\pi^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}) \\ \kappa_{\pi s_k}^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}) \end{pmatrix}.$$

By the induction hypothesis, we have $\mathbf{P}_\pi^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}) = \kappa_\pi^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z})$ and $\mathbf{P}_{\pi s_k}^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z}) = \kappa_{\pi s_k}^{\mathbf{i}, \mathfrak{p}, \sigma}(\mathbf{z})$, therefore we get that the left hand sides are equal, completing the induction step. $\square$

2.2. **Color-position symmetry.** Color-position symmetry in interacting particle systems has been studied in e.g. [AAV11, AHR09, BB19, BW18, Kua20]. As a warm up, we explain the recent results of [BB19] using the machinery of Hecke algebras. Let $\mathbf{i} = (i_1, i_2, \ldots, i_r)$ and $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r)$ be arbitrary. First, following [BB19], we would like to consider the case where the incoming colors are ordered, i.e., $\sigma = \mathrm{id}$. In this case, we usually omit σ from the notation and write $Y^{\mathbf{i}, \mathfrak{p}}$ and $\mathbf{P}_\pi^{\mathbf{i}, \mathfrak{p}}(\mathbf{z})$ instead of $Y^{\mathbf{i}, \mathfrak{p}, \mathrm{id}}$ and $\mathbf{P}_\pi^{\mathbf{i}, \mathfrak{p}, \mathrm{id}}(\mathbf{z})$.

For $\mathbf{i} = (i_1, i_2, \ldots, i_r)$ and $\mathfrak{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r)$, let us denote $\mathrm{rev}(\mathbf{i}) := (i_r, \ldots, i_2, i_1)$, $\mathrm{rev}(\mathfrak{p}) = (\mathfrak{p}_r, \ldots, \mathfrak{p}_2, \mathfrak{p}_1)$. The color-position symmetry of [BB19] amounts to the following statement.

Theorem 2.4 ([BB19, Theorem 2.2]). *For all $\mathbf{i}$, $\mathfrak{p}$, and $\pi \in S_n$, the coefficient of T_π in $Y^{\mathbf{i}, \mathfrak{p}}$ is equal to the coefficient of $T_{\pi^{-1}}$ in $Y^{\mathrm{rev}(\mathbf{i}), \mathrm{rev}(\mathfrak{p})}$:*

$$\mathbf{P}_\pi^{\mathbf{i}, \mathfrak{p}}(\mathbf{z}) = \mathbf{P}_{\pi^{-1}}^{\mathrm{rev}(\mathbf{i}), \mathrm{rev}(\mathfrak{p})}(\mathbf{z}).$$

Proof. Consider the anti-automorphism $D : \mathcal{H}_q(S_n; \mathbf{z}) \to \mathcal{H}_q(S_n; \mathbf{z})$ sending T_π to $T_{\pi^{-1}}$ for each $\pi \in S_n$. It reverses the order of multiplication, sending $Y^{\mathbf{i}, \mathfrak{p}} = R_{i_1}(\mathfrak{p}_1) R_{i_2}(\mathfrak{p}_2) \cdots R_{i_r}(\mathfrak{p}_r)$ to $Y^{\mathrm{rev}(\mathbf{i}), \mathrm{rev}(\mathfrak{p})} = R_{i_r}(\mathfrak{p}_r) \cdots R_{i_2}(\mathfrak{p}_2) R_{i_1}(\mathfrak{p}_1)$. The result follows. $\square$
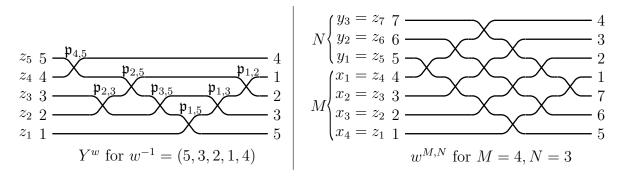
FIGURE 8. Left: the wiring diagram domain associated with a Yang–Baxter basis element Y^w. Here $\mathfrak{p}_{i,j} = \frac{z_j - z_i}{z_j - qz_i}$, see (2.6). Right: The case of a Grassmannian permutation $w^{M,N}$ from Example 2.9.

Remark 2.5. The above proof was also independently found by Bufetov [Buf20]. Connections between Hecke algebras and the ASEP have been discovered in the literature a number of times throughout the years, see e.g. [ADHR94, Can17, CdGW15, CMW18]. On the other hand, the equivalence between the SC6V model and the Yang–Baxter basis of [LLT97] discussed in Section 2.3 appears to have not been pointed out before.

Next, we would like to reduce the case of an arbitrary incoming color permutation σ to the above case $\sigma = $ id. Given $\sigma \in S_n$, a *reduced word* for σ is a sequence $\mathbf{j} = (j_1, j_2, \ldots, j_l)$ such that $\sigma = s_{j_1} s_{j_2} \cdots s_{j_l}$ and $l = \ell(\sigma)$. For two sequences $\mathbf{i} = (i_1, \ldots, i_r)$, $\mathbf{j} = (j_1, \ldots, j_l)$, let $(\mathbf{i}, \mathbf{j}) = (i_1, \ldots, i_r, j_1, \ldots, j_l)$ denote their concatenation. For $l \geq 1$, denote by $1^l := (1, 1, \ldots, 1)$ the sequence that consists of l ones.

Lemma 2.6. *Let* $\mathbf{i} = (i_1, i_2, \ldots, i_r)$, $\mathbf{p} = (\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r)$, *and* $\sigma \in S_n$ *be arbitrary, and choose a reduced word* $\mathbf{j} = (j_1, j_2, \ldots, j_l)$ *for* σ, *where* $l = \ell(\sigma)$. *Let* $\mathbf{i}' := (\mathbf{j}, \mathbf{i})$ *and* $\mathbf{p}' := (1^l, \mathbf{p})$. *Then*
$$Y^{\mathbf{i}, \mathbf{p}, \sigma} = Y^{\mathbf{i}', \mathbf{p}', \mathrm{id}}.$$

Proof. By definition, $Y^{\mathbf{i}', \mathbf{p}', \mathrm{id}} = R_{j_1}(1) \cdots R_{j_l}(1) Y^{\mathbf{i}, \mathbf{p}, \mathrm{id}}$. Since $R_k(1) = T_k$, we get $Y^{\mathbf{i}', \mathbf{p}', \mathrm{id}} = T_\sigma Y^{\mathbf{i}, \mathbf{p}, \mathrm{id}}$, which is by definition equal to $Y^{\mathbf{i}, \mathbf{p}, \sigma}$. $\square$

Remark 2.7. Lemma 2.6 yields a generalization of Theorem 2.4 to the case of arbitrary incoming colors: for the SC6V model inside $(\mathbf{i}, \mathbf{p})$ with incoming colors σ, the probability of observing π as the outgoing color permutation is equal to the probability of observing π^{-1} as the outgoing color permutation for the SC6V model inside $(\mathbf{i}', \mathbf{p}')$ with ordered incoming colors ($\sigma' = $ id). Here $\mathbf{i}' = (\mathrm{rev}(\mathbf{i}), \mathrm{rev}(\mathbf{j}))$, $\mathbf{j}$ is a reduced word for σ, and $\mathbf{p}' = (\mathbf{p}, 1^{\ell(\sigma)})$.

2.3. Yang–Baxter basis. The Hecke algebra $\mathcal{H}_q(S_n; \mathbf{z})$ has a basis called the *Yang–Baxter basis* $\{Y^w\}_{w \in S_n}$, introduced in [LLT97]. It is defined via the following recurrence relation. For the identity permutation id $\in S_n$, we set $Y^{\mathrm{id}} = T_{\mathrm{id}} = 1 \in \mathcal{H}_q(S_n; \mathbf{z})$. For any $w \in S_n$ and $k \in [1, n-1]$ such that $\ell(ws_k) = \ell(w) + 1$ (equivalently, such that $w^{-1}(k) < w^{-1}(k+1)$), we have

$$(2.6) \qquad Y^{ws_k} = Y^w \cdot R_k(\mathfrak{p}_{w^{-1}(k), w^{-1}(k+1)}), \quad \text{where} \quad \mathfrak{p}_{i,j} := \frac{z_j - z_i}{z_j - qz_i}.$$

This is a special case of the construction from the previous subsection, see Figure 8(left): we have $Y^w = Y^{\mathbf{i}, \mathbf{p}, \mathrm{id}}$ for a reduced word $\mathbf{i}$ of w and a particular choice of $\mathbf{p}$. The element Y^w can

be computed via this recurrence in various ways (which correspond to the various reduced words for w), but the result is uniquely determined because the elements $R_k(\mathfrak{p})$ satisfy the following *Yang–Baxter relation*: for any $a < b < c \in [1, n]$ and $k \in [1, n-2]$, we have

$$(2.7) \qquad R_k(\mathfrak{p}_{a,b})R_{k+1}(\mathfrak{p}_{a,c})R_k(\mathfrak{p}_{b,c}) = R_{k+1}(\mathfrak{p}_{b,c})R_k(\mathfrak{p}_{a,c})R_{k+1}(\mathfrak{p}_{a,b}).$$

Consider the entries $\mathbf{P}_\pi^w(\mathbf{z})$ of the transition matrix between the bases $\{Y^w\}$ and $\{T_\pi\}$:

$$(2.8) \qquad Y^w = \sum_{\pi \in S_n} \mathbf{P}_\pi^w(\mathbf{z}) T_\pi.$$

By Proposition 2.3, each of these coefficients equals the probability of observing π as an outgoing color permutation for the SC6V model associated with a reduced word for w.

2.4. **From skew domains to wiring diagram domains.** Let (P, Q) be a skew domain with $n = |P| = |Q|$. Recall that the column and row rapidities are given by $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$. The SC6V model inside (P, Q) gives rise to a probability distribution $(\mathbf{P}_\pi^{P,Q}(\mathbf{x}, \mathbf{y}))_{\pi \in S_n}$ on permutations. We claim that there exists a permutation $w^{P,Q} \in S_n$ and the values $\mathbf{z} = (z_1, z_2, \dots, z_n)$ such that for all $\pi \in S_n$, we have $\mathbf{P}_\pi^{w^{P,Q}}(\mathbf{z}) = \mathbf{P}_\pi^{P,Q}(\mathbf{x}, \mathbf{y})$. Indeed, recall that the steps of P and Q are given respectively by $P_1, P_2, \dots, P_n$ and $Q_1, Q_2, \dots, Q_n$. Let $i \in [1, n]$. Suppose that P_i is a vertical step located in row $r \in \mathbb{Z}_{\geq 1}$. Then there exists a unique vertical step Q_j of Q in the same row, and we set $z_i := y_r$ and $w^{P,Q}(i) := j$. Similarly, suppose that P_i is a horizontal step located in column $c \in \mathbb{Z}_{\geq 1}$. Then there exists a unique horizontal step Q_j of Q in the same column, and we set $z_i := x_c$ and $w^{P,Q}(i) := j$. Comparing the descriptions of the SC6V model in Sections 1.1 and 2.1, we find the following result.

Proposition 2.8. *For any skew domain (P, Q) and any $\pi \in S_n$, we have*

$$\mathbf{P}_\pi^{w^{P,Q}}(\mathbf{z}) = \mathbf{P}_\pi^{P,Q}(\mathbf{x}, \mathbf{y}).$$

A similar statement holds for the case of an arbitrary incoming color permutation σ.

Example 2.9. In the setting of Section 1.2, consider the SC6V model inside an $M \times N$-rectangular domain (P, Q). We have $n = M + N$, $\mathbf{z} = (x_M, \dots, x_1, y_1, \dots, y_N)$, and the permutation $w^{M,N} := w^{P,Q}$ is defined by $w^{M,N}(i) = i + N$ modulo n (thus $w^{M,N}$ is a *Grassmannian permutation* of length MN). We can write it as a product

$$(2.9) \qquad w^{M,N} = (s_M s_{M+1} \cdots s_{M+N-1}) \cdot (s_{M-1} s_M \cdots s_{M+N-2}) \cdots (s_1 s_2 \cdots s_N)$$

of MN simple transpositions. They correspond naturally to the cells in $[1, M] \times [1, N] = (P, Q)_{\mathbb{Z}}$. The case $M = 4$, $N = 3$ is shown in Figure 8(right).

3. Flip symmetry

Our goal is to prove Theorem 1.1 and state a more general version that will be used in the proof of Theorem 1.6. Throughout the first two subsections, we fix an $M \times N$-rectangular domain (P, Q), a vertical boundary condition $\mathbb{V} = \{(d_1, u_1), \dots, (d_v, u_v)\}$, and an integer h.

3.1. **Boundary conditions and the Hecke algebra.** Let $\mathbb{H} = \{(l_1, r_1), \ldots, (l_h, r_h)\}$ be a set of pairs. Let $\mathrm{SAT}(\mathbb{H}, \mathbb{V}) := \{\pi \in S_n \mid \mathbb{H}_\pi^{M,N} = \mathbb{H}$ and $\mathbb{V}_\pi^{M,N} = \mathbb{V}\}$ denote the set of all permutations $\pi \in S_n$ satisfying given horizontal and vertical boundary conditions $(\mathbb{H}, \mathbb{V})$. For an arbitrary element

$$(3.1) \qquad Y = \sum_{\pi \in S_n} \kappa_\pi T_\pi \in \mathcal{H}_q(S_n; \mathbf{z}), \quad \text{let} \quad Y^{\mathbb{H},\mathbb{V}} := \sum_{\pi \in \mathrm{SAT}(\mathbb{H}, \mathbb{V})} \kappa_\pi \in \mathbb{C}(q; \mathbf{z}).$$

For $\mathbb{H} = \{(l_1, r_1), \ldots, (l_h, r_h)\}$ and a permutation $\tau \in S_n$, we denote

$$\tau \cdot \mathbb{H} := \{(\tau(l_1), r_1), \ldots, (\tau(l_h), r_h)\} \quad \text{and} \quad \mathbb{H} \cdot \tau := \{(l_1, \tau(r_1)), \ldots, (l_h, \tau(r_h))\}.$$

Let $r \in [1, N-1]$. We write $\mathbb{H} < \mathbb{H} \cdot s_r$ if for all $\pi \in \mathrm{SAT}(\mathbb{H}, \mathbb{V})$, we have $\ell(\pi) < \ell(\pi s_r)$. Similarly, we write $\mathbb{H} > \mathbb{H} \cdot s_r$ if for all $\pi \in \mathrm{SAT}(\mathbb{H}, \mathbb{V})$, we have $\ell(\pi) > \ell(\pi s_r)$. It is easy to check that if neither $\mathbb{H} > \mathbb{H} \cdot s_r$ nor $\mathbb{H} < \mathbb{H} \cdot s_r$ is satisfied then we must have $\mathbb{H} = \mathbb{H} \cdot s_r$ as sets of pairs. Indeed, for each $\pi \in \mathrm{SAT}(\mathbb{H}, \mathbb{V})$, the condition that π satisfies $\mathbb{H}$ implies that either $\pi^{-1}(r) = l_j$ for some $j \in [1, h]$, or $\pi^{-1}(r) \in [1, M]$, and similarly for $\pi^{-1}(r+1)$. If both $\pi^{-1}(r), \pi^{-1}(r+1)$ belong to $[1, M]$ then $\mathbb{H} = \mathbb{H} \cdot s_r$. Otherwise, $\mathbb{H}$ determines whether $\pi^{-1}(r) < \pi^{-1}(r+1)$ or $\pi^{-1}(r) > \pi^{-1}(r+1)$, which determines respectively whether $\mathbb{H} \cdot s_r > \mathbb{H}$ or $\mathbb{H} \cdot s_r < \mathbb{H}$. Similarly, for $l \in [M+1, n-1]$, we write $\mathbb{H} < s_l \cdot \mathbb{H}$ (resp., $\mathbb{H} > s_l \cdot \mathbb{H}$) if for all $\pi \in \mathrm{SAT}(\mathbb{H}, \mathbb{V})$, we have $\ell(\pi) < \ell(s_l \pi)$ (resp., $\ell(\pi) > \ell(s_l \pi)$).

For an element $Y \in \mathcal{H}_q(S_n; \mathbf{z})$ and $r \in [1, N-1], l \in [M+1, n-1]$, (2.2)–(2.3) give

$$(3.2) \qquad (Y \cdot R_r(\mathfrak{p}))^{\mathbb{H},\mathbb{V}} = \begin{cases} Y^{\mathbb{H},\mathbb{V}}, & \text{if } \mathbb{H} \cdot s_r = \mathbb{H}, \\ \mathfrak{p} Y^{\mathbb{H} \cdot s_r, \mathbb{V}} + (1 - q\mathfrak{p}) Y^{\mathbb{H},\mathbb{V}}, & \text{if } \mathbb{H} \cdot s_r < \mathbb{H}, \\ q\mathfrak{p} Y^{\mathbb{H} \cdot s_r, \mathbb{V}} + (1 - \mathfrak{p}) Y^{\mathbb{H},\mathbb{V}}, & \text{if } \mathbb{H} \cdot s_r > \mathbb{H}; \end{cases}$$

$$(3.3) \qquad (R_l(\mathfrak{p}) \cdot Y)^{\mathbb{H},\mathbb{V}} = \begin{cases} Y^{\mathbb{H},\mathbb{V}}, & \text{if } s_l \cdot \mathbb{H} = \mathbb{H}, \\ \mathfrak{p} Y^{s_l \cdot \mathbb{H}, \mathbb{V}} + (1 - q\mathfrak{p}) Y^{\mathbb{H},\mathbb{V}}, & \text{if } s_l \cdot \mathbb{H} < \mathbb{H}, \\ q\mathfrak{p} Y^{s_l \cdot \mathbb{H}, \mathbb{V}} + (1 - \mathfrak{p}) Y^{\mathbb{H},\mathbb{V}}, & \text{if } s_l \cdot \mathbb{H} > \mathbb{H}. \end{cases}$$

Let us prove one of these identities, the other cases being completely analogous. Suppose that $\mathbb{H} \cdot s_r > \mathbb{H}$ and let $A := \mathrm{SAT}(\mathbb{H}, \mathbb{V})$. Then we may assume $Y = \sum_{\pi \in A} \kappa_\pi T_\pi + \sum_{\pi \in A} \kappa_{\pi s_r} T_{\pi s_r}$, so $Y^{\mathbb{H},\mathbb{V}} = \sum_{\pi \in A} \kappa_\pi$ and $Y^{\mathbb{H} \cdot s_r, \mathbb{V}} = \sum_{\pi \in A} \kappa_{\pi s_r}$. Applying (2.2), we find

$$Y \cdot R_r(\mathfrak{p}) = \sum_{\pi \in A} \kappa_\pi \left(\mathfrak{p} T_{\pi s_r} + (1 - \mathfrak{p}) T_\pi\right) + \sum_{\pi \in A} \kappa_{\pi s_r} \left(q\mathfrak{p} T_\pi + (1 - q\mathfrak{p}) T_{\pi s_r}\right)$$

$$= \sum_{\pi \in A} \left(\kappa_\pi (1 - \mathfrak{p}) + \kappa_{\pi s_r} q\mathfrak{p}\right) T_\pi + \sum_{\pi \in A} \left(\kappa_{\pi s_r}(1 - q\mathfrak{p}) + \kappa_\pi \mathfrak{p}\right) T_{\pi s_r}.$$

Thus $(Y \cdot R_r(\mathfrak{p}))^{\mathbb{H},\mathbb{V}} = \sum_{\pi \in A} \left(\kappa_\pi (1 - \mathfrak{p}) + \kappa_{\pi s_r} q\mathfrak{p}\right) = (1 - \mathfrak{p}) Y^{\mathbb{H},\mathbb{V}} + q\mathfrak{p} Y^{\mathbb{H} \cdot s_r, \mathbb{V}}$, which completes one of the cases in (3.2).

3.2. **Proof of Theorem 1.1.** We would like to show that for any horizontal boundary condition $\mathbb{H} = \{(l_1, r_1), \ldots, (l_h, r_h)\}$, we have $\mathbf{P}^{\mathbb{H},\mathbb{V}}(\mathbf{x}, \mathbf{y}) = \mathbf{P}^{180^\circ(\mathbb{H}),\mathbb{V}}(\mathbf{x}, \mathrm{rev}(\mathbf{y}))$. We will do this by induction, making extensive use of Equations (3.2)–(3.3). We will induct on the set $\mathcal{R}(\mathbb{H}) := \{r_1, r_2, \ldots, r_h\}$ using a certain partial order defined below.

Let $w := w^{M,N}$ be the Grassmannian permutation from Example 2.9, and let $Y^w \in \mathcal{H}_q(S_n; \mathbf{z})$ be the corresponding element of the Yang–Baxter basis. Equation (2.7) implies

$$(3.4) \qquad Y^w \cdot R_r(\mathfrak{p}_{l,l+1}) = R_l(\mathfrak{p}_{l,l+1}) \cdot s_{l;\mathbf{z}}(Y^w) \quad \text{for } r \in [1, N-1] \text{ and } l := r + M,$$

$$\mathbf{P}^{\mathbb{H}_0,\mathbb{V}}(\mathbf{x},\mathbf{y}) \quad = \quad \mathbf{P}^{180°(\mathbb{H}_0),\mathbb{V}}(\mathbf{x},\mathbf{y}') \quad = \quad \mathbf{P}^{180°(\mathbb{H}_0),\mathbb{V}}(\mathbf{x},\mathrm{rev}(\mathbf{y}))$$
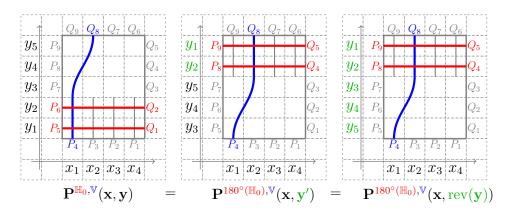
FIGURE 9. The base of the induction in the proof of Theorem 1.1. The first equality is true by definition, and the second equality follows by applying a sequence of Yang–Baxter moves to flip the order of y_3, y_4, y_5.

where $s_{l;\mathbf{z}}$ is the automorphism of $\mathcal{H}_q(S_n;\mathbf{z})$ that swaps z_l and z_{l+1}.

Let $\mathbb{H}_0 := \{(M+1,1),(M+2,2),\ldots,(M+h,h)\}$ be the horizontal boundary condition where the corresponding h colored paths are fully packed at the bottom of the rectangle. In this case, we have $180°(\mathbb{H}_0) = \{(n,N),(n-1,N-1),\ldots,(n-h+1,N-h+1)\}$, and the corresponding h paths are fully packed at the top of the rectangle.

Since $\mathbb{V},P,Q,\mathbf{x}$ are fixed, we denote $\mathbf{P}[\mathbb{H},\mathbf{y}] := \mathbf{P}^{\mathbb{H},\mathbb{V}}(\mathbf{x},\mathbf{y})$. The idea of the proof is to first show the result for $\mathbb{H}_0$ and then use (3.4) to express $\mathbf{P}[\mathbb{H},\mathbf{y}]$ inductively for any $\mathbb{H}$ in terms of $\mathbf{P}[\mathbb{H}_0,\mathbf{y}]$. Because of the 180°-symmetry of (3.2)–(3.4), it will follow that $\mathbf{P}[180°(\mathbb{H}),\mathrm{rev}(\mathbf{y})]$ can be expressed in terms of $\mathbf{P}[180°(\mathbb{H}_0),\mathrm{rev}(\mathbf{y})]$ in a symmetric way, which will imply $\mathbf{P}[\mathbb{H},\mathbf{y}] = \mathbf{P}[180°(\mathbb{H}),\mathrm{rev}(\mathbf{y})]$.

