TOTALLY NONNEGATIVE CRITICAL VARIETIES

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Abstract. We study totally nonnegative parts of critical varieties in the Grassmannian. We show that each totally nonnegative critical variety $\mathrm{Crit}_f^{\geqslant 0}$ is the image of an affine poset cyclohedron under a continuous map and use this map to define a boundary stratification of $\mathrm{Crit}_f^{\geqslant 0}$. For the case of the top-dimensional positroid cell, we show that the totally nonnegative critical variety $\mathrm{Crit}_{k,n}^{\geqslant 0}$ is homeomorphic to the second hypersimplex $\Delta_{2,n}$.

Introduction

The *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geqslant 0}(k,n)$ is a certain subset of the real Grassmannian introduced in [Pos06, Lus94, Lus98]. Recent years have revealed a variety of surprising connections between statistical mechanics and the structure of $\mathrm{Gr}_{\geqslant 0}(k,n)$; see e.g. [CW11, Lam18, GP20]. In a recent paper [Gal21a], we introduced *critical varieties* inside the Grassmannian, which may be considered “critical parts” of *positroid varieties* introduced in [KLS13]. The construction of critical varieties is based on Kenyon’s critical dimer model [Ken02] and simultaneously includes the embeddings of the critical Ising model and critical electrical networks into $\mathrm{Gr}_{\geqslant 0}(k,n)$ discovered in [Lam18, GP20].

Our aim in [Gal21a] was to develop a theory of critical varieties which would parallel the theory of positroid varieties. For example, we introduced complex-algebraic *open critical varieties* Crit_f° as well as their totally positive parts $\mathrm{Crit}_f^{>0}$ called *critical cells*. The goal of the present paper is to continue this program and study *totally nonnegative critical varieties* $\mathrm{Crit}_f^{\geqslant 0}$, defined as closures of critical cells $\mathrm{Crit}_f^{>0}$ inside $\mathrm{Gr}_{\geqslant 0}(k,n)$.

While investigating the structure of the spaces $\mathrm{Crit}_f^{\geqslant 0}$, we were led to consider several new families of polytopes generalizing *order polytopes* [Sta86], *associahedra* [Tam51, Sta63, Hai84, Lee89], and *cyclohedra* [BT94, Sim03]. We introduced *poset associahedra* and *affine poset cyclohedra* and explored their properties in [Gal21b]. An important result from the point of view of applications to critical varieties is that these polytopes arise as compactifications of certain configuration spaces of points on a line and on a circle, analogously to the cases of associahedra and cyclohedra [AS94, Sin04, LTV10].

The goal of this paper is to prove two results on totally nonnegative critical varieties $\mathrm{Crit}_f^{\geqslant 0}$. First, we show that each space $\mathrm{Crit}_f^{\geqslant 0}$ is the image of an affine poset cyclohedron under a surjective continuous map. This observation, which may be considered an analog of the results of [PSW09], allows us to introduce a boundary stratification of $\mathrm{Crit}_f^{\geqslant 0}$. (Unlike in the case of positroid cells, the boundary stratification of $\mathrm{Crit}_f^{\geqslant 0}$ is not merely obtained

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by intersecting $\mathrm{Crit}_f^{\geqslant 0}$ with various positroid cells; see Example 1.2.) Next, we concentrate on the special case of the totally nonnegative critical variety $\mathrm{Crit}_{k,n}^{\geqslant 0}$ corresponding to the *top-dimensional positroid cell* inside $\mathrm{Gr}_{\geqslant 0}(k,n)$. We show that $\mathrm{Crit}_{k,n}^{\geqslant 0}$ is homeomorphic to a polytope, namely, to the *second hypersimplex* $\Delta_{2,n}$, via a stratification-preserving homeomorphism.

As a surprising consequence, we see that $\mathrm{Crit}_{k