We start with the base case $\mathbf{P}[\mathbb{H}_0,\mathbf{y}] = \mathbf{P}[180°(\mathbb{H}_0),\mathrm{rev}(\mathbf{y})]$, illustrated in Figure 9. Recall from Figure 8(right) and Example 2.9 that the variables $\mathbf{x},\mathbf{y},\mathbf{z}$ are related as $\mathbf{z} = (x_M,x_{M-1},\ldots,x_1,y_1,y_2,\ldots,y_N)$. Let $\mathbf{y}' := (y_{h+1},y_{h+2},\ldots,y_N,y_h,y_{h-1},\ldots,y_1)$ and recall that $\mathrm{rev}(\mathbf{y}) = (y_N,y_{N-1},\ldots,y_1)$. It follows from the definition of the model that $\mathbf{P}[\mathbb{H}_0,\mathbf{y}] = \mathbf{P}[180°(\mathbb{H}_0),\mathbf{y}']$. To go from $\mathbf{y}'$ to $\mathrm{rev}(\mathbf{y})$, we use the standard application of the Yang–Baxter equation, see e.g. [BGW19, Eq. (5.18)]: it is clear that $\mathbb{H}_0' := 180°(\mathbb{H}_0)$ satisfies $\mathbb{H}_0' \cdot s_r = s_{r+M} \cdot \mathbb{H}_0' = \mathbb{H}_0$ for all $r \in [1,N-h-1]$. Thus (3.4) combined with (3.2)–(3.3) shows that $\mathbf{P}[180°(\mathbb{H}_0),\mathbf{y}']$ is symmetric in the variables $y_{h+1},y_{h+2},\ldots,y_N$ and therefore equals $\mathbf{P}[180°(\mathbb{H}_0),\mathrm{rev}(\mathbf{y})]$. This shows $\mathbf{P}[\mathbb{H}_0,\mathbf{y}] = \mathbf{P}[180°(\mathbb{H}_0),\mathrm{rev}(\mathbf{y})]$, finishing the induction base.

Recall that for $\mathbb{H} = \{(l_1,r_1),\ldots,(l_h,r_h)\}$, we denote $\mathcal{R}(\mathbb{H}) := \{r_1,r_2,\ldots,r_h\} \subseteq [1,N]$. For two h-element subsets $I = \{i_1 < i_2 < \cdots < i_h\}$ and $J = \{j_1 < j_2 < \cdots < j_h\}$ of $[1,N]$, we write $I \preceq J$ if $i_1 \leq j_1, i_2 \leq j_2, \ldots, i_h \leq j_h$. We also write $I \prec J$ if $I \preceq J$ but $I \neq J$. Note that if $\mathcal{R}(\mathbb{H}) = [1,h]$ is minimal in this order then either $\mathbb{H} = \mathbb{H}_0$ or $\mathbf{P}[\mathbb{H},\mathbf{y}] = 0$.

Let $\mathbb{H} = \{(l_1,r_1),\ldots,(l_h,r_h)\}$. Suppose that we have shown $\mathbf{P}[\mathbb{H}',\mathbf{y}] = \mathbf{P}[180°(\mathbb{H}'),\mathrm{rev}(\mathbf{y})]$ for all $\mathbb{H}'$ satisfying $\mathcal{R}(\mathbb{H}') \prec \mathcal{R}(\mathbb{H})$. If $\mathbb{H} = \mathbb{H}_0$ then we are done, otherwise $\mathcal{R}(\mathbb{H}) \neq [1,h]$, so let $r \in [1,N-1]$ be an index such that $r \notin \mathcal{R}(\mathbb{H})$ but $r+1 \in \mathcal{R}(\mathbb{H})$. Let $\mathbb{H}' := \mathbb{H} \cdot s_r > \mathbb{H}$ and denote $l := r + M$. Applying the map $Y \mapsto Y^{\mathbb{H}',\mathbb{V}}$ to both sides of (3.4) and using the relations (3.2)–(3.3), we get

$$(3.5) \qquad \kappa_1 (Y^w)^{\mathbb{H},\mathbb{V}} + \kappa_1' (Y^w)^{\mathbb{H}',\mathbb{V}} = \kappa_2' s_{l;\mathbf{z}} (Y^w)^{\mathbb{H}',\mathbb{V}} + \kappa_2'' s_{l;\mathbf{z}} (Y^w)^{\mathbb{H}'',\mathbb{V}},$$

$$\mathbb{H} \qquad \mathbb{H}' = \mathbb{H} \cdot s_r \qquad \mathbb{H}'' = s_l \cdot \mathbb{H} \cdot s_r$$
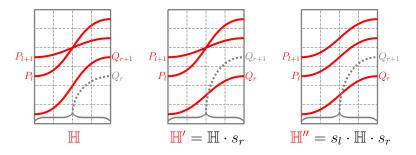
FIGURE 10. The induction step in the proof of Theorem 1.1: one can express $\mathbf{P}[\mathbb{H}, \mathbf{y}]$ recursively in terms of $\mathbf{P}[\mathbb{H}', \mathbf{y}]$ and $\mathbf{P}[\mathbb{H}'', \mathbf{y}]$.

for some coefficients κ_1, κ_1', κ_2', and κ_2'', where $\mathbb{H}'' := s_l \cdot \mathbb{H}' = s_l \cdot \mathbb{H} \cdot s_r$. See Figure 10.

Since $\mathbb{H}' \cdot s_r = \mathbb{H} < \mathbb{H}'$, we get $\kappa_1 = \mathfrak{p}_{l,l+1}$ and $\kappa_1' = (1 - q\mathfrak{p}_{l,l+1})$. The coefficients κ_2' and κ_2'' depend on whether $\mathbb{H}' < \mathbb{H}''$, $\mathbb{H}' = \mathbb{H}''$, or $\mathbb{H}' > \mathbb{H}''$. For example, Figure 10 illustrates the case $\mathbb{H}' > \mathbb{H}''$. Recall that $\mathbf{P}[\mathbb{H}, \mathbf{y}] = (Y^w)^{\mathbb{H}, \mathbb{V}}$ by Proposition 2.8. Thus (3.5) allows us to express $\mathbf{P}[\mathbb{H}, \mathbf{y}]$ as a linear combination of $\mathbf{P}[\mathbb{H}', \mathbf{y}]$, $s_{l;\mathbf{z}}\mathbf{P}[\mathbb{H}', \mathbf{y}]$, and $s_{l;\mathbf{z}}\mathbf{P}[\mathbb{H}'', \mathbf{y}]$.

Let $\bar{l} := n - r$, $\bar{r} := n - l$. Observe that $\mathbb{H} < \mathbb{H} \cdot s_r$ implies $180°(\mathbb{H}) < 180°(\mathbb{H} \cdot s_r)$, where $180°(\mathbb{H} \cdot s_r) = s_{\bar{l}} \cdot 180°(\mathbb{H})$. Applying the map $Y \mapsto Y^{180°(\mathbb{H}'), \mathbb{V}}$ to both sides of (3.4) yields

$$(3.6) \qquad \bar{\kappa}_2'(Y^w)^{180°(\mathbb{H}'), \mathbb{V}} + \bar{\kappa}_2''(Y^w)^{180°(\mathbb{H}''), \mathbb{V}} = \bar{\kappa}_1 s_{\bar{l};\mathbf{z}}(Y^w)^{180°(\mathbb{H}), \mathbb{V}} + \bar{\kappa}_1' s_{\bar{l};\mathbf{z}}(Y^w)^{180°(\mathbb{H}'), \mathbb{V}}$$

for some coefficients $\bar{\kappa}_1$, $\bar{\kappa}_1'$, $\bar{\kappa}_2'$, and $\bar{\kappa}_2''$. Because of the symmetry between Equations (3.2) and (3.3), we find that the tuple $(\bar{\kappa}_1, \bar{\kappa}_1', \bar{\kappa}_2', \bar{\kappa}_2'')$ is obtained from $(\kappa_1, \kappa_1', \kappa_2', \kappa_2'')$ by replacing $\mathfrak{p}_{l,l+1}$ with $\mathfrak{p}_{\bar{l}, \bar{l}+1}$. For instance, we have $\bar{\kappa}_1 = \mathfrak{p}_{\bar{l}, \bar{l}+1}$ and $\bar{\kappa}_1' = (1 - q\mathfrak{p}_{\bar{l}, \bar{l}+1})$.

Since $\mathfrak{p}_{l,l+1} = \left(s_{\bar{l};\mathbf{z}}(\mathfrak{p}_{\bar{l}, \bar{l}+1})\right)\big|_{\mathbf{y} \mapsto \mathrm{rev}(\mathbf{y})}$, the transformation $\bar{\kappa} \mapsto \left(s_{\bar{l};\mathbf{z}}(\bar{\kappa})\right)\big|_{\mathbf{y} \mapsto \mathrm{rev}(\mathbf{y})}$ takes $(\bar{\kappa}_1, \bar{\kappa}_1', \bar{\kappa}_2', \bar{\kappa}_2'')$ to $(\kappa_1, \kappa_1', \kappa_2', \kappa_2'')$. Applying $s_{\bar{l};\mathbf{z}}$ to both sides of (3.6) and substituting $\mathbf{y} \mapsto \mathrm{rev}(\mathbf{y})$ therefore gives

$$(3.7) \quad \begin{aligned} \kappa_2' s_{l;\mathbf{z}}\mathbf{P}[180°(\mathbb{H}'), \mathrm{rev}(\mathbf{y})] + \kappa_2'' s_{l;\mathbf{z}}\mathbf{P}[180°(\mathbb{H}''), \mathrm{rev}(\mathbf{y})] \\ = \kappa_1 \mathbf{P}[180°(\mathbb{H}), \mathrm{rev}(\mathbf{y})] + \kappa_1' \mathbf{P}[180°(\mathbb{H}'), \mathrm{rev}(\mathbf{y})]. \end{aligned}$$

Finally, notice that $\mathcal{R}(\mathbb{H}') = \mathcal{R}(\mathbb{H}'') \prec \mathcal{R}(\mathbb{H})$. Applying the induction hypothesis, we get

$$(3.8) \qquad \mathbf{P}[\mathbb{H}', \mathbf{y}] = \mathbf{P}[180°(\mathbb{H}'), \mathrm{rev}(\mathbf{y})] \quad \text{and} \quad \mathbf{P}[\mathbb{H}'', \mathbf{y}] = \mathbf{P}[180°(\mathbb{H}''), \mathrm{rev}(\mathbf{y})].$$

Combining (3.5), (3.7), and (3.8) with the fact that $\kappa_1 = \mathfrak{p}_{l,l+1} \neq 0$, we find $\mathbf{P}[\mathbb{H}, \mathbf{y}] = \mathbf{P}[180°(\mathbb{H}), \mathrm{rev}(\mathbf{y})]$, completing the induction step. $\square$

3.3. **Generalized flip theorem.** In order to pass from rectangles to arbitrary skew domains, we will need to state a certain generalization of the flip theorem to the case where the incoming colors are not necessarily ordered.

Fix two elements $\mathbf{i}, \mathbf{j} \in [1, n-1]$. For simplicity, let us first assume that $\mathbf{i} + \mathbf{j} = n$ so that the map $k \mapsto \mathbf{i} + \mathbf{j} - k$ is a bijection $[1, n-1] \to [1, n-1]$. (We explain how to lift this assumption in Remark 3.3.) Consider an involutive anti-automorphism $Y \mapsto (Y)^*_{\mathbf{i},\mathbf{j}}$ of $\mathcal{H}_q(S_n; \mathbf{z})$ sending $T_k \mapsto T_{\mathbf{i}+\mathbf{j}-k}$ for all $k \in [1, n-1]$, thus we have

$$(T_{k_1} T_{k_2} \cdots T_{k_r})^*_{\mathbf{i},\mathbf{j}} := T_{\mathbf{i}+\mathbf{j}-k_r} \cdots T_{\mathbf{i}+\mathbf{j}-k_2} T_{\mathbf{i}+\mathbf{j}-k_1}.$$
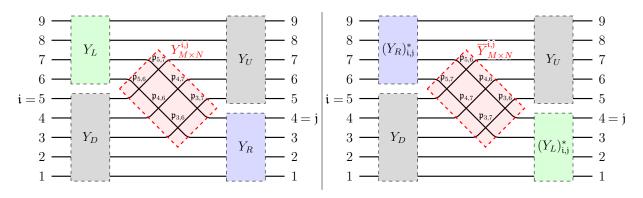
FIGURE 11. An application of the generalized flip theorem (Theorem 3.1).

Fix $M \in [1, \mathfrak{i}]$ and $N \in [1, \mathfrak{j}]$ satisfying $N + \mathfrak{i} = M + \mathfrak{j}$. Consider a permutation $w \in S_n$ defined for all $r \in [1, n]$ by

$$
w(r) := \begin{cases} r + M + \mathfrak{j} - \mathfrak{i}, & \text{if } r \in [\mathfrak{i} - M + 1, \mathfrak{i}], \\ r - N + \mathfrak{j} - \mathfrak{i}, & \text{if } r \in [\mathfrak{i} + 1, \mathfrak{i} + N], \\ r, & \text{otherwise.} \end{cases}
$$

We let $Y^{\mathfrak{i},\mathfrak{j}}_{M \times N} := Y^w$ be the corresponding Yang–Baxter basis element, see Figure 11(left). We also denote $\overline{Y}^{\mathfrak{i},\mathfrak{j}}_{M \times N} := Y^{\mathfrak{i},\mathfrak{j}}_{M \times N} \mid_{\mathbf{z} \mapsto \mathrm{rev}_{[\mathfrak{i}+1,\mathfrak{i}+N]}(\mathbf{z})}$, where the substitution $\mathbf{z} \mapsto \mathrm{rev}_{[\mathfrak{i}+1,\mathfrak{i}+N]}(\mathbf{z})$ is a shorthand for $z_{\mathfrak{i}+1} \mapsto z_{\mathfrak{i}+N}, z_{\mathfrak{i}+2} \mapsto z_{\mathfrak{i}+N-1}, \ldots, z_{\mathfrak{i}+N} \mapsto z_{\mathfrak{i}+1}$, see Figure 11(right).

Analogously to Section 1.2, given a permutation $\pi \in S_n$, we let

$$(3.9) \qquad \mathbb{H}^{\mathfrak{i},\mathfrak{j}}_\pi := \{(i, \pi(i)) \mid i > \mathfrak{i} \text{ and } \pi(i) \le \mathfrak{j}\}, \quad \mathbb{V}^{\mathfrak{i},\mathfrak{j}}_\pi := \{(i, \pi(i)) \mid i \le \mathfrak{i} \text{ and } \pi(i) > \mathfrak{j}\}.$$

For two sets $\mathbb{H} = \{(l_1, r_1), \ldots, (l_h, r_h)\}$ and $\mathbb{V} = \{(d_1, u_1), \ldots, (d_v, u_v)\}$ of pairs, we denote $\mathrm{SAT}_{\mathfrak{i},\mathfrak{j}}(\mathbb{H}, \mathbb{V}) := \{\pi \in S_n \mid \mathbb{H}^{\mathfrak{i},\mathfrak{j}}_\pi = \mathbb{H} \text{ and } \mathbb{V}^{\mathfrak{i},\mathfrak{j}}_\pi = \mathbb{V}\}$ and for $Y = \sum_{\pi \in S_n} \kappa_\pi T_\pi \in \mathcal{H}_q(S_n; \mathbf{z})$, we let $(Y)^{\mathbb{H},\mathbb{V}}_{\mathfrak{i},\mathfrak{j}} := \sum_{\pi \in \mathrm{SAT}_{\mathfrak{i},\mathfrak{j}}(\mathbb{H},\mathbb{V})} \kappa_\pi$ as in (3.1). Finally, we define

$$(3.10) \qquad 180^\circ_{\mathfrak{i},\mathfrak{j}}(\mathbb{H}) := \{(\mathfrak{i} + \mathfrak{j} + 1 - r_1, \mathfrak{i} + \mathfrak{j} + 1 - l_1), \ldots, (\mathfrak{i} + \mathfrak{j} + 1 - r_h, \mathfrak{i} + \mathfrak{j} + 1 - l_h)\}.$$

For $1 \le a \le b \le n$, we let $S_{[a,b]}$ denote the (parabolic) subgroup of S_n generated by $\{s_k \mid k \in [a, b-1]\}$, and we let $\mathcal{H}_q(S_{[a,b]}; \mathbf{z})$ be the subalgebra of $\mathcal{H}_q(S_n; \mathbf{z})$ generated by $\{T_\pi \mid \pi \in S_{[a,b]}\}$.

Theorem 3.1. *Fix $n, \mathfrak{i}, \mathfrak{j}, M, N$ as above and let*

$$Y_L \in \mathcal{H}_q(S_{[\mathfrak{i}+1,n]}; \mathbf{z}), \quad Y_D \in \mathcal{H}_q(S_{[1,\mathfrak{i}]}; \mathbf{z}), \quad Y_U \in \mathcal{H}_q(S_{[\mathfrak{j}+1,n]}; \mathbf{z}), \quad Y_R \in \mathcal{H}_q(S_{[1,\mathfrak{j}]}; \mathbf{z})$$

be arbitrary elements. Then for all $\mathbb{H}$ and $\mathbb{V}$, we have

$$(3.11) \qquad \left(Y_L \cdot Y_D \cdot Y^{\mathfrak{i},\mathfrak{j}}_{M \times N} \cdot Y_U \cdot Y_R \right)^{\mathbb{H},\mathbb{V}}_{\mathfrak{i},\mathfrak{j}} = \left((Y_R)^*_{\mathfrak{i},\mathfrak{j}} \cdot Y_D \cdot \overline{Y}^{\mathfrak{i},\mathfrak{j}}_{M \times N} \cdot Y_U \cdot (Y_L)^*_{\mathfrak{i},\mathfrak{j}} \right)^{180^\circ_{\mathfrak{i},\mathfrak{j}}(\mathbb{H}),\mathbb{V}}_{\mathfrak{i},\mathfrak{j}}.$$

See Figure 11 for an illustration when $n = 9$, $\mathfrak{i} = 5$, $\mathfrak{j} = 4$, $M = 3$, and $N = 2$.

Remark 3.2. The substitution $\mathbf{z} \mapsto \mathrm{rev}_{[\mathfrak{i}+1,\mathfrak{i}+N]}(\mathbf{z})$ is performed only for the element $Y^{\mathfrak{i},\mathfrak{j}}_{M \times N}$, the parameters appearing in the other four elements remain unchanged. Thus if the left hand side $Y_L \cdot Y_D \cdot Y^{\mathfrak{i},\mathfrak{j}}_{M \times N} \cdot Y_U \cdot Y_R$ belongs to the Yang–Baxter basis, the right hand side $(Y_R)^*_{\mathfrak{i},\mathfrak{j}} \cdot Y_D \cdot \overline{Y}^{\mathfrak{i},\mathfrak{j}}_{M \times N} \cdot Y_U \cdot (Y_L)^*_{\mathfrak{i},\mathfrak{j}}$ in general does not belong to the Yang–Baxter basis.

Proof. By the $\mathbb{C}(q; \mathbf{z})$-linearity of the maps $Y \mapsto (Y)_{i,j}^{\mathbb{H},\mathbb{V}}$ and $Y \mapsto (Y)_{i,j}^*$, it suffices to show the result for the case $Y_L = T_{w_L}$, $Y_D = T_{w_D}$, $Y_U = T_{w_U}$, $Y_R = T_{w_R}$ for some $w_L \in S_{[i+1,n]}$, $w_D \in S_{[1,i]}$, $w_U \in S_{[j+1,n]}$, $w_R \in S_{[1,j]}$. We do this by induction on the total length of these permutations.

The base case $\left(Y_{M \times N}^{i,j}\right)_{i,j}^{\mathbb{H},\mathbb{V}} = \left(\overline{Y}_{M \times N}^{i,j}\right)_{i,j}^{180_{i,j}^{\circ}(\mathbb{H}),\mathbb{V}}$ is the content of Theorem 1.1. For the induction step, observe that Y_L commutes with Y_D and Y_U commutes with Y_R. Let us for example consider the case $w_L = s_k w_L'$ with $\ell(w_L) = \ell(w_L') + 1$. Then $Y_L = T_k Y_L'$, where $Y_L' := T_{w_L'}$, and we also have $(Y_L)_{i,j}^* = (Y_L')_{i,j}^* T_{i+j-k}$. Recall that $T_k = R_k(1)$, thus the rules for how both sides of (3.11) change under multiplication by T_k are obtained from (3.2)–(3.3) by specializing $\mathfrak{p} = 1$. The induction step follows. The other three cases are handled similarly. $\qquad\square$

Remark 3.3. We have assumed above that $i + j = n$. For general i and j, let us introduce $k_{\min} := \max(1, i + j + 1 - n)$ and $k_{\max} := \min(n, i + j)$. Then the map $k \mapsto i + j - k$ is a bijection sending $[k_{\min}, k_{\max} - 1]$ to itself. In the statement of Theorem 3.1 we then additionally assume that the elements Y_L, $Y_{M \times N}^{i,j}$, and Y_R all belong to $\mathcal{H}_q(S_{[k_{\min}, k_{\max}]}; \mathbf{z})$. The proof of Theorem 3.1 remains the same in this more general case.

Remark 3.4. Substituting $Y_L = Y_U = Y_R = 1$ into (3.11), we find that the flip theorem (Theorem 1.1) holds more generally as described in Remark 1.5. Thus the two assumptions required for Theorem 1.1 to hold are that the incoming colors entering the rectangle from the left are increasing bottom-to-top and are all larger than the colors entering the rectangle from the bottom. Theorem 1.1 fails in general if one of these two assumptions is not satisfied.

4. Consequences of the flip theorem

Our proof of Theorem 1.6 will consist of two parts. In this section, we use the flip theorem to construct a "zoo" of transformations on tuples of (P, Q)-cuts that preserve the joint distribution of the associated vector of height functions. In the next section, we will show that, after restricting to *connected components*, any transformation satisfying the conditions of Theorem 1.6 can be obtained as a composition of transformations constructed in this section. We start by introducing some notation related to (P, Q)-cuts.

Definition 4.1. Let (P, Q) be a skew domain and consider a (P, Q)-cut $C = (l, d, u, r)$. Recall from Section 1.3 and Figure 4(middle) that we associate the numbers $1 \le i, j \le n$ to C so that i (resp., j) is the number of steps in P (resp., of Q) between its start and the bottom left (resp., top right) corner of the cell (l, d) (resp., (r, u)). We call the numbers (i, j) the *color cutoff levels* of C. Given another (P, Q)-cut $C' = (l', d', u', r') \ne C$, we say that C and C' *cross* if the closed line segments $[(l, d), (r, u)]$ and $[(l', d'), (r', u')]$ intersect in the plane. Equivalently, C and C' *cross* if $[l, r] \subseteq [l', r']$ and $[d, u] \supseteq [d', u']$ or vice versa.

For the rest of this section, we fix a skew domain (P, Q) and a tuple $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ of (P, Q)-cuts, where $C_i = (l_i, d_i, u_i, r_i)$ for $i \in [1, m]$. We let

$$\mathrm{Ht}^{P,Q}(\mathcal{C}; \mathbf{x}, \mathbf{y}) := \left(\mathrm{Ht}^{P,Q}(C_1; \mathbf{x}, \mathbf{y}), \mathrm{Ht}^{P,Q}(C_2; \mathbf{x}, \mathbf{y}), \ldots, \mathrm{Ht}^{P,Q}(C_m; \mathbf{x}, \mathbf{y})\right)$$

denote the associated vector of height functions.

4.1. **Admissible transformations: definitions.** Given two integers $l \leq r \in \mathbb{Z}_{\geq 1}$, we let $\mathrm{rev}_{[l,r]} : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ be the bijection defined by

$$\mathrm{rev}_{[l,r]}(k) = \begin{cases} l + r - k, & \text{if } k \in [l,r], \\ k, & \text{otherwise,} \end{cases} \qquad \text{for all } k \in \mathbb{Z}_{\geq 1}.$$

Definition 4.2. A *transformation* is a pair $\Phi := (\phi_H, \phi_V)$ of bijections $\phi_H, \phi_V : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$. We say that a transformation Φ is $\mathcal{C}$-*admissible* if there exists another skew domain (P', Q') and a tuple $\mathcal{C}' = (C'_1, C'_2, \ldots, C'_m)$ of (P', Q')-cuts with $C'_i = (l'_i, d'_i, u'_i, r'_i)$ such that for all $i \in [1, m]$, we have $\phi_H([l_i, r_i]) = [l'_i, r'_i]$ and $\phi_V([d_i, u_i]) = [d'_i, u'_i]$. In this case, we say that Φ is *strongly* $\mathcal{C}$-*admissible* if $\mathrm{Ht}^{P,Q}(\mathcal{C}; \mathbf{x}, \mathbf{y}) \overset{d}{=} \mathrm{Ht}^{P',Q'}(\mathcal{C}'; \mathbf{x}', \mathbf{y}')$, where $\mathbf{x}' := \phi_H^{-1}(\mathbf{x}) = (x_{\phi_H^{-1}(1)}, x_{\phi_H^{-1}(2)}, \ldots)$ and $\mathbf{y}' := \phi_V^{-1}(\mathbf{y}) = (y_{\phi_V^{-1}(1)}, y_{\phi_V^{-1}(2)}, \ldots)$.

Remark 4.3. Having $\phi_H([l_i, r_i]) = [l'_i, r'_i]$ is equivalent to having $\mathrm{supp}_H(C_i; \mathbf{x}) = \mathrm{supp}_H(C'_i; \mathbf{x}')$ for $\mathbf{x}' = \phi_H^{-1}(\mathbf{x})$: we have $\mathrm{supp}_H(C'_i; \mathbf{x}') = \{x'_s\}_{s \in [l'_i, r'_i]} = \{x'_{\phi_H(t)}\}_{t \in [l_i, r_i]}$, and this equals to $\mathrm{supp}_H(C_i; \mathbf{x}) = \{x_t\}_{t \in [l_i, r_i]}$ when we take $x'_{\phi_H(t)} := x_t$.

Remark 4.4. Given a $\mathcal{C}$-admissible transformation Φ, the above tuple $\mathcal{C}'$ is uniquely determined by the conditions $\phi_H([l_i, r_i]) = [l'_i, r'_i]$, $\phi_V([d_i, u_i]) = [d'_i, u'_i]$, thus we denote $\mathcal{C}' = \Phi(\mathcal{C})$. However, there are in general many choices for domains (P, Q) and (P', Q') such that $\mathcal{C}$ is a tuple of (P, Q)-cuts and $\mathcal{C}'$ is a tuple of (P', Q')-cuts. So it may appear that the notion of strong $\mathcal{C}$-admissibility depends on the choices of skew domains (P, Q) and (P', Q'), but actually this is not the case as we show in Corollary 4.13. Thus proving Theorem 1.6 amounts to showing that each $\mathcal{C}$-admissible transformation is strongly $\mathcal{C}$-admissible. We now give a list of strongly $\mathcal{C}$-admissible transformations.

4.2. **Admissible transformations: examples.** In this subsection, we state that several transformations are strongly $\mathcal{C}$-admissible. We will prove these statements later in Section 4.4. Fix some integers $M_\infty, N_\infty \in \mathbb{Z}_{\geq 1}$ satisfying $(P, Q)_\mathbb{Z} \subseteq [1, M_\infty] \times [1, N_\infty]$.

Color-position symmetry. Our first transformation comes from Theorem 2.4. In fact, this is the only transformation that is not a consequence of the flip theorem (although it is a trivial transformation from the point of view of Hecke algebras as explained in Section 2.2). Recall that we have fixed a skew domain (P, Q) and a tuple $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ of (P, Q)-cuts.

Lemma 4.5 (Color-position symmetry). $\Phi := \left(\mathrm{rev}_{[1, M_\infty]}, \mathrm{rev}_{[1, N_\infty]} \right)$ *is a strongly* $\mathcal{C}$-*admissible transformation.*

Under this transformation, the domain (P, Q) and all (P, Q)-cuts rotate by 180 degrees.

Global flips. Our next transformation is a direct application of Theorem 3.1. In fact, we describe two transformations, a *global* $\mathbb{H}$-*flip* and a *global* $\mathbb{V}$-*flip*. They are obtained from each other by interchanging the horizontal and vertical directions, i.e., reflecting along the line $y = x$. We thus describe only one of the two transformations.

Lemma 4.6 (Global $\mathbb{H}$-flip). *Suppose that $C = (l, d, u, r)$ is a (P, Q)-cut that crosses C_i for all $i \in [1, m]$. Then $\Phi := \left(\mathrm{rev}_{[1, M_\infty]} \circ \mathrm{rev}_{[l, r]}, \mathrm{rev}_{[d, u]} \right)$ is a strongly $\mathcal{C}$-admissible transformation.*

See Figure 12 for an example. The dashed rectangle represents the cut $C = (l, d, u, r)$.
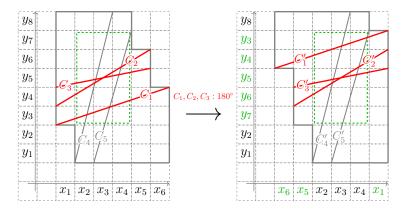
FIGURE 12.   A global $\mathbb{H}$-flip (Lemma 4.6).
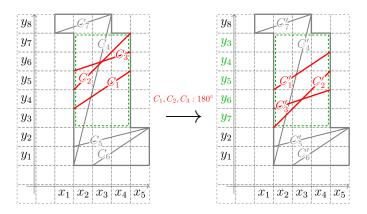
FIGURE 13.   A local $\mathbb{H}$-flip (Lemma 4.7).

Local flips. Similarly to global flips, we introduce *local* $\mathbb{H}$-*flips* and *local* $\mathbb{V}$-*flips*. For $d \leq u \in \mathbb{Z}_{\geq 1}$, we say that $\mathbb{Z}_{\geq 1} \times [d, u]$ is a *horizontal* $\mathcal{C}$-*strip* if there does not exist $i \in [1, m]$ such that either $d_i < d \leq u_i < u$ or $d < d_i \leq u < u_i$. In other words, for all $i \in [1, m]$, we have either $u_i < d$, or $u < d_i$, or $[d, u] \subseteq [d_i, u_i]$, or $[d_i, u_i] \subseteq [d, u]$.

Lemma 4.7 (Local $\mathbb{H}$-flip). *Suppose that $\mathbb{Z}_{\geq 1} \times [d, u]$ is a horizontal $\mathcal{C}$-strip. Let*

$$(4.1) \qquad J := \{ j \in [1, m] \mid [d_j, u_j] \subsetneq [d, u] \}, \quad K := \{ k \in [1, m] \mid [d, u] \subseteq [d_k, u_k] \}.$$

Suppose that there exist integers $l \leq r \in \mathbb{Z}_{\geq 1}$ satisfying $[l, r] = [l_j, r_j]$ and $[l_k, r_k] \subseteq [l, r]$ for all $j \in J$ and $k \in K$. Then $\Phi := (\mathrm{id}, \mathrm{rev}_{[d, u]})$ is a strongly $\mathcal{C}$-admissible transformation.

See Figure 13 for an example with $J = \{1, 2, 3\}$ and $K = \{4\}$. The dashed rectangle represents the cut (l, d, u, r).

Double flips and shifts. The last pair of local transformations that we consider are *double* $\mathbb{H}$-*flips* and *double* $\mathbb{V}$-*flips*.

Lemma 4.8 (Double $\mathbb{H}$-flip). *Let $\mathbb{Z}_{\geq 1} \times [d', u']$ and $\mathbb{Z}_{\geq 1} \times [d, u]$ be two horizontal $\mathcal{C}$-strips with $[d', u'] \subsetneq [d, u]$. Denote*

$$(4.2) \qquad\qquad I := \{ i \in [1, m] \mid [d_i, u_i] \subseteq [d', u'] \},$$

$$(4.3) \qquad\qquad J := \{ j \in [1, m] \mid [d_j, u_j] \subsetneq [d, u] \} \setminus I,$$

$$(4.4) \qquad\qquad K := \{ k \in [1, m] \mid [d, u] \subseteq [d_k, u_k] \}.$$

Suppose that there exist integers $l \leq r \in \mathbb{Z}_{\geq 1}$ satisfying $[l, r] \subseteq [l_i, r_i]$, $[l, r] = [l_j, r_j]$, and $[l_k, r_k] \subseteq [l, r]$ for all $i \in I$, $j \in J$, and $k \in K$. Then $\Phi := (\mathrm{id}, \mathrm{rev}_{[d,u]} \circ \mathrm{rev}_{[d',u']})$ is a strongly $\mathcal{C}$-admissible transformation.

See Figure 5 for an example with $d = 3$, $u = 9$, $d' = 5$, $u' = 6$, $l = 4$, $r = 7$, $I = \{1, 2, 3\}$, $J = \{4, 5\}$, and $K = \{6\}$.

Remark 4.9. Under the double $\mathbb{H}$-flip, the tuple $(C_j)_{j \in J}$ rotates by 180 degrees while the tuple $(C_i)_{i \in I}$ shifts up by $u + d - u' - d'$. The transformation makes sense when either I or J is empty. If I is empty then it is essentially a special case of a local $\mathbb{H}$-flip (or rather a composition of two local $\mathbb{H}$-flips one of which does not change the tuple of cuts). When J is empty, we call this transformation an $\mathbb{H}$-*shift*.

Remark 4.10. When $J = \{j\}$ consists of one element and $d' + u' = d + u$, the double $\mathbb{H}$-flip can be considered as an operation of shifting C_j relative to $(C_i)_{i \in [1,m] \setminus J}$. In this case, Lemma 4.8 generalizes the shift-invariance results of [BGW19]. Specifically, [BGW19, Theorems 1.2 and 4.13] correspond to the special case of Lemma 4.8 where $|J| = 1$, $(d', u') = (d + 1, u - 1)$, and $I \cup J \cup K = [1, m]$.

4.3. **Equivalence classes of pipe dreams.** In order to prove that the above transformations are strongly $\mathcal{C}$-admissible, we need to introduce a certain equivalence relation on the set of SC6V model configurations. Following the literature [BB93, FK96, KM05] on Schubert polynomials, we say that a (P, Q)-*pipe dream* (equivalently, a configuration of the SC6V model) is a diagram Π obtained by replacing each cell in $(P, Q)_{\mathbb{Z}}$ with either a crossing $+$ or an elbow $\asymp$. Thus the total number of (P, Q)-pipe dreams equals $2^{\#(P,Q)_{\mathbb{Z}}}$, and each (P, Q)-pipe dream Π is naturally a union of n colored paths, so we denote the corresponding color permutation by π_{Π}. For a (P, Q)-pipe dream Π, denote $\mathrm{Ht}_{\Pi}^{P,Q}(\mathcal{C}) := \left(\mathrm{Ht}_{\pi_{\Pi}}^{P,Q}(C_1), \ldots, \mathrm{Ht}_{\pi_{\Pi}}^{P,Q}(C_m) \right)$. When the skew domain (P, Q) is fixed, we usually omit the dependence on it from the notation.

The probability $\mathbf{P}(\Pi)$ of a given pipe dream Π is defined by

$$\mathbf{P}(\Pi) = \prod_{(i,j) \in (P,Q)_{\mathbb{Z}}} \mathrm{wt}_{\Pi}(i, j), \quad \text{where} \quad \mathrm{wt}_{\Pi}(i, j) \in \{\mathfrak{p}_{i,j}, q\mathfrak{p}_{i,j}, 1 - \mathfrak{p}_{i,j}, 1 - q\mathfrak{p}_{i,j}\}$$

according to the four possibilities in Figure 1. For all $\pi \in S_n$, we have $\mathbf{P}_{\pi}^{P,Q}(\mathbf{x}, \mathbf{y}) = \sum_{\Pi : \pi_{\Pi} = \pi} \mathbf{P}(\Pi)$.

Recall that we have fixed a tuple $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ of (P, Q)-cuts. For each $k \in [1, m]$, let $\mathfrak{i}_k, \mathfrak{j}_k \in [1, n]$ be the color cutoff levels of C_k (cf. Definition 4.1).

Definition 4.11. Given a pipe dream Π and a cell $(i, j) \in (P, Q)_{\mathbb{Z}}$, let $c_1 < c_2 \in [1, n]$ be the colors of the two paths entering this cell. For $k \in [1, m]$, we say that C_k *involves* (i, j) *in* Π if $(i, j) \in [l_k, r_k] \times [d_k, u_k]$ and $\mathfrak{i}_k \in [c_1, c_2 - 1]$, see Figure 14(left). We say that (i, j) is $(\Pi, \mathcal{C})$-*relevant* if there exists $k \in [1, m]$ such that C_k involves (i, j) in Π, otherwise we say that (i, j) is $(\Pi, \mathcal{C})$-*irrelevant*. Let $(P, Q)_{\mathbb{Z}} = \mathrm{Rel}(\Pi, \mathcal{C}) \sqcup \mathrm{Irrel}(\Pi, \mathcal{C})$ denote the sets of all $(\Pi, \mathcal{C})$-relevant and $(\Pi, \mathcal{C})$-irrelevant cells, respectively.

Lemma 4.12. *Let Π and Π' be pipe dreams such that Π' is obtained from Π by changing its values in some $(\Pi, \mathcal{C})$-irrelevant cells. Then*

$$\mathrm{Rel}(\Pi, \mathcal{C}) = \mathrm{Rel}(\Pi', \mathcal{C}), \quad \mathrm{Irrel}(\Pi, \mathcal{C}) = \mathrm{Irrel}(\Pi', \mathcal{C}), \quad \mathrm{Ht}_{\Pi}^{P,Q}(\mathcal{C}) = \mathrm{Ht}_{\Pi'}^{P,Q}(\mathcal{C}), \quad and$$

$$\mathrm{wt}_{\Pi}(i, j) = \mathrm{wt}_{\Pi'}(i, j) \quad \text{for all } (i, j) \in \mathrm{Rel}(\Pi, \mathcal{C}).$$
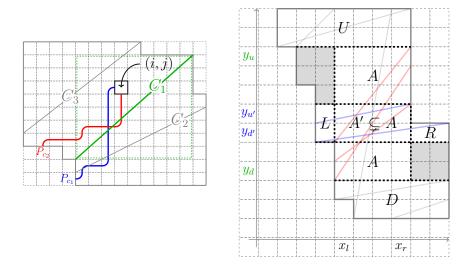
FIGURE 14. Left: The cut C_1 involves (i,j) in Π while C_2 and C_3 do not, see Definition 4.11. Right: The partition of the double $\mathbb{H}$-flip skew domain from Figure 5 as described in the proof of Lemma 4.8. The shaded areas contain no cells from $\operatorname{supp}(\mathcal{C})$ (and therefore no $(\Pi, \mathcal{C})$-relevant cells for any Π).

Proof. Without loss of generality we may assume that Π' is obtained from Π by changing the value in a single cell $(i,j) \in \operatorname{Irrel}(\Pi, \mathcal{C})$ from an elbow to a crossing or vice versa. Let p_1 and p_2 be the paths in Π that pass through the cell (i,j) and let $c_1 < c_2 \in [1,n]$ be their respective colors.

Let $(i',j') \in (P,Q)_{\mathbb{Z}}$ be some cell such that either $(i',j') \in \operatorname{Rel}(\Pi, \mathcal{C}) \cap \operatorname{Irrel}(\Pi', \mathcal{C})$ or $(i',j') \in \operatorname{Irrel}(\Pi, \mathcal{C}) \cap \operatorname{Rel}(\Pi', \mathcal{C})$. Then at least one of the paths p_1, p_2 has to enter the cell (i',j') after leaving (i,j), which implies $i \le i'$ and $j \le j'$.

Moreover, there must exist $k \in [1,m]$ such that $(i',j') \in [l_k, r_k] \times [d_k, u_k]$ and $\mathfrak{i}_k \in [c_1, c_2 - 1]$: the interval $[c_1, c_2 - 1]$ has to contain at least one such $\mathfrak{i}_k$ in order for the status of (i',j') to change between irrelevant and relevant. Because $\mathfrak{i}_k \in [c_1, c_2 - 1]$, the cell (l_k, d_k) has to be weakly below and to the left from (i,j), so $(i,j) \in [l_k, r_k] \times [d_k, u_k]$. It follows that C_k involves (i,j) in Π, so (i,j) is $(\Pi, \mathcal{C})$-relevant, a contradiction. This shows $\operatorname{Irrel}(\Pi, \mathcal{C}) = \operatorname{Irrel}(\Pi', \mathcal{C})$ and $\operatorname{Rel}(\Pi, \mathcal{C}) = \operatorname{Rel}(\Pi', \mathcal{C})$. It is also clear that $\operatorname{Ht}_\Pi^{P,Q}(\mathcal{C}) = \operatorname{Ht}_{\Pi'}^{P,Q}(\mathcal{C})$ because $\operatorname{Ht}_\Pi^{P,Q}(C_k) \ne \operatorname{Ht}_{\Pi'}^{P,Q}(C_k)$ can only happen when C_k involves (i,j) in Π.

Suppose now that $(i',j') \in \operatorname{Rel}(\Pi, \mathcal{C}) = \operatorname{Rel}(\Pi', \mathcal{C})$ is such that $\operatorname{wt}_\Pi(i',j') \ne \operatorname{wt}_{\Pi'}(i',j')$. As before, this implies that $i \le i'$ and $j \le j'$. Let $k \in [1,m]$ be such that C_k involves (i',j') in Π, and let $c_1' < c_2'$ be the colors of the paths entering (i',j') in Π, thus $\mathfrak{i}_k \in [c_1', c_2' - 1]$. In order for $\operatorname{wt}_\Pi(i',j')$ to change as we swap c_1 with c_2, it is necessary to have $\{c_1', c_2'\} \cap \{c_1, c_2\} \ne \emptyset$ and $[c_1', c_2' - 1] \subseteq [c_1, c_2 - 1]$. But then C_k involves (i,j) in Π, contradicting the assumption that $(i,j) \in \operatorname{Irrel}(\Pi, \mathcal{C})$. $\square$

Let us say that pipe dreams Π, Π' are $\mathcal{C}$-*equivalent* if they are obtained from each other by changing the values of $(\Pi, \mathcal{C})$-irrelevant cells. We denote by $[\Pi]$ the $\mathcal{C}$-equivalence class of Π. It consists of $2^{\#\operatorname{Irrel}(\Pi, \mathcal{C})}$ elements which all have the same height vector $\operatorname{Ht}_\Pi^{P,Q}(\mathcal{C})$. By Lemma 4.12, we have

$$(4.5) \qquad \mathbf{P}([\Pi]) := \sum_{\Pi' \in [\Pi]} \mathbf{P}(\Pi') = \prod_{(i,j) \in \operatorname{Rel}(\Pi, \mathcal{C})} \operatorname{wt}_\Pi(i,j),$$

where for $(i,j) \in \mathrm{Rel}(\Pi, \mathcal{C})$, $\mathrm{wt}_\Pi(i,j)$ does not depend on the choice of $\Pi \in [\Pi]$. Note that

$$(4.6) \qquad \mathrm{Rel}(\Pi, \mathcal{C}) \subseteq \mathrm{supp}(\mathcal{C}) := \bigcup_{i=1}^{m} \left([l_i, r_i] \times [d_i, u_i] \right),$$

where $\mathrm{supp}(\mathcal{C})$ does not depend on either Π or (P, Q). Thus, modulo $\mathcal{C}$-equivalence, we may restrict each (P, Q)-pipe dream only to the cells in $\mathrm{supp}(\mathcal{C})$ (or more generally only to the cells in some set A containing $\mathrm{supp}(\mathcal{C})$). One immediate application of this approach is that the distribution of the height vector $\mathrm{Ht}^{P,Q}(\mathcal{C}; \mathbf{x}, \mathbf{y})$ does not depend on the choice of (P, Q).

Corollary 4.13. *Let (P, Q) and (P', Q') be two skew domains, and let $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ be such that for all $i \in [1, m]$, C_i is both a (P, Q)-cut and a (P', Q')-cut. Then*

$$\mathrm{Ht}^{P,Q}(\mathcal{C}; \mathbf{x}, \mathbf{y}) \overset{d}{=} \mathrm{Ht}^{P',Q'}(\mathcal{C}; \mathbf{x}, \mathbf{y}).$$

Proof. The set of relevant cells is contained inside $\mathrm{supp}(\mathcal{C}) \subseteq (P, Q)_\mathbb{Z} \cap (P', Q')_\mathbb{Z}$, so (P, Q) and (P', Q') give rise to the same set of $\mathcal{C}$-equivalence classes of pipe dreams. $\qquad \square$

4.4. **Admissible transformations: proofs.** We are ready to prove Lemmas 4.5–4.8. We start with a tuple $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ of (P, Q)-cuts and then each lemma gives a transformation $\Phi = (\phi_H, \phi_V)$. It is straightforward to check in each case that this transformation is $\mathcal{C}$-admissible, and following Remark 4.4 we have another m-tuple $\mathcal{C}' = (C_1', C_2', \ldots, C_m') := \Phi(\mathcal{C})$ of (P', Q')-cuts for some other skew domain (P', Q'). As usual, for $k \in [1, m]$, we denote $C_k = (l_k, d_k, u_k, r_k)$, $C_k' = (l_k', d_k', u_k', r_k')$, and we let $(\mathfrak{i}_k, \mathfrak{j}_k)$ and $(\mathfrak{i}_k', \mathfrak{j}_k')$ be the color cutoff levels of C_k and C_k'. Recall also that we have $(P, Q)_\mathbb{Z} \subseteq [1, M_\infty] \times [1, N_\infty]$. We let $\mathbf{x}' := \phi_H^{-1}(\mathbf{x})$ and $\mathbf{y}' := \phi_V^{-1}(\mathbf{y})$, cf. Remark 4.3.

Proof of Lemma 4.5. In this case, (P', Q') is just a $180°$-rotation of (P, Q). Let $w_0 \in S_n$ be the permutation of maximal length, defined by $w_0(i) = n + 1 - i$ for all $i \in [1, n]$. We claim that the probability $\mathbf{P}_\pi^{P,Q}$ equals the probability $\mathbf{P}_{\pi'}^{P',Q'}$ for $\pi' := w_0 \pi^{-1} w_0$. This follows from the analog of Theorem 2.4 where instead of the anti-automorphism $T_\pi \mapsto T_{\pi^{-1}}$, one uses the anti-automorphism $T_\pi \mapsto T_{w_0 \pi^{-1} w_0}$. Observe also that $\mathfrak{j}_k' = n - \mathfrak{i}_k$ and $\mathfrak{i}_k' = n - \mathfrak{j}_k$ for all $k \in [1, m]$. We are done since by definition, $\mathrm{Ht}_\pi^{\mathfrak{i}_k, \mathfrak{j}_k} = \mathrm{Ht}_{\pi'}^{n - \mathfrak{j}_k, n - \mathfrak{i}_k}$. $\qquad \square$

For the remaining results, we need to discuss the further relationship between height functions, horizontal/vertical boundary conditions, skew domains, and Hecke algebra elements.

We may assume that the paths P and Q connect the bottom right and the top left vertices of the rectangle $[1, M_\infty] \times [1, N_\infty]$, in which case $n = M_\infty + N_\infty$. For a cell $(i, j) \in (P, Q)_\mathbb{Z}$, let $\mathrm{cont}_{i,j} := M_\infty + j - i \in [1, n-1]$ denote its *content*. For a subset $A \subseteq (P, Q)_\mathbb{Z}$, denote

$$(4.7) \qquad Y_A := \prod_{(i,j) \in A} R_{\mathrm{cont}_{i,j}}(\mathfrak{p}_{i,j}) \in \mathcal{H}_q(S_n; \mathbf{z}),$$

where the (non-commutative) product is taken in the "up-right reading order", so that the term corresponding to (i, j) appears before the term corresponding to (i', j') for all $i' \geq i$ and $j' \geq j$. The element $Y_A = \sum_{\pi \in S_n} \kappa_\pi^A T_\pi$ defines a probability distribution on S_n. For $k \in [1, m]$ and $\pi \in S_n$, we set $\mathrm{Ht}_\pi(C_k) := \mathrm{Ht}_\pi^{\mathfrak{i}_k, \mathfrak{j}_k}$, and this gives rise to a random variable $\mathrm{Ht}^{Y_A}(C_k)$: for $h \in \mathbb{Z}_{\geq 0}$, the probability that $\mathrm{Ht}^{Y_A}(C_k) = h$ equals $\sum_{\pi \in S_n : \mathrm{Ht}_\pi(C_k) = h} \kappa_\pi^A$. We let $\mathrm{Ht}^{Y_A}(\mathcal{C}) = \left(\mathrm{Ht}^{Y_A}(C_1), \ldots, \mathrm{Ht}^{Y_A}(C_m) \right)$. By Lemma 4.12, we have

$$(4.8) \qquad \mathrm{Ht}^{Y_A}(\mathcal{C}) \overset{d}{=} \mathrm{Ht}^{P,Q}(\mathcal{C}; \mathbf{x}, \mathbf{y}) \quad \text{if } \mathrm{supp}(\mathcal{C}) \subseteq A \subseteq (P, Q)_\mathbb{Z}.$$

For the next result, we borrow some notation from Section 3.3. We assume that all indices appearing in the statement belong to the set $[k_{\min}, k_{\max}]$ from Remark 3.3.

Lemma 4.14. *Let* $i, j \in [1, n]$. *Let* $\pi, \pi' \in S_n$ *be permutations satisfying*

$$\mathbb{H}_{\pi'}^{i,j} = 180_{i,j}^{\circ} \left(\mathbb{H}_{\pi}^{i,j} \right) \quad and \quad \mathbb{V}_{\pi'}^{i,j} = \mathbb{V}_{\pi}^{i,j}.$$

Then

(4.9) $\mathrm{Ht}_{\pi}^{i_0, j_1} = \mathrm{Ht}_{\pi'}^{i_0, j_1}$ *for all* $i_0 \leq i$ *and* $j_1 \geq j$;

(4.10) $\mathrm{Ht}_{\pi}^{i_1, j_0} = \mathrm{Ht}_{\pi'}^{i+j-j_0, i+j-i_1}$ *for all* $i_1 \geq i$ *and* $j_0 \leq j$.

Proof. The values $\mathrm{Ht}_{\pi}^{i_0, j_1}$ and $\mathrm{Ht}_{\pi}^{i_1, j_0}$ can be expressed in terms of $\mathbb{H}_{\pi}^{i,j}$ and $\mathbb{V}_{\pi}^{i,j}$:

$$\mathrm{Ht}_{\pi}^{i_0, j_1} = \#\{(d, u) \in \mathbb{V}_{\pi}^{i,j} \mid d \leq i_0 \text{ and } u > j_1\} + j_1 - i_0,$$
$$\mathrm{Ht}_{\pi}^{i_1, j_0} = \#\{(l, r) \in \mathbb{H}_{\pi}^{i,j} \mid l > i_1 \text{ and } r \leq j_0\}.$$

The first equation immediately implies (4.9), and comparing the second equation with the definition (3.10) of $180_{i,j}^{\circ}$, we see that (4.10) also follows. $\square$

Remark 4.15. Given two (P, Q)-cuts C, C' with color cutoff levels (i, j) and (i', j'), note that C and C' cross (in the sense of Definition 4.1) if and only if either $i' \geq i$ and $j' \leq j$ or $i' \leq i$ and $j' \geq j$, which is precisely when Equations (4.9) and (4.10) apply.

Proof of Lemma 4.6. Let $C = (l, d, u, r)$ be as in Lemma 4.6, and let i, j be its color cutoff levels. Let $L, D, A, U, R \subseteq (P, Q)_{\mathbb{Z}}$ be the intersections of $(P, Q)_{\mathbb{Z}}$ with $[1, l-1] \times [d, u]$, $[l, r] \times [1, d-1]$, $[l, r] \times [d, u]$, $[l, r] \times [u+1, N_{\infty}]$, and $[r+1, M_{\infty}] \times [d, u]$, respectively. The assumptions of Lemma 4.6 imply that $\mathrm{supp}(\mathcal{C}) \subseteq L \cup D \cup A \cup U \cup R$.

After possibly changing M_{∞} and shifting $(P, Q)_{\mathbb{Z}}$ inside $[1, M_{\infty}] \times [1, N_{\infty}]$, we may assume that:

- $l + r = 1 + M_{\infty}$,
- $(P', Q')_{\mathbb{Z}} \cap ([l, r] \times \mathbb{Z}_{\geq 1}) = (P, Q)_{\mathbb{Z}} \cap ([l, r] \times \mathbb{Z}_{\geq 1})$, and
- the sets $L' := (P', Q')_{\mathbb{Z}} \cap ([1, l-1] \times [d, u])$ and $R' := (P', Q')_{\mathbb{Z}} \cap ([r+1, M_{\infty}] \times [d, u])$ are the images of, respectively, R and L under the map $(i, j) \mapsto (l+r-i, u+d-j)$.

We find that

$$(Y_L)_{i,j}^* = Y_{R'} \mid_{(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')} \quad \text{and} \quad (Y_R)_{i,j}^* = Y_{L'} \mid_{(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')} .$$

Denote $\overline{Y}_A := Y_A \mid_{(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')}$ and let

$$Y := Y_L \cdot Y_D \cdot Y_A \cdot Y_U \cdot Y_R, \quad \text{and} \quad \overline{Y} := (Y_R)_{i,j}^* \cdot Y_D \cdot \overline{Y}_A \cdot Y_U \cdot (Y_L)_{i,j}^*.$$

By Theorem 3.1, for all $\mathbb{H}, \mathbb{V}$, we have $(Y)_{i,j}^{\mathbb{H}, \mathbb{V}} = (\overline{Y})_{i,j}^{180_{i,j}^{\circ}(\mathbb{H}), \mathbb{V}}$. By (4.8), we get $\mathrm{Ht}^Y(\mathcal{C}) \overset{d}{=} \mathrm{Ht}^{P,Q}(\mathcal{C}; \mathbf{x}, \mathbf{y})$ and $\mathrm{Ht}^{\overline{Y}}(\mathcal{C}') \overset{d}{=} \mathrm{Ht}^{P',Q'}(\mathcal{C}'; \mathbf{x}', \mathbf{y}')$. Finally, we have assumed that the cut C crosses all elements of $\mathcal{C}$, and therefore all elements of $\mathcal{C}'$. Applying Lemma 4.14, we find $\mathrm{Ht}^Y(\mathcal{C}) \overset{d}{=} \mathrm{Ht}^{\overline{Y}}(\mathcal{C}')$. $\square$

Proof of Lemma 4.7. Let l, d, u, r, J, K be as in Lemma 4.7. Let $D, A, U \subseteq (P, Q)_{\mathbb{Z}}$ be the intersections of $(P, Q)_{\mathbb{Z}}$ with $\mathbb{Z}_{\geq 1} \times [1, d-1]$, $\mathbb{Z}_{\geq 1} \times [d, u]$, and $\mathbb{Z}_{\geq 1} \times [u+1, N_{\infty}]$, respectively. We may assume that $A = [l, r] \times [d, u]$ is a rectangle, in which case $C := (l, d, u, r)$ is a (P, Q)-cut and we denote by i, j its color cutoff levels.

Given a (P, Q)-pipe dream Π, let us denote by

$$(4.11) \qquad \mathbb{V}_\Pi^A := \{(d_1^{A,\Pi}, u_1^{A,\Pi}), \ldots, (d_v^{A,\Pi}, u_v^{A,\Pi})\}$$

the set of all pairs $d_i^{A,\Pi}, u_i^{A,\Pi} \in [l, r]$ such that the path in Π that enters A from below in column $d_i^{A,\Pi}$ exits A from above in column $u_i^{A,\Pi}$.

Let Π and Π' be two (P, Q)-pipe dreams. We say that Π is $(\mathcal{C}, C)$-*equivalent* to Π' if their restrictions to $D \cup (\mathrm{Rel}(\Pi, \mathcal{C}) \cap U)$ coincide and in addition $\mathbb{V}_\Pi^A = \mathbb{V}_{\Pi'}^A$. Our first goal is to show that $\mathrm{Rel}(\Pi, \mathcal{C}) \cap U = \mathrm{Rel}(\Pi', \mathcal{C}) \cap U$, and thus Π' is $(\mathcal{C}, C)$-equivalent to Π. Along the way, we will also see that for each cell $(i, j) \in \mathrm{Rel}(\Pi, \mathcal{C}) \cap U$, we have $\mathrm{wt}_\Pi(i, j) = \mathrm{wt}_{\Pi'}(i, j)$.

For a (P, Q)-pipe dream Π, $k \in [1, m]$, and $(i, j) \in [l_k, r_k] \times [d_k, u_k]$, let $\mathcal{D}_{i,j}^{C_k}(\Pi) \in \{1, 0\}$ (resp., $\mathcal{L}_{i,j}^{C_k}(\Pi) \in \{1, 0\}$) be equal to 1 if the path in Π entering (i, j) from below (resp., from the left) enters the rectangle $[l_k, r_k] \times [d_k, u_k]$ from the left, and to 0 if it enters the rectangle $[l_k, r_k] \times [d_k, u_k]$ from below. Thus C_k involves (i, j) in Π iff $\mathcal{D}_{i,j}^{C_k}(\Pi) \neq \mathcal{L}_{i,j}^{C_k}(\Pi)$.

Suppose that Π is $(\mathcal{C}, C)$-equivalent to Π'. Our goal is to show that in this case, for all $k \in [1, m]$ and all $(i, j) \in U \cap ([l_k, r_k] \times [d_k, u_k])$, we have $\mathcal{D}_{i,j}^{C_k}(\Pi) = \mathcal{D}_{i,j}^{C_k}(\Pi')$ and $\mathcal{L}_{i,j}^{C_k}(\Pi) = \mathcal{L}_{i,j}^{C_k}(\Pi')$. Otherwise, choose $(i, j) \in U \cap ([l_k, r_k] \times [d_k, u_k])$ with the minimal possible value of $i + j$ such that either $\mathcal{D}_{i,j}^{C_k}(\Pi) \neq \mathcal{D}_{i,j}^{C_k}(\Pi')$ or $\mathcal{L}_{i,j}^{C_k}(\Pi) \neq \mathcal{L}_{i,j}^{C_k}(\Pi')$. Suppose first that $\mathcal{D}_{i,j}^{C_k}(\Pi) \neq \mathcal{D}_{i,j}^{C_k}(\Pi')$. If $(i, j - 1) \in U \cap ([l_k, r_k] \times [d_k, u_k])$ then the restrictions of Π and Π' to $(i, j - 1)$ must be different, so $(i, j - 1) \notin \mathrm{Rel}(\Pi, \mathcal{C})$. Therefore C_k does not involve $(i, j - 1)$ in Π, so $\mathcal{D}_{i,j-1}^{C_k}(\Pi) = \mathcal{L}_{i,j-1}^{C_k}(\Pi) = \mathcal{D}_{i,j}^{C_k}(\Pi)$. By the minimality of $i + j$, we must have $\mathcal{D}_{i,j-1}^{C_k}(\Pi') = \mathcal{D}_{i,j-1}^{C_k}(\Pi)$ and $\mathcal{L}_{i,j-1}^{C_k}(\Pi') = \mathcal{L}_{i,j-1}^{C_k}(\Pi)$, which implies $\mathcal{D}_{i,j}^{C_k}(\Pi) = \mathcal{D}_{i,j}^{C_k}(\Pi')$, a contradiction. Thus we must have $(i, j - 1) \notin U \cap ([l_k, r_k] \times [d_k, u_k])$. If $(i, j - 1) \notin [l_k, r_k] \times [d_k, u_k]$ then by definition we have $\mathcal{D}_{i,j}^{C_k}(\Pi) = \mathcal{D}_{i,j}^{C_k}(\Pi') = 0$. The only other option is that $(i, j-1) \in ([l_k, r_k] \times [d_k, u_k]) \setminus U$, which implies that $(i, j-1) \in A$ and $k \in K$. We know that $\mathbb{V}_\Pi^A = \mathbb{V}_{\Pi'}^A$ and that the restrictions of Π and Π' to D coincide, so $\mathcal{D}_{i,j}^{C_k}(\Pi) = \mathcal{D}_{i,j}^{C_k}(\Pi')$. Suppose now that $\mathcal{L}_{i,j}^{C_k}(\Pi) \neq \mathcal{L}_{i,j}^{C_k}(\Pi')$. An argument similar to the one above shows that we cannot have $(i - 1, j) \in U \cap ([l_k, r_k] \times [d_k, u_k])$. But then $(i - 1, j) \notin [l_k, r_k] \times [d_k, u_k]$, so $\mathcal{L}_{i,j}^{C_k}(\Pi) = \mathcal{L}_{i,j}^{C_k}(\Pi') = 1$.

We have shown that $\mathcal{D}_{i,j}^{C_k}(\Pi) = \mathcal{D}_{i,j}^{C_k}(\Pi')$ and $\mathcal{L}_{i,j}^{C_k}(\Pi) = \mathcal{L}_{i,j}^{C_k}(\Pi')$ for all $(i, j) \in U \cap ([l_k, r_k] \times [d_k, u_k])$. This implies both of our desired statements: that $\mathrm{Rel}(\Pi, \mathcal{C}) \cap U = \mathrm{Rel}(\Pi', \mathcal{C}) \cap U$ and that $\mathrm{wt}_\Pi(i, j) = \mathrm{wt}_{\Pi'}(i, j)$ for all $(i, j) \in \mathrm{Rel}(\Pi, \mathcal{C}) \cap U$.

Fix some (P, Q)-pipe dream Π_0 and denote by $[\Pi_0]_{\mathcal{C},C}$ the $(\mathcal{C}, C)$-equivalence class of Π_0. Denote by $\sigma_D \in S_{[1,\mathrm{i}]}$ the color permutation induced by the restriction of Π_0 to D on the top right boundary of D. Applying Theorem 3.1 (cf. Remark 3.4) to the element $T_{\sigma_D} \cdot Y_A$, we find that for any set $\mathbb{H}$ of pairs, we have

$$\sum_{\Pi \in [\Pi_0]_{\mathcal{C},C}:\ \mathbb{H}_{\pi_\Pi}^{\mathrm{i},\mathrm{j}} = \mathbb{H}} \mathbf{P}(\Pi) = \sum_{\Pi \in [\Pi_0]_{\mathcal{C},C}:\ \mathbb{H}_{\pi_\Pi}^{\mathrm{i},\mathrm{j}} = 180_{\mathrm{i},\mathrm{j}}^\circ(\mathbb{H})} \mathbf{P}(\Pi)\Big|_{(\mathbf{x},\mathbf{y}) \mapsto (\mathbf{x}',\mathbf{y}')}.$$

Summing over all possible classes $[\Pi_0]_{\mathcal{C},C}$ and applying Lemma 4.14, the result follows. $\square$

Proof of Lemma 4.8. We proceed similarly to the proof of Lemma 4.7, except that now we want to apply two $\mathbb{H}$-flips. Let $L, D, A, A', U, R \subseteq (P, Q)_\mathbb{Z}$ be the intersections of $(P, Q)_\mathbb{Z}$ with $[1, l - 1] \times [d', u']$, $[l, M_\infty] \times [1, d - 1]$, $[l, r] \times [d, u]$, $[l, r] \times [d', u']$, $[1, r] \times [u + 1, N_\infty]$, and $[r + 1, M_\infty] \times [d', u']$ respectively, see Figure 14(right). Note that the shaded areas $[1, l - 1] \times [u' + 1, u]$ and $[r + 1, M_\infty] \times [d, d' - 1]$ contain no cells in $\mathrm{supp}(\mathcal{C})$, so $\mathrm{supp}(\mathcal{C}) \subseteq$

$L \cup D \cup A \cup U \cup R$. We may assume that $A' \subsetneq A$ are rectangles: $A' = [l, r] \times [d', u']$ and $A = [l, r] \times [d, u]$.

Introduce (P, Q)-cuts $C := (l, d, u, r)$ and $C' := (l, d', u', r)$ with color cutoff levels i, j and i', j', respectively. As above, fix a (P, Q)-pipe dream Π_0. The domain (P, Q) changes into (P', Q'), however, the subdomains D, A, U stay invariant. We denote by $[\Pi_0]_{\mathcal{C}, C}^{P, Q}$ the set of (P, Q)-pipe dreams that are $(\mathcal{C}, C)$-equivalent to Π_0. We also denote by $[\Pi_0]_{\mathcal{C}, C}^{P', Q'}$ the set of all (P', Q')-pipe dreams Π' such that $\mathbb{V}_{\Pi_0}^A = \mathbb{V}_{\Pi'}^A$ and the restrictions of Π_0 and Π' to $D \cup (\mathrm{Rel}(\Pi_0, \mathcal{C}) \cap U)$ coincide. Recall that our probability space consists of $2^{\#(P,Q)_{\mathbb{Z}}}$ (P, Q)-pipe dreams. The class $[\Pi_0]_{\mathcal{C}, C}^{P, Q}$ is considered an *event*, and for $i \in [1, m]$, we let $\mathrm{Ht}(C_i; \mathbf{x}, \mathbf{y})|_{[\Pi_0]_{\mathcal{C}, C}^{P, Q}}$ denote the random variable $\mathrm{Ht}^{P, Q}(C_i; \mathbf{x}, \mathbf{y})$ conditioned on this event. Let

$$\mathrm{Ht}(\mathcal{C}; \mathbf{x}, \mathbf{y})|_{[\Pi_0]_{\mathcal{C}, C}^{P, Q}} := \left(\mathrm{Ht}(C_i; \mathbf{x}, \mathbf{y})|_{[\Pi_0]_{\mathcal{C}, C}^{P, Q}} \right)_{i \in [1, m]}.$$

Let $\sigma_D \in S_{[1, \mathsf{i}]}$ be the color permutation induced by the restriction of Π_0 to D on the top right boundary of D, and denote $Y := T_{\sigma_D} \cdot Y_L \cdot Y_A \cdot Y_R$. Consider subsets $L', R' \subseteq (P', Q')_{\mathbb{Z}}$ obtained respectively from L and R via the map sending $(i, j) \mapsto (i, j + u + d - u' - d')$. We may assume that L' and R' are the intersections of $(P', Q')_{\mathbb{Z}}$ with $[1, l-1] \times [d'', u'']$ and $[r+1, M_\infty] \times [d'', u'']$, where $u'' := u + d - d'$, $d'' := u + d - u'$. Thus $\mathrm{supp}(\mathcal{C}') \subseteq L' \cup D \cup A \cup U \cup R'$. Let $Y' := (T_{\sigma_D} \cdot Y_{L'} \cdot Y_A \cdot Y_{R'})|_{(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')}$. We claim that Y' is obtained from Y via two applications of Theorem 3.1, first at $(\mathsf{i}', \mathsf{j}')$ and then at (i, j). To see that, let us write $A = A_D \sqcup A' \sqcup A_U$, where A_D and A_U are the intersections of A with $[l, r] \times [d, d'-1]$ and $[l, r] \times [u'+1, u]$, respectively. Thus $Y_A = Y_{A_D} \cdot Y_{A'} \cdot Y_{A_U}$ and T_{σ_D}, Y_{A_D} commute with Y_L while Y_{A_U} commutes with Y_R. Applying Theorem 3.1 at $(\mathsf{i}', \mathsf{j}')$ sends

$$(4.12) \qquad Y_L \cdot (T_{\sigma_D} Y_{A_D}) \cdot Y_{A'} \cdot Y_{A_U} \cdot Y_R \mapsto (Y_R)_{\mathsf{i}', \mathsf{j}'}^* \cdot (T_{\sigma_D} Y_{A_D}) \cdot \overline{Y_{A'}} \cdot Y_{A_U} \cdot (Y_L)_{\mathsf{i}', \mathsf{j}'}^*.$$

(In (4.12) and (4.13), the five terms on each side, separated by the $\cdot$ symbol, appear in the same order as the corresponding five terms on each side of (3.11).) We may now apply Theorem 3.1 at (i, j), sending

$$(4.13) \quad (Y_R)_{\mathsf{i}', \mathsf{j}'}^* \cdot T_{\sigma_D} \cdot \left(Y_{A_D} \overline{Y_{A'}} Y_{A_U} \right) \cdot 1 \cdot (Y_L)_{\mathsf{i}', \mathsf{j}'}^* \mapsto ((Y_L)_{\mathsf{i}', \mathsf{j}'}^*)_{\mathsf{i}, \mathsf{j}}^* \cdot T_{\sigma_D} \cdot \left(\overline{Y_{A_D} \overline{Y_{A'}} Y_{A_U}} \right) \cdot 1 \cdot ((Y_R)_{\mathsf{i}', \mathsf{j}'}^*)_{\mathsf{i}, \mathsf{j}}^*.$$

The terms $((Y_L)_{\mathsf{i}', \mathsf{j}'}^*)_{\mathsf{i}, \mathsf{j}}^*$, $\overline{Y_{A_D} \overline{Y_{A'}} Y_{A_U}}$, and $((Y_R)_{\mathsf{i}', \mathsf{j}'}^*)_{\mathsf{i}, \mathsf{j}}^*$ are obtained respectively from $Y_{L'}$, Y_A, and $Y_{R'}$ by substituting $(\mathbf{x}, \mathbf{y}) \mapsto (\mathbf{x}', \mathbf{y}')$. Each of the cuts C' and C (at which we applied Theorem 3.1) crosses C_k all $k \in I \cup J \cup K$. By Lemma 4.14, we get

$$(4.14) \qquad\qquad \mathrm{Ht}(\mathcal{C}; \mathbf{x}, \mathbf{y})|_{[\Pi_0]_{\mathcal{C}, C}^{P, Q}} \stackrel{d}{=} \mathrm{Ht}(\mathcal{C}'; \mathbf{x}', \mathbf{y}')|_{[\Pi_0]_{\mathcal{C}, C}^{P', Q'}}.$$

Summing over all possible classes $[\Pi_0]_{\mathcal{C}, C}^{P, Q}$, the result follows. $\square$

5. Proof of Theorem 1.6

Recall from Remark 4.4 that our goal is to show that each $\mathcal{C}$-admissible transformation is strongly $\mathcal{C}$-admissible. We showed above that several $\mathcal{C}$-admissible transformations are strongly $\mathcal{C}$-admissible. The purpose of this section is to show that, after passing to *connected components* defined below, any $\mathcal{C}$-admissible transformation can be represented as a composition of transformations introduced in Section 4.2.

Throughout, we fix the following data:

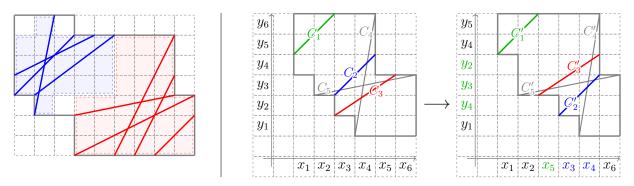- two skew domains (P, Q) and (P', Q');

FIGURE 15. Left: a tuple $\mathcal{C}$ of cuts with disconnected overlap graph $G^{\mathrm{ov}}(\mathcal{C})$. The sets $\mathrm{supp}(\mathcal{C}_I)$ and $\mathrm{supp}(\mathcal{C}_J)$ are shaded. Right: a $\mathcal{C}$-admissible transformation that gives a counterexample to Lemma 5.9 when the assumption that $G^{\mathrm{ov}}(\mathcal{C})$ is connected is not satisfied.

- a tuple $\mathcal{C} := (C_1, C_2, \ldots, C_m)$ of (P, Q)-cuts with $C_i := (l_i, d_i, u_i, r_i)$ for $i \in [1, m]$;
- a tuple $\mathcal{C}' := (C_1', C_2', \ldots, C_m')$ of (P', Q')-cuts with $C_i' := (l_i', d_i', u_i', r_i')$ for $i \in [1, m]$;
- a $\mathcal{C}$-admissible transformation $\Phi := (\phi_H, \phi_V)$ satisfying $\mathcal{C}' = \Phi(\mathcal{C})$ (cf. Remark 4.4).

We assume that $C_i \neq C_j$ for $i \neq j$.

5.1. **Connected components.** Clearly, if C_i, C_j satisfy $[l_i, r_i] \cap [l_j, r_j] = [d_i, u_i] \cap [d_j, u_j] = \emptyset$ then the random variables $\mathrm{Ht}^{P,Q}(C_i; \mathbf{x}, \mathbf{y})$ and $\mathrm{Ht}^{P,Q}(C_j; \mathbf{x}, \mathbf{y})$ are independent. One can check that this remains true when only one of the two intersections is empty. A generalization of this to joint distributions is given in Proposition 5.3 below.

Definition 5.1. We say that two cuts C_i and C_j *have overlapping rectangles* if $[l_i, r_i] \cap [l_j, r_j] \neq \emptyset$ and $[d_i, u_i] \cap [d_j, u_j] \neq \emptyset$. We denote by $G^{\mathrm{ov}}(\mathcal{C})$ the *overlap graph* of $\mathcal{C}$: the vertex set of $G^{\mathrm{ov}}(\mathcal{C})$ is $[1, m]$, and i and j are connected by an edge if and only if C_i and C_j have overlapping rectangles.

Remark 5.2. Since Φ is $\mathcal{C}$-admissible, the graphs $G^{\mathrm{ov}}(\mathcal{C})$ and $G^{\mathrm{ov}}(\mathcal{C}')$ coincide.

Proposition 5.3. *Suppose that the set $[1, m]$ is partitioned into two nonempty subsets $[1, m] = I \sqcup J$ such that $G^{\mathrm{ov}}(\mathcal{C})$ contains no edges connecting a vertex in I to a vertex in J. Then the height vectors $\left(\mathrm{Ht}^{P,Q}(C_i; \mathbf{x}, \mathbf{y})\right)_{i \in I}$ and $\left(\mathrm{Ht}^{P,Q}(C_j; \mathbf{x}, \mathbf{y})\right)_{j \in J}$ are independent (as random variables).*

See Figure 15(left) for an example.

Proof. Denote $\mathcal{C}_I := (C_i)_{i \in I}$ and $\mathcal{C}_J := (C_j)_{j \in J}$, and let $\mathrm{supp}(\mathcal{C}_I), \mathrm{supp}(\mathcal{C}_J) \subseteq (P, Q)_{\mathbb{Z}}$ be the corresponding supports defined in (4.6). Consider a pipe dream Π and a cell $(a, b) \in \mathrm{Rel}(\Pi, \mathcal{C})$. By (4.6), we must have either $(a, b) \in \mathrm{Rel}(\Pi, \mathcal{C}_I) \subseteq \mathrm{supp}(\mathcal{C}_I)$ or $(a, b) \in \mathrm{Rel}(\Pi, \mathcal{C}_J) \subseteq \mathrm{supp}(\mathcal{C}_J)$. Suppose that, say, $(a, b) \in \mathrm{Rel}(\Pi, \mathcal{C}_I)$ and let $i \in I$ be such that C_i involves (a, b) in Π. Let p_1, p_2 be the two paths of colors $c_1 < c_2$ entering (a, b) in Π. Then p_1 must enter the rectangle $[l_i, r_i] \times [d_i, u_i]$ from the bottom while p_2 must enter the same rectangle from the left. Thus changing the values of Π inside the cells of $\mathrm{supp}(\mathcal{C}_J)$ preserves $\mathrm{Rel}(\Pi, \mathcal{C}_I)$, $\mathrm{wt}_\Pi(a, b)$ for $(a, b) \in \mathrm{Rel}(\Pi, \mathcal{C}_I)$, and $\mathrm{Ht}_\Pi^{P,Q}(\mathcal{C}_I)$. Similarly, changing the values of Π inside the cells of $\mathrm{supp}(\mathcal{C}_I)$ preserves $\mathrm{Rel}(\Pi, \mathcal{C}_J)$, $\mathrm{wt}_\Pi(a, b)$ for $(a, b) \in \mathrm{Rel}(\Pi, \mathcal{C}_J)$, and $\mathrm{Ht}_\Pi^{P,Q}(\mathcal{C}_J)$. We are done by (4.5). $\square$

In view of Proposition 5.3, **we assume from now on that the graphs $G^{\mathrm{ov}}(\mathcal{C}) = G^{\mathrm{ov}}(\mathcal{C}')$ are connected**.

5.2. **Non-crossing cut poset.**

Definition 5.4. We write $C_i \leq C_j$ if C_i and C_j do not cross (cf. Definition 4.1) and in addition C_j is *weakly up-left* from C_i, that is, $l_i \geq l_j$, $r_i \geq r_j$, $d_i \leq d_j$, and $u_i \leq u_j$. We write $C_i < C_j$ if $C_i \leq C_j$ and $C_i \neq C_j$. We denote by $\mathcal{P}_\mathcal{C} = (\mathcal{C}, \leq)$ the associated poset (partially ordered set).

As follows from our examples in Section 4.2, the posets $\mathcal{P}_\mathcal{C}$ and $\mathcal{P}_{\mathcal{C}'}$ in general need not coincide. For example, the color-position symmetry (Lemma 4.5) reverses all relations in $\mathcal{P}_\mathcal{C}$. However, for certain *pairs of relations* in $\mathcal{P}_\mathcal{C}$, we can show that their "relative orientation" is preserved.

Definition 5.5. We say that the relations $C_{i_1} < C_{j_1}$ and $C_{i_2} < C_{j_2}$ in $\mathcal{P}_\mathcal{C}$ are Φ-*linked* if either $C'_{i_1} < C'_{j_1}$ and $C'_{i_2} < C'_{j_2}$ or $C'_{i_1} > C'_{j_1}$ and $C'_{i_2} > C'_{j_2}$.

Thus when Φ is the color-position symmetry of Lemma 4.5, any two relations $C_{i_1} < C_{j_1}$ and $C_{i_2} < C_{j_2}$ are Φ-linked. The following simple observation gives rise to a family of pairs of relations that are Φ-linked regardless of the choice of Φ.

Lemma 5.6. *Suppose that $i, j, k \in [1, m]$ are such that $C_i < C_j$ and $C_i < C_k$. Suppose in addition that C_j and C_k have overlapping rectangles. Then the relations $C_i < C_j$ and $C_i < C_k$ are Φ-linked.*

Remark 5.7. Recall that if C_j crosses C_k then they have overlapping rectangles, so the above lemma applies in this important special case.

Proof. Suppose first that C_j crosses C_k. Then C'_j and C'_k cross but none of them crosses C'_i, which immediately implies the result. Suppose now that C_j does not cross C_k. Since they have overlapping rectangles, the (P, Q)-cuts $C_{j,k} := (l_j, d_j, u_k, r_k)$ and $C_{k,j} := (l_k, d_k, u_j, r_j)$ cross and satisfy $C_i < C_{j,k}$ and $C_i < C_{k,j}$. The overlapping rectangles condition also implies that both the intersection and the union of $[l_j, r_j]$ and $[l_k, r_k]$ is a single interval that is sent by ϕ_H to some other interval. This shows that Φ sends $C_{j,k}, C_{k,j}$ to (P', Q')-cuts $C'_{j,k} := (l'_j, d'_j, u'_k, r'_k)$ and $C'_{k,j} := (l'_k, d'_k, u'_j, r'_j)$ that cross each other but do not cross C'_i. Since each of them crosses both C'_j and C'_k, the result follows. $\square$

Our next goal is to describe more Φ-linked pairs of relations. We will repeatedly make use of the *transitivity* property of Φ-linked relations: if $C_{i_1} < C_{j_1}$ and $C_{i_2} < C_{j_2}$ are Φ-linked and $C_{i_2} < C_{j_2}$ and $C_{i_3} < C_{j_3}$ are Φ-linked then $C_{i_1} < C_{j_1}$ and $C_{i_3} < C_{j_3}$ are Φ-linked.

5.3. **Rigid relations and indecomposable components.**
As it is apparent from local and double $\mathbb{H}$- and $\mathbb{V}$-flips (Lemmas 4.7 and 4.8), one needs to take special care of relations between (P, Q)-cuts whose horizontal or vertical projections coincide.

Definition 5.8. We say that a relation $C_i < C_j$ in $\mathcal{P}_\mathcal{C}$ is *rigid* if $[l_i, r_i] \neq [l_j, r_j]$ and $[d_i, u_i] \neq [d_j, u_j]$. In this case, we write $C_i \overset{\text{\textregistered}}{<} C_j$.

One immediate property of rigid relations is that having $C_i \leq C_j \overset{\text{\textregistered}}{<} C_k$ or $C_i \overset{\text{\textregistered}}{<} C_j \leq C_k$ implies $C_i \overset{\text{\textregistered}}{<} C_k$. The following lemma is a special case of Proposition 5.12 below.

Lemma 5.9. *Suppose that $i, j, k \in [1, m]$ satisfy $C_i \leq C_j \overset{®}{<} C_k$. Then the rigid relations $C_i \overset{®}{<} C_k$ and $C_j \overset{®}{<} C_k$ are Φ-linked.*

Proof. We will use the following observation:

(5.1)   if C'_s, C'_t do not cross and $a \in [l'_s, r'_s] \setminus [l'_t, r'_t]$, then $C'_s < C'_t$ (resp., $C'_s > C'_t$) if and only if a is larger (resp., smaller) than all elements of $[l'_t, r'_t]$.

Our assumptions on i, j, k imply that C'_k does not cross C'_i and C'_j.

Let us assume first that $[l_i, r_i] \cap [l_j, r_j] \neq \emptyset$, which in view of $C_i \leq C_j$ is equivalent to $l_i \leq r_j$. We consider three cases: $[l_i, r_i] \cap [l_j, r_j] \nsubseteq [l_k, r_k]$, $[l_k, r_k] \nsubseteq [l_i, r_i] \cup [l_j, r_j]$ and $[l_i, r_i] \cap [l_j, r_j] \subseteq [l_k, r_k] \subseteq [l_i, r_i] \cup [l_j, r_j]$. In the first case, let $a \in [l_i, r_i] \cap [l_j, r_j]$ be such that $a \notin [l_k, r_k]$. Then $\phi_H(a)$ is either strictly to the left or strictly to the right of $\phi_H([l_k, r_k]) = [l'_k, r'_k]$, therefore by (5.1), we have either $C'_i, C'_j > C'_k$ or $C'_i, C'_j < C'_k$, respectively. Thus the relations $C_i \overset{®}{<} C_k$ and $C_j \overset{®}{<} C_k$ are Φ-linked. Similarly, in the second case, let $b \notin [l_i, r_i] \cup [l_j, r_j]$ be such that $b \in [l_k, r_k]$. Because the intersection $[l_i, r_i] \cap [l_j, r_j]$ is nonempty, the image of the union $[l_i, r_i] \cup [l_j, r_j]$ under ϕ_H is a single interval $[l', r'] = [l'_i, r'_i] \cup [l'_j, r'_j]$. Then $\phi_H(b) \notin [l', r']$ is either to the left or to the right of this interval, so we again find that the relations $C_i \overset{®}{<} C_k$ and $C_j \overset{®}{<} C_k$ are Φ-linked. In the third case, we have

$$([l_i, r_i] \cap [l_j, r_j]) \subseteq [l_k, r_k] \subseteq ([l_i, r_i] \cup [l_j, r_j]) \quad \Longleftrightarrow \quad [l_i, r_j] \subseteq [l_k, r_k] \subseteq [l_j, r_i].$$

This yields $l_j \leq l_k \leq l_i \leq r_j \leq r_k \leq r_i$. On the other hand, $C_j < C_k$ implies $l_k \leq l_j$ and $r_k \leq r_j$, so we get $l_j = l_k$ and $r_j = r_k$. This leads to a contradiction since $C_j \overset{®}{<} C_k$ requires $[l_j, r_j] \neq [l_k, r_k]$. We have completed the proof in the case $[l_i, r_i] \cap [l_j, r_j] \neq \emptyset$.

A similar argument finishes the proof in the case $[d_i, u_i] \cap [d_j, u_j] \neq \emptyset$. Suppose now that $[l_i, r_i] \cap [l_j, r_j] = [d_i, u_i] \cap [d_j, u_j] = \emptyset$. In this case, we finish the proof by induction on the graph distance $\mathrm{dist}(i, j)$ between i and j in $G^{\mathrm{ov}}(\mathcal{C})$, which we have assumed to be connected. The conditions $[l_i, r_i] \cap [l_j, r_j] = [d_i, u_i] \cap [d_j, u_j] = \emptyset$ imply that $\mathrm{dist}(i, j) \geq 2$. Consider the shortest path from i to j in $G^{\mathrm{ov}}(\mathcal{C})$, and let $i' \neq i, j$ be the first vertex after i on this path. By definition, C_i and $C_{i'}$ have overlapping rectangles. On the other hand, it is easy to see that $C_{i'}$ does not cross C_j because of the condition $[l_i, r_i] \cap [l_j, r_j] = [d_i, u_i] \cap [d_j, u_j] = \emptyset$. Thus $C_{i'} < C_j \overset{®}{<} C_k$ and $\mathrm{dist}(i', j) < \mathrm{dist}(i, j)$, so we apply the induction hypothesis to conclude that the relations $C_{i'} \overset{®}{<} C_k$ and $C_j \overset{®}{<} C_k$ are Φ-linked. Since C_i and $C_{i'}$ have overlapping rectangles, Lemma 5.6 shows that the relations $C_{i'} \overset{®}{<} C_k$ and $C_i \overset{®}{<} C_k$ are Φ-linked, so by transitivity, $C_i \overset{®}{<} C_k$ and $C_j \overset{®}{<} C_k$ are Φ-linked. $\square$

Example 5.10. Lemma 5.9 does not hold without our running assumption that the graph $G^{\mathrm{ov}}(\mathcal{C})$ is connected, see Figure 15(right).

The following notion is motivated by global $\mathbb{H}$- and $\mathbb{V}$-flips described in Lemma 4.6.

Definition 5.11. We say that $\mathcal{C}$ is *indecomposable* if there does not exist a partition $[1, m] = I \sqcup J$ into nonempty subsets such that C_i crosses C_j for all $i \in I$ and $j \in J$.

Proposition 5.12. *Suppose that $\mathcal{C}$ is indecomposable. Then any two rigid relations $C_{i_1} \overset{®}{<} C_{j_1}$ and $C_{i_2} \overset{®}{<} C_{j_2}$ are Φ-linked.*

Proof. Assume that $C_{i_1} \leq C_{i_2}$. By Lemma 5.9, the relations $C_{i_1} \overset{\text{ⓡ}}{<} C_{j_2}$ and $C_{i_2} \overset{\text{ⓡ}}{<} C_{j_2}$ are Φ-linked. If C_{j_1} and C_{j_2} cross then by Remark 5.7, the relations $C_{i_1} < C_{j_2}$ and $C_{i_1} < C_{j_1}$ are Φ-linked, and by transitivity this implies the result. If C_{j_1} and C_{j_2} do not cross then since the relations $C_{i_1} \overset{\text{ⓡ}}{<} C_{j_2}$ and $C_{i_1} \overset{\text{ⓡ}}{<} C_{j_1}$ are both rigid, Lemma 5.9 shows that they are Φ-linked, so we are done by transitivity. We have shown the result in the case $C_{i_1} \leq C_{i_2}$.

The cases $C_{i_2} \leq C_{i_1}$, $C_{j_1} \leq C_{j_2}$, and $C_{j_2} \leq C_{j_1}$ are completely similar. Thus we may assume that C_{i_1} crosses C_{i_2} and C_{j_1} crosses C_{j_2}.

Since C_{i_1} crosses C_{i_2} and $C_{i_1} < C_{j_1}$, we see that either $C_{i_2} < C_{j_1}$ or C_{i_2} crosses C_{j_1}. If $C_{i_2} < C_{j_1}$ then by Remark 5.7, the relations $C_{i_2} < C_{j_1}$ and $C_{i_1} \overset{\text{ⓡ}}{<} C_{j_1}$ are Φ-linked, and by the same remark, the relations $C_{i_2} < C_{j_1}$ and $C_{i_2} \overset{\text{ⓡ}}{<} C_{j_2}$ are Φ-linked, so the result follows. The case $C_{i_1} < C_{j_2}$ is handled similarly, thus we may assume that each of C_{i_1}, C_{j_1} crosses each of C_{i_2}, C_{j_2}.

Since $\mathcal{C}$ is indecomposable, there exists a sequence $k_0, k_1, \ldots, k_t, k_{t+1}$ satisfying $\{k_0, k_1\} = \{i_1, j_1\}$, $\{k_t, k_{t+1}\} = \{i_2, j_2\}$, and such that for all $s \in [0, t]$ we have either $C_{k_s} < C_{k_{s+1}}$ or $C_{k_s} > C_{k_{s+1}}$. Out of all such sequences, choose the one with minimal possible t. Because of the above assumptions, we must have $t \geq 3$.

Since t is minimal possible, we find that for any $s \in [1, t-2]$, C_{k_s} must cross $C_{k_{s+2}}$. Thus by Remark 5.7, the relations between $C_{k_s}, C_{k_{s+1}}$ and $C_{k_{s+1}}, C_{k_{s+2}}$ are Φ-linked for all $s \in [1, t-2]$. By transitivity, the relations between C_{k_1}, C_{k_2} and $C_{k_{t-1}}, C_{k_t}$ are Φ-linked.

Our next goal is to show that the relations between C_{k_0}, C_{k_1} and C_{k_1}, C_{k_2} are Φ-linked. Note that C_{k_3} must cross both C_{i_1} and C_{j_1}, otherwise we could remove k_2 from the sequence (after possibly swapping k_0 and k_1) thus decreasing t. Because C_{k_2} does not cross C_{k_3}, we cannot have $C_{i_1} \leq C_{k_2} \leq C_{j_1}$. Recall also that C_{k_2} does not cross C_{k_1}. Thus either C_{k_2} crosses C_{k_0} (in which case the relations between C_{k_0}, C_{k_1} and C_{k_1}, C_{k_2} are Φ-linked by Remark 5.7), or $C_{k_2} < C_{i_1} \overset{\text{ⓡ}}{<} C_{j_1}$, or $C_{i_1} \overset{\text{ⓡ}}{<} C_{j_1} < C_{k_2}$. In the latter two cases, after possibly swapping k_0 and k_1, we see that the relations between C_{k_0}, C_{k_1} and C_{k_1}, C_{k_2} are Φ-linked by Lemma 5.9. Similarly, we show that the relations between $C_{k_{t-1}}, C_{k_t}$ and $C_{k_t}, C_{k_{t+1}}$ are Φ-linked. The result follows by transitivity. □

5.4. Finishing the proof.

So far our main focus has been on structural properties of $\mathcal{C}$-admissible transformations. Next, we describe how each such transformation can be represented as a composition of strongly $\mathcal{C}$-admissible transformations constructed in Section 4.2.

Definition 5.13. Two $\mathcal{C}$-admissible transformations are called *flip-equivalent* if they can be obtained from each other by composing with the transformations described in Lemmas 4.5–4.8.

Recall that we have fixed a particular $\mathcal{C}$-admissible transformation Φ.

Lemma 5.14. Φ *is flip-equivalent to a transformation that preserves all rigid relations. More specifically, after possibly applying the color-position symmetry (Lemma 4.5) and several global $\mathbb{H}$-flips (Lemma 4.6), we may assume that for all $C_i \overset{\text{ⓡ}}{<} C_j$ we have $C_i' < C_j'$.*

Proof. Suppose that $C_i \overset{\text{ⓡ}}{<} C_j$ is such that $C_i' > C_j'$. If $\mathcal{C}$ is indecomposable then after applying the color-position symmetry (Lemma 4.5), we find $C_i' < C_j'$, and for any other rigid relation $C_a \overset{\text{ⓡ}}{<} C_b$, we get $C_a' < C_b'$ by Proposition 5.12. If $\mathcal{C}$ is not indecomposable then

we can split it into indecomposable components: $[1, m] = I_1 \sqcup I_2 \sqcup \cdots \sqcup I_s$. We assume that these indecomposable components are ordered "by slope" so that for any $i, j \in [1, s]$, $i < j$, and any $a \in I_i$, $b \in I_j$, we have $[l_a, r_a] \supseteq [l_b, r_b]$ and $[d_a, u_a] \subseteq [d_b, u_b]$. It is then clear that for each $p \in [1, s]$, one choose a cut $C = (l, d, u, r)$ crossing all cuts in $\mathcal{C}$ in such a way that $[l_a, r_a] \supseteq [l, r] \supseteq [l_b, r_b]$ and $[d_a, u_a] \subseteq [d, u] \subseteq [d_b, u_b]$ for all $a \in [1, p]$ and $b \in [p + 1, s]$. Applying a global $\mathbb{H}$-flip at C will flip the first p components while leaving the rest unchanged. Similarly, applying a second global $\mathbb{H}$-flip, we can flip the first $p - 1$ components back. Thus each individual component I_p can be flipped using a composition of two $\mathbb{H}$-flips. $\qquad\square$

Definition 5.15. We say that Φ is *orientation-preserving* if for any relation $C_i < C_j$, we have $C'_i < C'_j$.

Proposition 5.16. Φ *is flip-equivalent to an orientation-preserving transformation.*

Proof. By Lemma 5.14, we may assume that for all $C_i \overset{\circledR}{<} C_j$ we have $C'_i < C'_j$. Let us say that (a, b) is a *bad pair* if $C_a < C_b$ and $C'_a > C'_b$. For each bad pair (a, b), we must have either $[l_a, r_a] = [l_b, r_b]$ or $[d_a, u_a] = [d_b, u_b]$, so assume that $l \le r$ are such that $[l_a, r_a] = [l_b, r_b] = [l, r]$ for some bad pair (a, b).

Let $\mathcal{B}(l, r)$ denote the set of all bad pairs (a, b) satisfying $[l_a, r_a] = [l_b, r_b] = [l, r]$. Choose $(a, b) \in \mathcal{B}(l, r)$ to be the pair which maximizes the value $u_b - d_a$, and let $d := d_a$, $u := u_b$. We claim that $\mathbb{Z}_{\ge 1} \times [d, u]$ is a horizontal $\mathcal{C}$-strip. Indeed, otherwise there would exist $i \in [1, m]$ such that either $d_i < d \le u_i < u$ or $d < d_i \le u < u_i$. Suppose for example that $d_i < d \le u_i < u$. These conditions imply that $C_i < C_b$. Observe that $d \in [d_i, u_i] \setminus [d_b, u_b]$ and also $d \in [d_a, u_a] \setminus [d_b, u_b]$. Since $C'_a > C'_b$, (5.1) shows that $C'_i > C'_b$ so (i, b) is a bad pair, and because $[d_i, u_i] \ne [d_b, u_b]$, we must have $[l_i, r_i] = [l_b, r_b] = [l, r]$. This implies $(i, b) \in \mathcal{B}(l, r)$, and since we have $u_b - d_i > u_b - d_a$, we get a contradiction. The case $d < d_i \le u < u_i$ is handled similarly, thus we have shown that $\mathbb{Z}_{\ge 1} \times [d, u]$ is a horizontal $\mathcal{C}$-strip.

Our goal is to apply either a local $\mathbb{H}$-flip or a double $\mathbb{H}$-flip inside $\mathbb{Z}_{\ge 1} \times [d, u]$ so that (a, b) would stop being a bad pair. We note that this operation may introduce new bad pairs (a', b') but for each of them, the quantity $u_{b'} - d_{a'}$ will be strictly smaller than $u - d = u_b - d_a$. Indeed, our flip only changes the orientation of the pairs (a', b') such that $[d_{a'}, u_{b'}] \subseteq [d, u]$, and for any pair satisfying $[l_{a'}, r_{a'}] = [l_{b'}, r_{b'}] = [l, r]$, $d_{a'} = d$, $u_{b'} = u$, and $C_{a'} < C_{b'}$, we see that $C_{a'}$ crosses C_a, $C_{b'}$ crosses C_b, $C_{a'} < C_b$, and $C_a < C_{b'}$. By Remark 5.7, we get that $C'_{a'} > C'_{b'}$, so (a', b') is already a bad pair and will stop being a bad pair after we apply the flip. We have shown that it is enough to apply either Lemma 4.7 or Lemma 4.8 inside $\mathbb{Z}_{\ge 1} \times [d, u]$ so that, in the notation of (4.1)–(4.2), we would have $a, b \in J$.

Following (4.1), let

$$J' := \{j \in [1, m] \mid [d_j, u_j] \subsetneq [d, u]\}, \quad K := \{k \in [1, m] \mid [d, u] \subseteq [d_k, u_k]\}.$$

Since $d_a < d_b$ and $u_a < u_b$, we find that for all $k \in K$, C_k crosses both C_a and C_b and therefore satisfies $[l_k, r_k] \subseteq [l, r]$. Thus if $[l_j, r_j] = [l, r]$ for all $j \in J'$ then we can apply a local $\mathbb{H}$-flip (Lemma 4.7) and finish the proof. Otherwise, let $I' := \{i \in J' \mid [l_i, r_i] \ne [l, r]\}$. We need to apply a double $\mathbb{H}$-flip for a careful choice of $[d', u'] \subsetneq [d, u]$.

First, we would like to show that for all $i \in I'$, C_i crosses both C_a and C_b. To see that, recall that $[d_i, u_i] \subsetneq [d, u]$ and $[l_i, r_i] \ne [l, r]$. Thus either $d \notin [d_i, u_i]$ or $u \notin [d_i, u_i]$ (or both), so assume $d \notin [d_i, u_i]$. Then we cannot have $C_i < C_a$. If C_i crosses both C_a and C_b then we are done. If C_i crosses C_a but not C_b then $[d_i, u_i] \ne [d_b, u_b]$ so we must have $C_i \overset{\circledR}{<} C_b$. This

contradicts Remark 5.7 since $C_i' < C_b'$ but $C_a' > C_b'$. Thus we may assume that C_i does not cross C_a, and since $d \notin [d_i, u_i]$, we get $C_a \overset{\circledR}{<} C_i$. We cannot have $C_a < C_i < C_b$ because then we would have $[l_i, r_i] = [l, r]$. Thus $C_i \not< C_b$, so there exists $c \in ([d_b, u_b] \cap [d_i, u_i]) \setminus [d_a, u_a]$, in which case we arrive at a contradiction via (5.1). The case $u \notin [d_i, u_i]$ is handled similarly. We have shown that for all $i \in I'$, C_i crosses both C_a and C_b.

We now proceed as in the proof of Proposition 5.12. An *alternating $\{a, b\}$-path* is a sequence $k_0, k_1, \ldots, k_t$ of elements of J' such that $\{k_0, k_1\} = \{a, b\}$, and for all $s \in [0, t-1]$, we have either $C_{k_s} < C_{k_{s+1}}$ or $C_{k_s} > C_{k_{s+1}}$. We say that such a path *ends at k_t*. We let

$$(5.2) \qquad J := \{j \in J' \mid \text{there exists an alternating } \{a, b\}\text{-path ending at } j\}.$$

Clearly this set contains both a and b. We claim that $J \cap I' = \emptyset$, in other words, that for all $j \in J$, we have $[l_j, r_j] = [l, r]$. Otherwise, consider an alternating $\{a, b\}$-path of smallest possible length t such that $k_t = j \in J \cap I'$. We showed above that C_j crosses both C_a and C_b, so $t \geq 3$. Since t is minimal possible, we see that C_{k_s} crosses $C_{k_{s+2}}$ for all $s \in [1, t-2]$, and in addition C_{k_3} crosses both C_{k_0} and C_{k_1}. By construction, C_{k_2} crosses neither C_{k_1} nor C_{k_3}. If it also does not cross C_{k_0} then we must have $C_a < C_{k_2} < C_b$ (because $[d_{k_2}, u_{k_2}] \subseteq [d, u]$ and $[l_{k_2}, r_{k_2}] = [l, r]$). This contradicts the fact that C_{k_3} crosses both C_{k_0}, C_{k_1} but not C_{k_2}. Thus C_{k_s} crosses $C_{k_{s+2}}$ for all $s \in [0, t-2]$. In particular, for all $s \in [0, t-2]$, the relations between $C_{k_s}, C_{k_{s+1}}$ and $C_{k_{s+1}}, C_{k_{s+2}}$ are Φ-linked. Since t is minimal and $k_t \in I'$, we find $[l_{k_{t-1}}, r_{k_{t-1}}] = [l, r] \neq [l_{k_t}, r_{k_t}]$. We showed above that C_{k_t} must cross both C_a and C_b. Since $[d_{k_t}, u_{k_t}] \subsetneq [d, u]$, we get $[l, r] \subsetneq [l_{k_t}, r_{k_t}]$. By construction, $C_{k_{t-1}}$ does not cross C_{k_t}. Together with $[l, r] \subsetneq [l_{k_t}, r_{k_t}]$, this implies $[d_{k_{t-1}}, u_{k_{t-1}}] \neq [d_{k_t}, u_{k_t}]$, so the relation between $C_{k_{t-1}}$ and C_{k_t} must be rigid. We arrive at a contradiction since the relation between C_{k_0}, C_{k_1} (i.e., between C_a and C_b) was reversed by Φ while the Φ-linked relation between $C_{k_{t-1}}, C_{k_t}$ was preserved by Φ due to being rigid. We have shown that $J \cap I' = \emptyset$.

Consider a graph G with vertex set J' and edges $\{i, j\}$ for all $i, j \in J'$ such that C_i and C_j do not cross. Then it is easy to check from (5.2) that J is actually the connected component of G that contains the edge $\{a, b\}$. Thus letting $I := \{J' \setminus J\}$, we find that for all $i \in I$ and $j \in J$, C_i crosses C_j. Set $d' := \min_{i \in I} d_i$ and $u' := \max_{i \in I} u_i$. We see that all conditions of Lemma 4.8 are satisfied, and that moreover $a, b \in J$. Applying the double $\mathbb{H}$-flip, we proceed by induction on $u_b - d_a$ as described above. $\square$

Proof of Theorem 1.6. By Proposition 5.16, we may assume that Φ is orientation-preserving. We will show that Φ is a composition of $\mathbb{H}$- and $\mathbb{V}$-shifts as described in Remark 4.9. Specifically, we would like to apply Lemma 4.8 when $d' = d + 1$ and $u' = u$, in which case (4.3) gives $J = \emptyset$ and the transformation $(\mathrm{id}, \mathrm{rev}_{[d,u]} \circ \mathrm{rev}_{[d+1,u]})$ swaps y_d and $(y_d + 1, \ldots, y_u)$, preserving the order of the latter.

Let $a \in [1, N_\infty - 1]$ be such that $\phi_V(a) > \phi_V(a+1)$. If there exist indices $i, j \in [1, m]$ such that $u_i = a$ and $d_j = a + 1$ then $C_i < C_j$ and $C_i' > C_j'$ by (5.1), which contradicts the assumption that Φ is orientation-preserving. Thus let us assume that there does not exist $i \in [1, m]$ such that $u_i = a$. We set $d := a$.

We construct $u \geq d + 1$ by the following algorithm. First, set $u := d + 1$. If we have found $i \in [1, m]$ such that $d < d_i \leq u < u_i$, we increase u by setting $u := u_i$, and repeat this procedure until there is no i satisfying this assumption. It is straightforward to see that for each $a' \in [d+1, u]$ we must have $\phi_V(a') < \phi_V(d)$.

We claim that $\mathbb{Z}_{\geq 1} \times [d, u]$ and $\mathbb{Z}_{\geq 1} \times [d+1, u]$ are both horizontal $\mathcal{C}$-strips. By construction, there is no $i \in [1, m]$ satisfying $d < d_i \leq u < u_i$. Suppose that we have found $j \in [1, m]$

satisfying $d_j < d \le u_j < u$. We cannot have $u_j = d$, thus $d_j < d + 1 \le u_j < u$. Because of the way we constructed the interval $[d + 1, u]$, there must exist some $i \in I$ such that $d_j < d_i \le u_j < u_i$. This implies $C_i > C_j$, and since $d \in [d_j, u_j] \setminus [d_i, u_i]$, we get a contradiction by (5.1). We have shown that $\mathbb{Z}_{\ge 1} \times [d, u]$ and $\mathbb{Z}_{\ge 1} \times [d + 1, u]$ are horizontal $\mathcal{C}$-strips.

Let $I := \{i \in [1, m] \mid [d_i, u_i] \subseteq [d + 1, u]\}$ and $K := \{k \in [1, m] \mid d \in [d_k, u_k]\}$. By our assumption, we have $d + 1 \in [d_k, u_k]$ for all $k \in K$. Because $\mathbb{Z}_{\ge 1} \times [d, u]$ and $\mathbb{Z}_{\ge 1} \times [d + 1, u]$ are horizontal $\mathcal{C}$-strips, we get that $[d, u] \subseteq [d_k, u_k]$ for all $k \in K$. We claim that C_i crosses C_k for all $i \in I$ and $k \in K$. Indeed, if otherwise C_i does not cross C_k for some $i \in I$ and $k \in K$, we must have $C_i > C_k$ by (5.1). But we have already shown that $\phi_V(a') < \phi_V(d)$ for $a' \in [d + 1, u]$, which by (5.1) contradicts the fact that Φ is orientation-preserving. Thus indeed C_i crosses C_k for all $i \in I$ and $k \in K$, so there exist $l \le r$ such that $[l_k, r_k] \subseteq [l, r] \subseteq [l_i, r_i]$ for all $i \in I$ and $k \in K$. We may now apply the corresponding $\mathbb{H}$-shift of Lemma 4.8 with $d' = d + 1$, $u' = u$, and $J = \emptyset$. We have already observed that $\phi_V(a') < \phi_V(d)$ for all $a' \in [d + 1, u]$. Therefore this $\mathbb{H}$-shift reduces the number of inversions of ϕ_V.

We have assumed above that $u_i \ne a$ for all $i \in [1, m]$. If this is not the case then we must have $d_i \ne a + 1$ for all $i \in [1, m]$. This case is treated similarly – we apply an $\mathbb{H}$-shift with $d' = d$ (for some choice of $d \le a$), $u' = a$, and $u = a + 1$. Applying these $\mathbb{H}$-shifts repeatedly, we arrive at the case $\phi_V = \mathrm{id}$, and then we apply a sequence of $\mathbb{V}$-shifts until $\phi_H = \mathrm{id}$. $\qquad\square$

6. APPLICATIONS

We degenerate the SC6V model proceeding step-by-step in the order specified in [BGW19, Sections 6 and 7]. The nature of the limiting procedure in [BGW19] requires us to only consider tuples of (P, Q)-cuts all of whose left endpoints belong to the same vertical line.

Definition 6.1. We say that $\mathcal{C} = (C_1, \ldots, C_m)$ with $C_i = (l_i, d_i, u_i, r_i)$ is a *left-aligned tuple of cuts* if $l_i = 1$ for all $i \in [1, m]$ and there exists an up-left path Q that contains the top right corners of the cells (r_i, u_i) for all $i \in [1, m]$. Given a bijection $\phi_V : \mathbb{Z}_{\ge 1} \to \mathbb{Z}_{\ge 1}$ and another left-aligned tuple of cuts $\mathcal{C}' = (C_1', \ldots, C_m')$, we write $\mathcal{C}' = \phi_V(\mathcal{C})$ if $l_i' = l_i = 1$, $r_i' = r_i$, and $\phi_V([d_i, u_i]) = [d_i', u_i']$ for all $i \in [1, m]$.

Remark 6.2. In the language of Definition 4.2, (id, ϕ_V) is a $\mathcal{C}$-admissible transformation. It is thus a composition of double $\mathbb{H}$-flips, $\mathbb{H}$-shifts, and local $\mathbb{H}$-flips described in Section 4.2.

In all cases discussed below, the convergence in finite-dimensional distributions was shown in [BGW19] in order to prove the shift-invariance property, and we rely on the same convergence result to take the limit of Theorem 1.6.

6.1. **Continuous model.** Recall that the *Beta distribution* $B(a, b)$ with parameters $a, b \in \mathbb{R}_{>0}$ is supported on a line segment $(0, 1) \subseteq \mathbb{R}$ and has density

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} x^{a-1}(1 - x)^{b-1}, \quad 0 < x < 1.$$

Fix two families $\boldsymbol{\sigma} = (\sigma_i)_{i \in \mathbb{Z}_{\ge 0}}$ and $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{Z}_{\ge 1}}$ of real numbers satisfying $0 < \rho_j < \sigma_i$ for all $(i, j) \in \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 1}$. The following description[4] of the *continuous SC6V model* can be found in [BGW19, Section 6.4].

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[4]The description in [BGW19] is stated for the case when all ρ_j are equal to a single value ρ, but their limiting results, specifically, [BGW19, Corollary 6.21] are valid for an arbitrary choice of $(\rho_j)_{j \in \mathbb{Z}_{\ge 1}}$.

- For each $(i,j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$, we sample (independently) a Beta-distributed random variable $\eta_{i,j} \sim B(\sigma_i - \rho_j, \rho_j)$.
- We consider the vertices in $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$. To each of the four edges adjacent to a vertex of $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ we will assign a (random) vector $(m_1, m_2, \dots)$ of nonnegative real numbers. We refer to m_c as the *mass* of color c passing through this edge.
- For $i \in \mathbb{Z}_{\geq 1}$, the edge entering the vertex $(i, 1)$ from the bottom has all masses equal to 0.
- For $j \in \mathbb{Z}_{\geq 1}$, the masses $(m_1, m_2, \dots)$ assigned to the edge entering the vertex $(1, j)$ from the left are sampled as follows. We set $m_c = 0$ for all $c \neq j$ and let $m_j := -\ln(\eta_{0,j})$.
- Suppose that $(i, j) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$ has incoming masses $(\alpha_1, \alpha_2, \dots)$ and $(\beta_1, \beta_2, \dots)$ assigned to the bottom and left edges, respectively. The outgoing masses $(\gamma_1, \gamma_2, \dots)$ and $(\delta_1, \delta_2, \dots)$ assigned respectively to the top and right outgoing edges are defined by

$$\exp(-\delta_{\geq c}) = \exp(-\alpha_{\geq c}) + (1 - \exp(-\alpha_{\geq c}))\eta_{i,j}, \quad c = 1, 2, \dots,$$

where $\delta_{\geq c} := \sum_{c'=c}^{\infty} \delta_{c'}$ and $\alpha_{\geq c} := \sum_{c'=c}^{\infty} \alpha_{c'}$. We define the remaining masses via the mass conservation law: for $c = 1, 2, \dots$, we set $\gamma_c := \alpha_c + \beta_c - \delta_c$.

Given a left-aligned tuple $\mathcal{C} = (C_1, \dots, C_m)$ of cuts with $C_i = (1, d_i, u_i, r_i)$, we will be interested in the random vector $(\mathrm{Ht}(C_1; \boldsymbol{\sigma}, \boldsymbol{\rho}), \dots, \mathrm{Ht}(C_m; \boldsymbol{\sigma}, \boldsymbol{\rho}))$ of height functions, where each individual height function $\mathrm{Ht}(C_i; \boldsymbol{\sigma}, \boldsymbol{\rho})$ is defined as the total mass of color $\geq d_i$ passing through the right outgoing edges of the vertices $(r_i, d_i), (r_i, d_i + 1), \dots, (r_i, u_i)$. For a bijection $\phi_V : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$, we denote $\phi_V^{-1}(\boldsymbol{\rho}) := (\rho_{\phi_V^{-1}(1)}, \rho_{\phi_V^{-1}(2)}, \dots)$ (cf. Remark 4.3).

Theorem 6.3. *In the above setting of the continuous SC6V model, consider two left-aligned tuples $\mathcal{C} = (C_1, \dots, C_m)$ and $\mathcal{C}' = (C_1', \dots, C_m')$ of cuts. Suppose that we have a bijection $\phi_V : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ such that $\phi_V(\mathcal{C}) = \mathcal{C}'$. Then*

$$(\mathrm{Ht}(C_1; \boldsymbol{\sigma}, \boldsymbol{\rho}), \dots, \mathrm{Ht}(C_m; \boldsymbol{\sigma}, \boldsymbol{\rho})) \stackrel{d}{=} (\mathrm{Ht}(C_1'; \boldsymbol{\sigma}, \boldsymbol{\rho}'), \dots, \mathrm{Ht}(C_m'; \boldsymbol{\sigma}, \boldsymbol{\rho}')), \quad \text{where } \boldsymbol{\rho}' := \phi_V^{-1}(\boldsymbol{\rho}).$$

Proof. The continuous SC6V model is obtained as the following limit. First, consider the original SC6V model. Choose $M_0, M_1, \dots \in \mathbb{Z}_{\geq 1}$ and $N_1, N_2, \dots \in \mathbb{Z}_{\geq 1}$ and subdivide the positive quadrant into rectangles of sizes $M_i \times N_j$ for all $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. One then combines each rectangle into a single *fused vertex* by specializing the corresponding row and column rapidities to geometric progressions in q of lengths M_i and N_j. The fact that the height functions of the resulting *fused SC6V model* are specializations of the height functions of the original SC6V model is shown in [BGW19, Theorem 6.2]. (It is stated for the case when all N_j are equal but translates verbatim to the case of arbitrary N_j.) For each cut $C_i = (1, d_i, u_i, r_i)$, we can consider the *unmerged cut* $\overline{C}_i = (1, \overline{d}_i, \overline{u}_i, \overline{r}_i)$ given by $\overline{d}_i = N_1 + N_2 + \dots + N_{d_i - 1} + 1$, $\overline{u}_i = N_1 + N_2 + \dots + N_{u_i}$, and $\overline{r}_i = M_1 + M_2 + \dots + M_{r_i}$. The unmerged versions $\overline{\phi_V}$ of ϕ_V and $\overline{C}_i'$ of C_i' are defined similarly. The two resulting tuples of unmerged cuts satisfy the assumptions of Theorem 1.6 and therefore the corresponding vectors of height functions have the same distribution modulo replacing the row rapidities $\mathbf{y} = (y_1, y_2, \dots)$ with $(\overline{\phi_V})^{-1}(\mathbf{y})$. Thus after specializing the rapidities to geometric progressions and merging each $M_i \times N_j$ rectangle into a single fused vertex, we see that the two height vectors have the same distribution in the setting of the fused SC6V model. A similar argument can be found in the proof of [BGW19, Theorem 6.3]. The result now follows from [BGW19, Corollary 6.21] which states that vertex weights of the continuous SC6V model are obtained as particular limits of the fused SC6V model vertex weights. $\qquad \square$

6.2. **Beta polymer.** Our next object is the *directed Beta polymer* [BC17] which is essentially just a restatement of the continuous SC6V model from the previous subsection. We again fix two families $\boldsymbol{\sigma} = (\sigma_i)_{i \in \mathbb{Z}_{\geq 0}}$ and $\boldsymbol{\rho} = (\rho_j)_{j \in \mathbb{Z}_{\geq 1}}$ as above. We first describe the model following [BGW19, Section 7.1].

- For each $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$, we sample (independently) a Beta-distributed random variable $\eta_{i,j} \sim B(\sigma_i - \rho_j, \rho_j)$.
- We consider the vertices of the grid $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with diagonal and vertical edges, that is, for each $(i, j) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$, we introduce diagonal edges connecting $(i - 1, j - 1)$ to (i, j) and vertical edges connecting $(i, j - 1)$ to (i, j). For each $j \in \mathbb{Z}_{\geq 1}$, we also have a vertical edge connecting $(0, j - 1)$ to $(0, j)$.
- To each edge e we assign a *weight* $\mathrm{wt}(e)$: if e is a vertical edge connecting $(i, j - 1)$ to (i, j) then we set $\mathrm{wt}(e) := \eta_{i,j}$, and if e is a diagonal edge connecting $(i - 1, j - 1)$ to (i, j) then we set $\mathrm{wt}(e) := 1 - \eta_{i,j}$.
- For $l, d, u, r \in \mathbb{Z}_{\geq 0}$ such that $l \leq r$ and $r - l \leq u - d$, the *delayed Beta polymer partition function* $\mathfrak{Z}^{B;\boldsymbol{\sigma},\boldsymbol{\rho}}_{(l,d) \to (r,u)}$ is defined by

$$\mathfrak{Z}^{B;\boldsymbol{\sigma},\boldsymbol{\rho}}_{(l,d) \to (r,u)} = \sum_{(l,d) = \pi_d \to \pi_{d+1} \to \cdots \to \pi_u = (r,u)} \prod_{k=f(\pi)}^{u} \mathrm{wt}(\pi_{k-1} \to \pi_k),$$

where $\pi_k - \pi_{k-1} \in \{(1, 1), (0, 1)\}$ for all k and $f(\pi) := \min\{k \mid \pi_k - \pi_{k-1} = (0, 1)\}$. In other words, we take the product of weights of all edges of our lattice path that occur weakly after its first vertical edge.

Definition 6.4. For vectors $\mathbf{d}, \mathbf{u}, \mathbf{r} \in (\mathbb{Z}_{\geq 0})^m$, we denote $\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r}) := (C_1, \ldots, C_m)$, where $C_i := (1, d_i + 1, u_i, r_i)$ for all $i \in [1, m]$.

Remark 6.5. We put $d_i + 1$ above in order to account for the following discrepancy: for $C = (1, d, u, r)$, $\mathrm{Ht}(C; \boldsymbol{\sigma}, \boldsymbol{\rho})$ depends on $\sigma_0, \ldots, \sigma_r$ and $\rho_d, \ldots, \rho_u$, while $\mathfrak{Z}^{B;\boldsymbol{\sigma},\boldsymbol{\rho}}_{(0,d) \to (r,u)}$ depends on $\sigma_0, \ldots, \sigma_r$ and $\rho_{d+1}, \ldots, \rho_u$. It is thus natural to identify $\mathfrak{Z}^{B;\boldsymbol{\sigma},\boldsymbol{\rho}}_{(0,d) \to (r,u)}$ with the cut $(1, d+1, u, r)$.

The following result confirms the prediction of [BGW19, Remark 7.4].

Theorem 6.6. *Consider vectors* $\mathbf{d}, \mathbf{u}, \mathbf{d}', \mathbf{u}', \mathbf{r} \in (\mathbb{Z}_{\geq 0})^m$ *such that both* $\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})$ *and* $\mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r})$ *are left-aligned tuples of cuts. Suppose that we have a bijection* $\phi_V : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ *such that* $\phi_V(\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})) = \mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r})$ *and set* $\boldsymbol{\rho}' := \phi_V^{-1}(\boldsymbol{\rho})$. *Then*

$$(6.1) \qquad \left(\mathfrak{Z}^{B;\boldsymbol{\sigma},\boldsymbol{\rho}}_{(0,d_1) \to (r_1,u_1)}, \ldots, \mathfrak{Z}^{B;\boldsymbol{\sigma},\boldsymbol{\rho}}_{(0,d_m) \to (r_m,u_m)} \right) \overset{d}{=} \left(\mathfrak{Z}^{B;\boldsymbol{\sigma},\boldsymbol{\rho}'}_{(0,d_1') \to (r_1,u_1')}, \ldots, \mathfrak{Z}^{B;\boldsymbol{\sigma},\boldsymbol{\rho}'}_{(0,d_m') \to (r_m,u_m')} \right).$$

Proof. By [BGW19, Proposition 7.2], the joint distribution of the left hand side of (6.1) coincides with the distribution of the vector $\left(\exp\left(- \mathrm{Ht}(C_1; \boldsymbol{\sigma}, \boldsymbol{\rho}) \right), \ldots, \exp\left(- \mathrm{Ht}(C_m; \boldsymbol{\sigma}, \boldsymbol{\rho}) \right) \right)$ of exponentiated height functions $\mathrm{Ht}(C_i; \boldsymbol{\sigma}, \boldsymbol{\rho})$ from Section 6.1. Here $(C_1, \ldots, C_m) = \mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})$ as in Definition 6.4. The result now follows from Theorem 6.3. $\square$

6.3. **Intersection matrices.** When taking the limits in the next subsections, we will specialize to the *homogeneous case* where all σ_i are equal to a single value σ and all ρ_j are equal to a single value ρ satisfying $0 < \rho < \sigma$. In view of this, let us discuss intersection matrices introduced in Definition 1.8.

Proposition 6.7. *Consider vectors* $\mathbf{d}, \mathbf{u}, \mathbf{d}', \mathbf{u}', \mathbf{r} \in (\mathbb{Z}_{\geq 0})^m$ *and let* $\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r}) = (C_1, \ldots, C_m)$ *and* $\mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r}) = (C_1', \ldots, C_m')$ *be as in Definition 6.4. Assume that both* $\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})$ *and*

$\mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r})$ *are left-aligned tuples of cuts. Then there exists a bijection* $\phi_V : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ *satisfying* $\phi_V(\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})) = \mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r})$ *if and only if* $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$.

Proof. In order to distinguish between subsets of $\mathbb{Z}$ and subsets of $\mathbb{R}$, let us denote $[d, u]_{\mathbb{Z}} := \{j \in \mathbb{Z} \mid d \leq j \leq u\}$ and $[d, u]_{\mathbb{R}} := \{y \in \mathbb{R} \mid d \leq y \leq u\}$. Recall that $C_i = (1, d_i + 1, u_i, r_i)$ and $C_i' = (1, d_i' + 1, u_i', r_i')$ for all $i \in [1, m]$. Recall also that the entries of $\mathrm{IM}^+(\mathbf{d}, \mathbf{u})$ are given by $\mathrm{IM}_{i,j}^+ = \max(0, \min(u_i, u_j) - \max(d_i, d_j))$, which is the cardinality of $[d_i + 1, u_i]_{\mathbb{Z}} \cap [d_j + 1, u_j]_{\mathbb{Z}}$.

If ϕ_V exists then it sends $[d_i + 1, u_i]_{\mathbb{Z}} \cap [d_j + 1, u_j]_{\mathbb{Z}}$ bijectively to $[d_i' + 1, u_i']_{\mathbb{Z}} \cap [d_j' + 1, u_j']_{\mathbb{Z}}$, thus $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$. Conversely, it suffices to show that for any $I \subseteq [1, m]$, the intersections $\bigcap_{i \in I}[d_i + 1, u_i]_{\mathbb{Z}}$ and $\bigcap_{i \in I}[d_i' + 1, u_i']_{\mathbb{Z}}$ have the same cardinality. Indeed, we have

$$\# \bigcap_{i \in I}[d_i + 1, u_i]_{\mathbb{Z}} = \max\left(0, \min\{u_i \mid i \in I\} - \max\{d_j \mid j \in I\}\right) = \min\left\{\mathrm{IM}_{i,j}^+ \big| i, j \in I\right\}.$$

Thus the assumption $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$ implies the result for all $I \subseteq [1, m]$. $\square$

6.4. **Gamma polymer.** Recall that the Gamma distribution with parameter $\kappa > 0$ is supported on $\mathbb{R}_{>0}$ with density

$$\frac{1}{\Gamma(\kappa)} x^{\kappa - 1} \exp(-x), \quad x > 0.$$

Let us now fix some $\kappa > 0$ and describe the Gamma polymer [CSS15, OO15].
- For each $(i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$, we sample an independent random variable $\eta_{i,j}$ which is Gamma-distributed with parameter κ.
- We again deal with the grid $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ with vertical and diagonal edges. The weight of each vertical edge connecting $(i, j - 1)$ to (i, j) is equal to $\eta_{i,j}$, and the weights of all diagonal edges are equal to 1.
- For $l, d, u, r \in \mathbb{Z}_{\geq 0}$ such that $l \leq r$ and $r - l \leq u - d$, the *Gamma polymer partition function* $\mathfrak{Z}_{(l,d) \to (r,u)}^{\Gamma}$ is defined by

$$\mathfrak{Z}_{(l,d) \to (r,u)}^{\Gamma} = \sum_{(l,d) = \pi_d \to \pi_{d+1} \to \cdots \to \pi_u = (r,u)} \prod_{k=d+1}^{u} \mathrm{wt}(\pi_{k-1} \to \pi_k),$$

where $\pi_k - \pi_{k-1} \in \{(1, 1), (0, 1)\}$ for all k. Note that since the diagonal edges have weight 1, only the vertical edges contribute to the product.

Theorem 6.8. *Consider vectors* $\mathbf{d}, \mathbf{u}, \mathbf{d}', \mathbf{u}', \mathbf{r} \in (\mathbb{Z}_{\geq 0})^m$ *such that both* $\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})$ *and* $\mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r})$ *are left-aligned tuples of cuts. If* $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$ *then*

$$\left(\mathfrak{Z}_{(0,d_1) \to (r_1,u_1)}^{\Gamma}, \ldots, \mathfrak{Z}_{(0,d_m) \to (r_m,u_m)}^{\Gamma}\right) \overset{d}{=} \left(\mathfrak{Z}_{(0,d_1') \to (r_1,u_1')}^{\Gamma}, \ldots, \mathfrak{Z}_{(0,d_m') \to (r_m,u_m')}^{\Gamma}\right).$$

Proof. By Proposition 6.7, there exists a bijection $\phi_V : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ sending $\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})$ to $\mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r})$. The result now follows from Theorem 6.6 by substituting $\rho_j = \varepsilon^{-1}$, $\sigma_i = \varepsilon^{-1} + \kappa$ and taking a limit as $\varepsilon \to 0$, see [BGW19, Section 7.2]. $\square$

6.5. **O'Connell–Yor polymer.** Our next limiting transition leads to the following model.
- For each $n \in \mathbb{Z}_{\geq 0}$, let $B_n(t)$, $t \geq 0$, be an independent standard Brownian motion.
- For each $l, r \in \mathbb{Z}_{\geq 0}$ and $d, u \in \mathbb{R}_{\geq 0}$ satisfying $l \leq r$ and $d \leq u$, define

$$\mathfrak{Z}_{(l,d) \to (r,u)}^{OY} = \int_{d = t_l < t_{l+1} < \cdots < t_{r+1} = u} \exp\left[\sum_{i=l}^{r} B_i(t_{i+1}) - B_i(t_i)\right] dt_{l+1} \ldots dt_r.$$

We extend the definition of a cut $C = (l, d, u, r)$ to the case where $d \leq u$ are real numbers. In this case, we say that $\mathcal{C} = (C_1, \ldots, C_m)$ is a *left-aligned tuple of cuts* if $l_i = 1$ for all $i \in [1, m]$ and for any $i, j \in [1, m]$, we have either $r_i \leq r_j$ and $u_i \geq u_j$ or $r_i \geq r_j$ and $u_i \leq u_j$.

Theorem 6.9. *Consider vectors* $\mathbf{d}, \mathbf{u}, \mathbf{d}', \mathbf{u}' \in (\mathbb{R}_{\geq 0})^m$ *and* $\mathbf{r} \in (\mathbb{Z}_{\geq 0})^m$ *such that both* $\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})$ *and* $\mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r})$ *are left-aligned tuples of cuts. If* $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$ *then*

$$\left(\mathfrak{Z}^{OY}_{(0,d_1)\to(r_1,u_1)}, \ldots, \mathfrak{Z}^{OY}_{(0,d_m)\to(r_m,u_m)} \right) \stackrel{d}{=} \left(\mathfrak{Z}^{OY}_{(0,d_1')\to(r_1,u_1')}, \ldots, \mathfrak{Z}^{OY}_{(0,d_m')\to(r_m,u_m')} \right).$$

Proof. As explained in [BGW19, Section 7.3], one obtains $\mathfrak{Z}^{OY}_{(l,d)\to(r,u)}$ as a properly rescaled limit of $\mathfrak{Z}^{\Gamma}_{(l,Ld)\to(r,Lu)}$ as $L \to \infty$. It is easy to check that there exists a sequence $\mathbf{d}_L, \mathbf{u}_L, \mathbf{d}'_L, \mathbf{u}'_L \in (\mathbb{Z}_{\geq 0})^m$ for $L = 1, 2, \ldots$ such that for all L,

- $\mathcal{C}(\mathbf{d}_L, \mathbf{u}_L, \mathbf{r})$ and $\mathcal{C}(\mathbf{d}'_L, \mathbf{u}'_L, \mathbf{r})$ are left-aligned tuples of cuts,
- $\mathrm{IM}^+(\mathbf{d}_L, \mathbf{u}_L) = \mathrm{IM}^+(\mathbf{d}'_L, \mathbf{u}'_L)$,
- the limit of $\frac{1}{L}(\mathbf{d}_L, \mathbf{u}_L, \mathbf{d}'_L, \mathbf{u}'_L)$ as $L \to \infty$ equals $(\mathbf{d}, \mathbf{u}, \mathbf{d}', \mathbf{u}')$, and
- the limit of $\frac{1}{L}\mathrm{IM}^+(\mathbf{d}_L, \mathbf{u}_L)$ as $L \to \infty$ equals $\mathrm{IM}^+(\mathbf{d}, \mathbf{u})$.

The result now follows by applying Theorem 6.8 and taking the limit as $L \to \infty$. $\square$

We are ready to consider the models described in Section 1.4.

6.6. Brownian last passage percolation. Recall that $\mathfrak{Z}_{(l,d)\to(r,u)}$ has been defined in (1.6). Let us give a generalization of Theorem 1.9.

Theorem 6.10. *Consider vectors* $\mathbf{d}, \mathbf{u}, \mathbf{d}', \mathbf{u}' \in (\mathbb{R}_{\geq 0})^m$ *and* $\mathbf{r} \in (\mathbb{Z}_{\geq 0})^m$ *such that both* $\mathcal{C}(\mathbf{d}, \mathbf{u}, \mathbf{r})$ *and* $\mathcal{C}(\mathbf{d}', \mathbf{u}', \mathbf{r})$ *are left-aligned tuples of cuts. If* $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$ *then*

$$\left(\mathfrak{Z}_{(0,d_1)\to(r_1,u_1)}, \ldots, \mathfrak{Z}_{(0,d_m)\to(r_m,u_m)} \right) \stackrel{d}{=} \left(\mathfrak{Z}_{(0,d_1')\to(r_1,u_1')}, \ldots, \mathfrak{Z}_{(0,d_m')\to(r_m,u_m')} \right).$$

Proof. As explained in [BGW19, Section 7.4], $\mathfrak{Z}_{(l,d)\to(r,u)}$ is obtained as a scaling limit of the logarithm of $\mathfrak{Z}^{OY}_{(l,dL)\to(r,uL)}$ as $L \to \infty$. The result thus follows from Theorem 6.9. $\square$

6.7. KPZ equation.

Proof of Theorem 1.10. The argument in [BGW19, Section 7.5] shows that one can obtain $\mathcal{Z}^{(y)}(t, x)$ as a scaling limit of $\mathfrak{Z}^{OY}_{(0,y)\to(tL, t\sqrt{L}+x)}$ as $L \to \infty$. Let L be such that tL is an integer and set $\mathbf{r} = (tL, tL, \ldots, tL) \in (\mathbb{Z}_{\geq 0})^m$, $\mathbf{r}' = (t\sqrt{L}, t\sqrt{L}, \ldots, t\sqrt{L}) \in (\mathbb{R}_{\geq 0})^m$. We introduce $\mathbf{d}, \mathbf{d}', \mathbf{u}, \mathbf{u}' \in (\mathbb{R}_{\geq 0})^m$ by $\mathbf{d} := \mathbf{y}$, $\mathbf{d}' := \mathbf{y}'$, $\mathbf{u} := \mathbf{r}' + \mathbf{x}$, and $\mathbf{u}' := \mathbf{r}' + \mathbf{x}'$. Let $\mathbb{1}_{m \times m}$ denote the $m \times m$ matrix with all entries equal to 1. If L is large enough so that $t\sqrt{L} > d_i$ for all $i \in [1, m]$ then we have

$$(6.2) \qquad \mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = t\sqrt{L}\mathbb{1}_{m \times m} + \mathrm{IM}(\mathbf{y}, \mathbf{x}) = t\sqrt{L}\mathbb{1}_{m \times m} + \mathrm{IM}(\mathbf{y}', \mathbf{x}') = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}').$$

Therefore Theorem 6.9 applies and we obtain the result by sending $L \to \infty$. $\square$

6.8. Airy sheet. In view of the recent results [DOV18], one can define the Airy sheet $\mathcal{A}(x, y)$ as a limit of the Brownian last passage percolation, cf. [BGW19, Equation (7.14)]:

$$(6.3) \qquad \frac{\mathfrak{Z}_{(0,2xn^{2/3})\to(n, n+2yn^{2/3})} - 2n - 2n^{2/3}(y - x) + (x - y)^2 n^{1/3}}{n^{1/3}} \to \mathcal{A}(x, y)$$

for $x, y \in \mathbb{R}$ as $n \to \infty$.

Proof of Theorem 1.11. We proceed as in the proof of Theorem 1.10: for an integer $n \in \mathbb{Z}_{\geq 1}$, set $\mathbf{r} = (n, n, \dots, n) \in (\mathbb{Z}_{\geq 0})^m$, $\mathbf{d} := 2n^{2/3}\mathbf{x}$, $\mathbf{d}' := 2n^{2/3}\mathbf{x}'$, $\mathbf{u} := \mathbf{r} + 2n^{2/3}\mathbf{y}$, and $\mathbf{u}' := \mathbf{r} + 2n^{2/3}\mathbf{y}'$. As in (6.2), we find $\mathrm{IM}^+(\mathbf{d}, \mathbf{u}) = \mathrm{IM}^+(\mathbf{d}', \mathbf{u}')$ and then deduce the result from Theorem 6.10 by sending $n \to \infty$.   □

7. Arbitrary permutations, Kazhdan–Lusztig polynomials, and positroid varieties

We discuss a surprising connection between the SC6V model and a family of algebraic and combinatorial objects: Kazhdan–Lusztig R-polynomials (see [KL79] or [BB05, Chapter 5]), Deodhar's *distinguished subexpressions* [Deo85, MR04], and the positroid decomposition [Pos06, BGY06, KLS13] of the Grassmannian. As we speculate in Section 7.3, these observations suggest that there could be a common generalization of the SC6V model and the combinatorics of positroid varieties.

In the last subsection, we give a counterexample to a natural extension of Theorem 1.6 to arbitrary wiring diagram domains but give a conjectural generalization of the shift invariance of [BGW19] to arbitrary domains.

First, we concentrate on the limit of the SC6V model as $y_1, y_2, \dots \to 0$, in which case all parameters $\mathfrak{p}_{i,j}$ defined in (1.1) become equal to $1/q$. More generally, for an arbitrary Yang–Baxter element Y^w defined for $w \in S_n$ in (2.6), we consider a limiting regime where $z_1 \gg z_2 \gg \cdots \gg z_n$, i.e., we assume that $z_j/z_i \to 0$ for all $1 \leq i < j \leq n$. In the limit, we get $\mathfrak{p}_{i,j} = 1/q$ for all $1 \leq i < j \leq n$. We denote the resulting element by

$$Y^w|_{z_1 \gg \cdots \gg z_n} = R_{i_1}(1/q) R_{i_2}(1/q) \cdots R_{i_r}(1/q),$$

where $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced word.

7.1. **Kazhdan–Lusztig R-polynomials.** For all pairs $\pi, w \in S_n$ such that $\pi \leq w$ in the Bruhat order, the associated *Kazhdan–Lusztig R-polynomials* $R^{\pi,w}(q)$ are defined uniquely by the condition that for all $w \in S_n$, we have

(7.1)
$$q^{\ell(w)}(T_{w^{-1}})^{-1} = \sum_{\pi \leq w} R^{\pi,w}(q) T_\pi.$$

The following result was shown in [BN19] by induction on $\ell(w)$. We will prove it bijectively in Proposition 7.3 below.

Lemma 7.1. *For all $w \in S_n$, we have*
$$q^{\ell(w)} Y^w|_{z_1 \gg \cdots \gg z_n} = \sum_{\pi \leq w} R^{\pi,w}(q) T_\pi.$$

Proof. Indeed, we have $R_k(1/q) = 1/q T_k + (1 - 1/q)$ which by (2.1) is equal to T_k^{-1}. Thus $Y^w|_{z_1 \gg \cdots \gg z_n} = T_{i_1}^{-1} \cdots T_{i_r}^{-1} = (T_{w^{-1}})^{-1}$. The result follows from (7.1).   □

By Proposition 2.3, this gives an interpretation of each Kazhdan–Lusztig R-polynomial in terms of the SC6V model: $q^{-\ell(w)} R^{\pi,w}(q)$ equals the probability of observing π as the color permutation of the SC6V model inside a wiring diagram for w with all $\mathfrak{p}$-parameters equal to $1/q$. Note that one of the vertex weights (namely, $1 - q\mathfrak{p}$) in Figure 1 becomes equal to zero, however, the resulting "five-vertex model" is very different from the standard five-vertex model studied e.g. in [BBBG19, BSW19, dGKW18]. Our goal is to show that after sending all $\mathfrak{p}$-parameters to $1/q$, the configurations of the SC6V model with nonzero weight
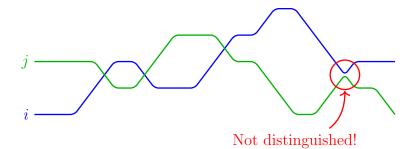
FIGURE 16. The distinguished condition (Definition 7.2) states that two paths cannot touch after they have crossed an odd number of times. This is precisely the condition describing SC6V configurations that remain after we send the weight $1 - q\mathfrak{p}$ to 0.

admit a weight-preserving bijection with well-studied combinatorial objects called *Deodhar's distinguished subexpressions* [Deo85].

Definition 7.2. A *reduced expression* for $w \in S_n$ is a sequence $\mathbf{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_r})$ such that $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a reduced word. A *subexpression* for $\pi \in S_n$ inside $\mathbf{w}$ is a sequence $\boldsymbol{\pi} = (s_{i_1}^{\boldsymbol{\pi}}, s_{i_2}^{\boldsymbol{\pi}}, \ldots, s_{i_r}^{\boldsymbol{\pi}})$ such that $\pi = s_{i_1}^{\boldsymbol{\pi}} s_{i_2}^{\boldsymbol{\pi}} \cdots s_{i_r}^{\boldsymbol{\pi}}$ and each $s_{i_j}^{\boldsymbol{\pi}}$ is equal to either s_{i_j} or id. We introduce partial products $\pi_{(j)} := s_{i_1}^{\boldsymbol{\pi}} s_{i_2}^{\boldsymbol{\pi}} \cdots s_{i_j}^{\boldsymbol{\pi}}$. A subexpression $\boldsymbol{\pi}$ is called *distinguished* (see Figure 16) if whenever $\ell(\pi_{(j-1)} s_{i_j}) = \ell(\pi_{(j-1)}) - 1$, we have $s_{i_j}^{\boldsymbol{\pi}} = s_{i_j}$. We denote $J_{\boldsymbol{\pi}}^\circ := \{j \in [1, r] \mid s_{i_j}^{\boldsymbol{\pi}} = \mathrm{id}\}$, $J_{\boldsymbol{\pi}}^- := \{j \in [1, r] \mid \ell(\pi_{(j-1)} s_{i_j}^{\boldsymbol{\pi}}) = \ell(\pi_{(j-1)}) - 1\}$ and define the *weight* of $\boldsymbol{\pi}$ by $(q-1)^{|J_{\boldsymbol{\pi}}^\circ|} q^{|J_{\boldsymbol{\pi}}^-|}$.

When $q \geq 2$ is a prime power,[5] the above weights have the following geometric meaning: the R-polynomial $R^{\pi,w}(q)$ equals the number of points over $\mathbb{F}_q$ inside an *(open) Richardson variety* $\mathcal{R}^{\pi,w}$. Richardson varieties give a natural stratification of the *flag variety* G/B, where $G = \mathrm{SL}_n(\mathbb{C})$ and B is the subgroup of G consisting of upper triangular matrices. (Thus G/B is the space of *complete flags* ($V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n$) of linear subspaces of $\mathbb{C}^n$ with $\dim V_i = i$ for all i.) We have $G/B = \bigsqcup_{\pi \leq w} \mathcal{R}^{\pi,w}$. Deodhar [Deo85] constructed a decomposition of each Richardson variety $\mathcal{R}^{\pi,w}$ into finer pieces corresponding to distinguished subexpressions of π inside a reduced expression $\mathbf{w}$ for w. Each piece is isomorphic to $(\mathbb{C}^*)^{|J_{\boldsymbol{\pi}}^\circ|} \times \mathbb{C}^{|J_{\boldsymbol{\pi}}^-|}$. The same decomposition works over finite fields, thus the number of $\mathbb{F}_q$-points inside the corresponding piece equals $(q-1)^{|J_{\boldsymbol{\pi}}^\circ|} q^{|J_{\boldsymbol{\pi}}^-|}$. In particular, we have

$$R^{\pi,w}(q) = \sum_{\substack{\text{distinguished subexpressions } \boldsymbol{\pi} \\ \text{for } \pi \text{ inside } \mathbf{w}}} (q-1)^{|J_{\boldsymbol{\pi}}^\circ|} q^{|J_{\boldsymbol{\pi}}^-|}.$$

The following observation gives a bijective proof of Lemma 7.1.

Proposition 7.3. *Let $\pi, w \in S_n$ and choose a reduced expression $\mathbf{w}$ for w. Then distinguished subexpressions for π inside $\mathbf{w}$ are in a natural bijection with configurations of the SC6V model inside a wiring diagram corresponding to $\mathbf{w}$ with all $\mathfrak{p}$-parameters set to $1/q$. For a given subexpression $\boldsymbol{\pi}$, the SC6V-weight of the corresponding configuration equals $q^{-\ell(w)} \cdot (q-1)^{|J_{\boldsymbol{\pi}}^\circ|} q^{|J_{\boldsymbol{\pi}}^-|}$.*

Proof. This is obvious from the definitions, see Figure 16. $\square$

---

[5]Note that for $q \geq 1$ and $\mathfrak{p} := 1/q$, the probabilities $\mathfrak{p}$, $1 - \mathfrak{p}$, $q\mathfrak{p}$, and $1 - q\mathfrak{p}$ all belong to $[0, 1]$.

7.2. **Grassmannian interpretation of the flip theorem.** The flip theorem (Theorem 1.1) gives a non-trivial relation between partition functions of the SC6V model associated with a wiring diagram of a "rectangular" permutation $w := w^{M,N}$ from Example 2.9. Therefore specializing to $\mathbf{y} = 0$, we obtain a relation between R-polynomials $R^{\pi,w}(q)$ for various π satisfying horizontal and vertical boundary conditions.

Recall that we set $n := M + N$. When $w = w^{M,N}$ is a rectangular permutation and $\pi \leq w$, the Richardson variety $\mathcal{R}^{\pi,w}$ projects isomorphically onto a *positroid variety* inside the *Grassmannian* $\mathrm{Gr}(M, n)$ of complex M-dimensional linear subspaces of $\mathbb{C}^n$. The positroid decomposition of $\mathrm{Gr}(M, n)$ was constructed by Knutson–Lam–Speyer [KLS13] building on the work of Postnikov [Pos06]. Theorem 1.1 implies that the number of points over $\mathbb{F}_q$ in one subvariety $\Pi^{\mathbb{H},\mathbb{V}}$ of $\mathrm{Gr}(M, n)$ equals the number of points in another subvariety $\Pi^{180°(\mathbb{H}),\mathbb{V}}$ of $\mathrm{Gr}(M, n)$. Our goal is to describe the subvarieties $\Pi^{\mathbb{H},\mathbb{V}}$ and $\Pi^{180°(\mathbb{H}),\mathbb{V}}$ and give a simple bijection between their points over any field. This gives a "lift" of the $\mathbf{y} = 0$ specialization of Theorem 1.1 to $\mathrm{Gr}(M, n)$. One might wonder whether the whole Theorem 1.1, as well as other properties and objects related to the SC6V model, can be lifted to the level of $\mathrm{Gr}(M, n)$. We discuss this further in Section 7.3.

Let $\mathrm{Bound}(M, n)$ denote the set of *bounded affine permutations* which are bijections $f : \mathbb{Z} \to \mathbb{Z}$ such that

- $f(i + n) = f(i) + n$ for all $i \in \mathbb{Z}$,
- $\frac{1}{n} \sum_{i=1}^{n} (f(i) - i) = M$, and
- $i \leq f(i) \leq i + n$ for all $i \in \mathbb{Z}$.

An element $X \in \mathrm{Gr}(M, n)$ can be viewed as a row span of a full rank complex $M \times n$ matrix A. Denote the columns of A by $A_1, A_2, \ldots, A_n \in \mathbb{C}^M$. We extend this uniquely to all $i \in \mathbb{Z}$ by requiring that $A_{i+n} = A_i$ for all $i \in \mathbb{Z}$. We define $f_A : \mathbb{Z} \to \mathbb{Z}$ by

$$(7.2) \qquad f_A(i) = \min \{j \geq i \mid A_i \in \mathrm{Span}\,(A_{i+1}, A_{i+2}, \ldots, A_j)\} \quad \text{for } i \in \mathbb{Z}.$$
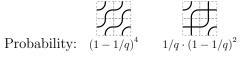
It is not hard to see that $f_A \in \mathrm{Bound}(M, n)$ and that it only depends on the row span X of A, thus we denote $f_X := f_A$ and consider a map $\mathrm{Gr}(M, n) \to \mathrm{Bound}(M, n)$ sending $X \mapsto f_X$. Denoting $\Pi_f^\circ := \{X \in \mathrm{Gr}(M, n) \mid f_X = f\}$, we obtain the *positroid decomposition* of $\mathrm{Gr}(M, n)$ given by $\mathrm{Gr}(M, n) = \bigsqcup_{f \in \mathrm{Bound}(M,n)} \Pi_f^\circ$.

A permutation $u \in S_n$ is called (M, N)-*Grassmannian* if $u(1) < u(2) < \cdots < u(N)$ and $u(N + 1) < u(N + 2) < \cdots < u(n)$. Let $t_{\omega_M} \in \mathrm{Bound}(M, n)$ denote the map sending $i \mapsto i + n$ for $i \in [1, M]$ and $i \mapsto i$ for $i \in [M + 1, n]$ (this defines $t_{\omega_M}(i)$ uniquely for all other $i \in \mathbb{Z}$). Extend each permutation $\pi \in S_n$ to a map $\pi : \mathbb{Z} \mapsto \mathbb{Z}$ satisfying $\pi(i + n) = \pi(i) + n$. Then it is well known [KLS13, Proposition 3.15] that the map $(\pi, w) \mapsto f_{\pi,w} := \pi t_{\omega_M} w^{-1}$ (where multiplication is given by composition) gives a bijection between pairs (π, w) such that $\pi \leq w$ and w is (M, N)-Grassmannian and the set $\mathrm{Bound}(M, n)$. See [GKL19, Figure 2] for a pictorial representation of this correspondence. By [KLS13, Proposition 5.4], the Richardson variety $\mathcal{R}^{\pi,w}$ is isomorphic to the positroid variety $\Pi_{f_{\pi,w}}^\circ$, thus the number of points in $\Pi_{f_{\pi,w}}^\circ$ over $\mathbb{F}_q$ is counted by the R-polynomial $R^{\pi,w}(q)$. When $w = w^{M,N}$, we have $f_{\pi,w} = \pi \mathrm{id}_M$, where $\mathrm{id}_M : \mathbb{Z} \to \mathbb{Z}$ sends $i \mapsto i + M$ for all $i \in \mathbb{Z}$.

Example 7.4. Let $w = w^{M,N}$ and $\pi = \mathrm{id}$. In this case, $f_{\pi,w} = \mathrm{id}_M$ corresponds to the *top-dimensional positroid variety* $\Pi_{\mathrm{id}_M}^\circ$. The row span of a $k \times n$ matrix A belongs to $\Pi_{\mathrm{id}_M}^\circ$ if and only if all of its *Plücker coordinates* corresponding to cyclic intervals are nonzero:

$$\Delta_{1,2,\ldots,M}(A) \neq 0, \quad \Delta_{2,3,\ldots,M+1}(A) \neq 0, \quad \ldots, \quad \Delta_{n,1,2,\ldots,M-1}(A) \neq 0.$$

In other words, the row span of A belongs to $\Pi^\circ_{\mathrm{id}_M}$ if and only if the vectors $A_{i+1}, A_{i+2}, \ldots, A_{i+M}$ form a basis of $\mathbb{C}^M$ for all $i \in \mathbb{Z}$, as clearly follows from (7.2). Let us now consider the case $M = N = 2$ and count the number of points inside $\Pi^\circ_{\mathrm{id}_M}$ over $\mathbb{F}_q$ in two different ways. Having $\pi = \mathrm{id}$ corresponds to the following two SC6V model configurations:

Probability: $(1 - 1/q)^4$ $1/q \cdot (1 - 1/q)^2$

By Proposition 7.3, multiplying the probabilities by $q^{\ell(w)}$, we get $R^{\pi,w}(q) = (q-1)^4 + q(q-1)^2$. On the other hand, applying row operations, we find that $\Pi^\circ_{\mathrm{id}_M}$ consists of (row spans of) matrices $\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$ such that $a \neq 0$, $d \neq 0$, and $ad \neq bc$. The number of such matrices over $\mathbb{F}_q$ equals $(q - 1)^4 + q(q - 1)^2$, as expected.

Recall from Section 1.2 that we have defined the quantities $\mathbb{H} := \mathbb{H}^{M,N}_\pi$ and $\mathbb{V} := \mathbb{V}^{M,N}_\pi$ associated to $\pi \in S_n$. We see that if $\mathbb{H} = \{(l_1, r_1), \ldots, (l_h, r_h)\}$ and $\mathbb{V} = \{(d_1, u_1), \ldots, (d_v, u_v)\}$ then $f := f_{\pi,w} = \pi\mathrm{id}_M$ satisfies the following conditions:

(7.3) $$\{(i, f(i)) \mid i, f(i) \in [1, N]\} = \{(l_1 - M, r_1), \ldots, (l_h - M, r_h)\};$$

(7.4) $$\{(i, f(i)) \mid i \in [1 - M, 0] \text{ and } f(i) \in [N + 1, n]\} = \{(d_1 - M, u_1), \ldots, (d_v - M, u_v)\}.$$

Denote the left and right hand sides of (7.3) (resp., (7.4)) by $\widetilde{\mathbb{H}}_f$ and $\widetilde{\mathbb{H}}$ (resp., $\widetilde{\mathbb{V}}_f$ and $\widetilde{\mathbb{V}}$).

Let $X \in \mathrm{Gr}(M, n)$ be the row span of a matrix A and consider the pair (π_X, w_X) such that $f_X = f_{\pi_X, w_X}$. It is not hard to see that $w_X = w^{M,N}$ if and only if the vectors $A_{N+1}, A_{N+2}, \ldots, A_n$ form a basis of $\mathbb{C}^M$. We denote by $\Omega_{[N+1,n]} \subseteq \mathrm{Gr}(M, n)$ the set of (row spans of) such matrices A, which is usually called an *opposite Schubert cell*. We let

$$\Pi^{\mathbb{H},\mathbb{V}} := \{X \in \Omega_{[N+1,n]} \mid \widetilde{\mathbb{H}}_{f_X} = \widetilde{\mathbb{H}} \text{ and } \widetilde{\mathbb{V}}_{f_X} = \widetilde{\mathbb{V}}\}.$$

Finally, define the map $\mathrm{rev}_{[1,N]} : \mathrm{Gr}(M, n) \to \mathrm{Gr}(M, n)$ sending (the row span of) a matrix with columns $A_1, A_2, \ldots, A_n$ to (the row span of) the matrix with columns

$$A_N, A_{N-1}, \ldots, A_1, A_{N+1}, A_{N+2}, \ldots, A_n.$$

The $\mathbf{y} \to 0$ limit of Theorem 1.1 states that the varieties $\Pi^{\mathbb{H},\mathbb{V}}$ and $\Pi^{180^\circ(\mathbb{H}),\mathbb{V}}$ contain the same number of $\mathbb{F}_q$-points. In fact, a much stronger statement holds.

Proposition 7.5. *For all $\mathbb{H}, \mathbb{V}$, the map $\mathrm{rev}_{[1,N]}$ gives an isomorphism of varieties:*

$$\mathrm{rev}_{[1,N]} : \Pi^{\mathbb{H},\mathbb{V}} \xrightarrow{\sim} \Pi^{180^\circ(\mathbb{H}),\mathbb{V}}.$$

Proof. Let $X \in \Pi^{\mathbb{H},\mathbb{V}}$ be the row span of A, $f := f_X$, $Y := \mathrm{rev}_{[1,N]}(X)$, $g := f_Y$. Clearly, $Y \in \Omega_{[N+1,n]}$. By (7.2), we see that if $f(i) = j$ then A_j appears with a nonzero coefficient in the expansion of A_i as a linear combination of $A_{i+1}, \ldots, A_j$. In particular, $\max\{i' \leq j \mid A_j \in \mathrm{Span}(A_{j-1}, A_{j-2}, \ldots, A_{i'})\}$ is equal to i. This shows that $\widetilde{\mathbb{H}}_g = \widetilde{180^\circ(\mathbb{H})}$. Since $\mathrm{Span}(A_1, \ldots, A_N) = \mathrm{Span}(A_N, \ldots, A_1)$, we get $\widetilde{\mathbb{V}}_g = \widetilde{\mathbb{V}}$, thus indeed $Y \in \Pi^{180^\circ(\mathbb{H}),\mathbb{V}}$. $\square$

7.3. **Common generalizations?** The main message suggested by the above observations is the following. The $\mathbf{y} = 0$ specialization of the SC6V model recovers well studied objects such as R-polynomials, which are "shadows" of geometric objects such as positroid varieties. The latter can be parametrized by other combinatorial objects such as planar bipartite graphs of [Pos06].

Question 7.6. Does there exist a common generalization of the stochastic colored six-vertex model in a rectangle (with arbitrary $\mathbf{x}, \mathbf{y}$) and positroid varieties in the Grassmannian?

For example, it would be interesting to give an interpretation of the SC6V model in terms of *planar bipartite graphs* of [Pos06]. Surprisingly, the recurrence (3.2)–(3.4) used in the proof of Theorem 1.1 has already appeared in the literature precisely in the language of such graphs. We thank Thomas Lam for pointing out the following remark to us.

Remark 7.7. The Grassmannian lift of the $\mathbf{y} = 0$ specialization of (3.2)–(3.4) was used in [MS16, Section 4] to show local acyclicity of cluster algebras associated to Postnikov's planar bipartite graphs. Applying either recurrence to R-polynomials and distinguished subexpressions leads to unexpected phenomena related to rational Catalan combinatorics which will be explored in future work with Thomas Lam.

The above local acyclicity property was used in [GL19] to show that positroid varieties are cluster varieties, which allows one to study their cohomology via the associated *mixed Hodge tables*, see [LS16]. The number of points over $\mathbb{F}_q$ can be calculated from the mixed Hodge table, thus one possible direction is to understand the relation between the SC6V model and the cohomology of positroid varieties.

Kazhdan–Lusztig R-polynomials arise in several other contexts. For example, they are used to express *Kazhdan–Lusztig P-polynomials* [KL79, KL80], which have nonnegative integer coefficients and are of great interest in representation theory and algebraic geometry. One can define P-polynomials through an involution on $\mathcal{H}_q(S_n; \mathbf{z})$ that is very similar (but different) from the one used in the proof of Theorem 2.4. We thank Pavlo Pylyavskyy for discussions related to the following question.

Question 7.8. Determine the relation between Kazhdan–Lusztig P-polynomials and the SC6V model. For example, is there a probabilistic interpretation of their nonnegative integer coefficients?

By Proposition 2.3, the SC6V model is literally equivalent to the expansion of the Yang–Baxter basis $\{Y^w\}$ in the $\{T_\pi\}$ basis. The Yang–Baxter basis has several interesting properties. For example, the *orthogonality relations*, discovered already in [LLT97, Theorem 5.1], are yet to be understood from the SC6V model point of view.

One other direction that we think is worth exploring is to understand the appearance of pipe dreams both in the SC6V model dynamics and in Schubert calculus. There are several classes of pipe dreams arising in Schubert calculus, for example, ordinary pipe dreams [BB93, FK96, KM05] or *bumpless* pipe dreams recently introduced in [LLS18]. One may consider *reduced* pipe dreams (where two paths can intersect at most once) or *non-reduced* pipe dreams which appear e.g. in the study of Grothendieck polynomials [FK94, LRS06, Las02].

Question 7.9. Determine which of the above classes of pipe dreams are related to the SC6V model or its specializations.

An example of a relation like this between *bumpless pipe dreams* [LLS18] and the six-vertex model was recently pointed out by Anna Weigandt [Wei20]. These objects are closely related to alternating sign matrices [Las02, Kup96]. See also [BBBG19, BSW19] for related work.

7.4. **Counting pipe dreams.** We give a curious enumerative specialization of Theorem 1.1. The number of configurations of the SC6V model inside an $M \times N$-rectangular domain is equal to 2^{MN}, and for each configuration Π, we denote the associated color permutation by π_Π (as in Section 4.3). Given $\mathbb{H}, \mathbb{V}$, we let $\mathcal{PD}^{\mathbb{H},\mathbb{V}} := \{\Pi \mid \mathbb{H}^{M,N}_{\pi_\Pi} = \mathbb{H} \text{ and } \mathbb{V}^{M,N}_{\pi_\Pi} = \mathbb{V}\}$.

Corollary 7.10. *For any $\mathbb{H}, \mathbb{V}$, we have*

$$(7.5) \qquad \#\mathcal{PD}^{\mathbb{H},\mathbb{V}} = \#\mathcal{PD}^{180°(\mathbb{H}),\mathbb{V}}.$$

Proof. For a variable t, denote $y := \frac{1-qt}{1-t}x$ and substitute $y_j := y$ and $x_i := x$ for all $i \in [1, M]$ and $j \in [1, N]$. Then each parameter $\mathfrak{p}_{i,j}$ given by (1.1) becomes equal to t. For a pipe dream Π, let $\mathrm{xing}(\Pi)$ denote the number of cells of Π that contain a crossing, thus $0 \leq \mathrm{xing}(\Pi) \leq MN$. Applying Theorem 1.1 and sending $q \to 1$, we get

$$(7.6) \qquad \sum_{\Pi \in \mathcal{PD}^{\mathbb{H},\mathbb{V}}} t^{\mathrm{xing}(\Pi)}(1-t)^{MN-\mathrm{xing}(\Pi)} = \sum_{\Pi \in \mathcal{PD}^{180°(\mathbb{H}),\mathbb{V}}} t^{\mathrm{xing}(\Pi)}(1-t)^{MN-\mathrm{xing}(\Pi)}.$$

Substituting $t = \frac{1}{2}$, the result follows. $\qquad\qquad\square$

Remark 7.11. Dividing both sides of (7.6) by $(1-t)^{MN}$ and replacing $\frac{t}{1-t}$ with q yields a q-analog of (7.5):

$$(7.7) \qquad \sum_{\Pi \in \mathcal{PD}^{\mathbb{H},\mathbb{V}}} q^{\mathrm{xing}(\Pi)} = \sum_{\Pi \in \mathcal{PD}^{180°(\mathbb{H}),\mathbb{V}}} q^{\mathrm{xing}(\Pi)}.$$

Problem 7.12. Give a non-recursive bijective proof of (7.5) or (7.7).

7.5. **Arbitrary permutations.** We finish with a discussion of more general wiring diagram domains associated to arbitrary Yang–Baxter basis elements.

The statement of Theorem 1.6 extends perfectly well to arbitrary $w \in S_n$: one may consider two families of height functions $(\mathrm{Ht}(\mathfrak{i}_1, \mathfrak{j}_1), \ldots, \mathrm{Ht}(\mathfrak{i}_m, \mathfrak{j}_m))$, $(\mathrm{Ht}(\mathfrak{i}'_1, \mathfrak{j}'_1), \ldots, \mathrm{Ht}(\mathfrak{i}'_m, \mathfrak{j}'_m))$ defined in (1.3). They are considered as random vectors with respect to the probability distribution arising from the SC6V model associated with Y^w. Suppose that a permutation $\mathbf{z}' = (z'_1, \ldots, z'_n)$ of the variables in $\mathbf{z}$ is such that all marginal distributions are preserved:

$$\mathrm{Ht}(\mathfrak{i}_k, \mathfrak{j}_k) \overset{d}{=} \mathrm{Ht}(\mathfrak{i}'_k, \mathfrak{j}'_k) \mid_{\mathbf{z} \mapsto \mathbf{z}'} \quad \text{for all } k \in [1, m].$$

Does this imply that

$$(\mathrm{Ht}(\mathfrak{i}_1, \mathfrak{j}_1), \ldots, \mathrm{Ht}(\mathfrak{i}_m, \mathfrak{j}_m)) \overset{d}{=} (\mathrm{Ht}(\mathfrak{i}'_1, \mathfrak{j}'_1), \ldots, \mathrm{Ht}(\mathfrak{i}'_m, \mathfrak{j}'_m)) \mid_{\mathbf{z} \mapsto \mathbf{z}'}?$$

When w is associated with a skew domain as in Section 2.4 (such permutations are called *fully commutative*), the answer is positive since this is precisely the subject of Theorem 1.6. For general w, we give a counterexample.

Example 7.13. Let $n = 4$, $w = s_3 s_2 s_3 = s_2 s_3 s_2$ and consider height functions $\mathrm{Ht}(2, 2)$ and $\mathrm{Ht}(3, 3)$. Suppose $\pi \in S_n$ has nonzero probability (i.e., $\pi \leq w$ in the Bruhat order). Then

$$\mathrm{Ht}_\pi(2, 2) = \begin{cases} 0, & \text{if } \pi(2) = 2, \\ 1, & \text{otherwise;} \end{cases} \quad \mathrm{Ht}_\pi(3, 3) = \begin{cases} 0, & \text{if } \pi(4) = 4, \\ 1, & \text{otherwise.} \end{cases}$$

Considering the wiring diagram associated with the reduced word $w = s_3 s_2 s_3$, we see that the probability that $\mathrm{Ht}(2, 2) = 0$ is equal to $1 - \mathfrak{p}_{2,4}$. If we instead use the Yang–Baxter relation and consider the reduced word $w = s_2 s_3 s_2$, we similarly find that the probability

that $\mathrm{Ht}(3,3) = 0$ is also equal to $1 - \mathfrak{p}_{2,4}$. In particular, the distributions of $\mathrm{Ht}(2,2)$ and $\mathrm{Ht}(3,3)$ are the same and are invariant under swapping z_1 and z_3. However, their joint distribution is not invariant under this substitution: for example, the only permutation π satisfying $\mathrm{Ht}_\pi(2,2) = 0$ and $\mathrm{Ht}_\pi(3,3) = 1$ is the simple transposition $\pi = s_3$ which appears with probability $(1 - \mathfrak{p}_{2,3})(1 - \mathfrak{p}_{3,4})\mathfrak{p}_{2,4}$. This quantity depends on z_3 but not on z_1 so it is not invariant under swapping these two variables. Therefore Theorem 1.6 does not extend to arbitrary permutations w.

Despite this example, we can still give a conjectural generalization of the shift-invariance results of [BGW19] to arbitrary w. Fix $\alpha, \delta \in [1, n]$, and for each $\pi \in S_n$, define

$$\mathring{\mathbb{H}}_\pi^{\alpha,\delta} := \{(i, \pi(i)) \mid i > \alpha \text{ and } \pi(i) < \delta\}, \quad \mathring{\mathbb{V}}_\pi^{\alpha,\delta} := \{(i, \pi(i)) \mid i < \alpha \text{ and } \pi(i) > \delta\}.$$

This differs from $\mathbb{H}_\pi^{\alpha,\delta}$ and $\mathbb{V}_\pi^{\alpha,\delta}$ defined in (3.9) by passing from weak to strong inequalities.

Fix two sets $\mathbb{H}, \mathbb{V}$ of pairs and denote $\mathring{\mathrm{SAT}}_{\alpha,\delta}(\mathbb{H}, \mathbb{V}) := \{\pi \in S_n \mid \mathring{\mathbb{H}}_\pi^{\alpha,\delta} = \mathbb{H}, \mathring{\mathbb{V}}_\pi^{\alpha,\delta} = \mathbb{V}\}$. Also let us fix $w \in S_n$ and consider the probability distribution associated with Y^w. We let $\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}$ denote the probability that $\pi \in \mathring{\mathrm{SAT}}_{\alpha,\delta}(\mathbb{H}, \mathbb{V})$. We further introduce probabilities $\mathring{\mathbf{P}}_{\alpha\nearrow}^{\mathbb{H,V}}, \mathring{\mathbf{P}}_{\alpha\rightarrow}^{\mathbb{H,V}}, \mathring{\mathbf{P}}_{\alpha\searrow}^{\mathbb{H,V}}$ that π belongs to $\mathring{\mathrm{SAT}}_{\alpha,\delta}(\mathbb{H}, \mathbb{V})$ and satisfies, respectively, $\pi(\alpha) > \delta$, $\pi(\alpha) = \delta$, or $\pi(\alpha) < \delta$. Similarly, we let $\mathring{\mathbf{P}}_{\searrow\delta}^{\mathbb{H,V}}, \mathring{\mathbf{P}}_{\rightarrow\delta}^{\mathbb{H,V}}, \mathring{\mathbf{P}}_{\nearrow\delta}^{\mathbb{H,V}}$ denote the probabilities that π belongs to $\mathring{\mathrm{SAT}}_{\alpha,\delta}(\mathbb{H}, \mathbb{V})$ and satisfies, respectively, $\pi^{-1}(\delta) > \alpha$, $\pi^{-1}(\delta) = \alpha$, or $\pi^{-1}(\delta) < \alpha$. The probabilities $\mathring{\mathbf{P}}_{\alpha\rightarrow}^{\mathbb{H,V}}$ and $\mathring{\mathbf{P}}_{\rightarrow\delta}^{\mathbb{H,V}}$ are equal and denoted by $\mathring{\mathbf{P}}_{\alpha\rightarrow\delta}^{\mathbb{H,V}}$. We have

$$\mathring{\mathbf{P}}_{\alpha\nearrow}^{\mathbb{H,V}} + \mathring{\mathbf{P}}_{\alpha\rightarrow\delta}^{\mathbb{H,V}} + \mathring{\mathbf{P}}_{\alpha\searrow}^{\mathbb{H,V}} = \mathring{\mathbf{P}}_{\searrow\delta}^{\mathbb{H,V}} + \mathring{\mathbf{P}}_{\alpha\rightarrow\delta}^{\mathbb{H,V}} + \mathring{\mathbf{P}}_{\nearrow\delta}^{\mathbb{H,V}} = \mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}.$$

Let $\beta := w^{-1}(\delta)$ and $\gamma := w(\alpha)$. We say that $\{\alpha, \beta\}$ is an *inversion of w* if either $\alpha < \beta$ and $\gamma > \delta$ or $\alpha > \beta$ and $\gamma < \delta$. (In particular, $\{\alpha, \beta\}$ is not an inversion when $\alpha = \beta$.) The following conjecture has been verified for $n \leq 6$.

Conjecture 7.14. Let $\alpha, \delta \in [1, n]$, $w \in S_n$, and $\beta := w^{-1}(\delta)$ be such that $\{\alpha, \beta\}$ is not an inversion of w. Then for any $\mathbb{H}, \mathbb{V}$, we have

$$(7.8) \quad \frac{\mathring{\mathbf{P}}_{\alpha\nearrow}^{\mathbb{H,V}}}{\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}} = \frac{\mathring{\mathbf{P}}_{\nearrow\delta}^{\mathbb{H,V}}}{\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}}\Bigg|_{z_\alpha \leftrightarrow z_\beta}, \quad \frac{\mathring{\mathbf{P}}_{\alpha\rightarrow\delta}^{\mathbb{H,V}}}{\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}} = \frac{\mathring{\mathbf{P}}_{\alpha\rightarrow\delta}^{\mathbb{H,V}}}{\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}}\Bigg|_{z_\alpha \leftrightarrow z_\beta}, \quad \text{and} \quad \frac{\mathring{\mathbf{P}}_{\alpha\searrow}^{\mathbb{H,V}}}{\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}} = \frac{\mathring{\mathbf{P}}_{\searrow\delta}^{\mathbb{H,V}}}{\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}}\Bigg|_{z_\alpha \leftrightarrow z_\beta}.$$

Remark 7.15. Both sides of (7.8) can be interpreted as *conditional* probabilities and can be restated in the language of height functions. If the denominator $\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}$ is symmetric in z_α, z_β then the equalities in (7.8) hold for just the numerators. This is the case for skew domains, and therefore it is straightforward to check that Conjecture 7.14 implies [BGW19, Theorems 1.2 and 4.13]. However, $\mathring{\mathbf{P}}_{\alpha\blacksquare\delta}^{\mathbb{H,V}}$ is not symmetric in z_α, z_β for other choices of w as Example 7.17 demonstrates.

Remark 7.16. When $\{\alpha, \beta\}$ is an inversion of w, Conjecture 7.14 does not hold: in the limit regime where $z_1 \gg \cdots \gg z_n$, the $\mathfrak{p}$-parameters become equal to $1/q$, and then sending $q \to 1$, we obtain T_w as the limit of Y^w. In this case, w appears as the color permutation with probability 1. If, say, $\alpha > \beta$ then we get $\mathring{\mathbf{P}}_{\alpha\searrow}^{\mathbb{H,V}} = \mathring{\mathbf{P}}_{\nearrow\delta}^{\mathbb{H,V}} = 1$ while $\mathring{\mathbf{P}}_{\alpha\nearrow}^{\mathbb{H,V}} = \mathring{\mathbf{P}}_{\searrow\delta}^{\mathbb{H,V}} = 0$, violating the first and the third equalities in (7.8).
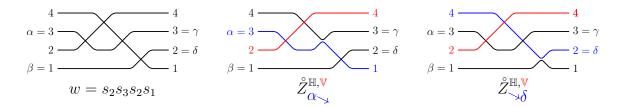
FIGURE 17. In this case, Conjecture 7.14 does not hold without the denominators, see Example 7.17.

Example 7.17. Let $n = 4$, $w = s_2 s_3 s_2 s_1$, $\alpha = 3$, $\beta = 1$, $\gamma = 3$, $\delta = 2$, $\mathbb{H} = \emptyset$, and $\mathbb{V} = \{(2,4)\}$, see Figure 17. For such $\mathbb{H}$ and $\mathbb{V}$, we find $\mathring{\mathbf{P}}^{\mathbb{H},\mathbb{V}}_{\alpha \blacksquare \delta} = \mathfrak{p}_{23}\mathfrak{p}_{24}(1 - \mathfrak{p}_{34}\mathfrak{p}_{14})$. The permutations contributing to $\mathring{\mathbf{P}}^{\mathbb{H},\mathbb{V}}_{\alpha \searrow}$ and $\mathring{\mathbf{P}}^{\mathbb{H},\mathbb{V}}_{\searrow \delta}$ are shown in Figure 17, so (7.8) reads $\frac{\mathfrak{p}_{23}\mathfrak{p}_{24}(1-\mathfrak{p}_{34})\mathfrak{p}_{14}}{\mathfrak{p}_{23}\mathfrak{p}_{24}(1-\mathfrak{p}_{34}\mathfrak{p}_{14})} = \frac{\mathfrak{p}_{23}\mathfrak{p}_{24}\mathfrak{p}_{34}(1-\mathfrak{p}_{14})}{\mathfrak{p}_{23}\mathfrak{p}_{24}(1-\mathfrak{p}_{34}\mathfrak{p}_{14})}\big|_{z_1 \leftrightarrow z_3}$. This identity is true, but would be false without the denominators.

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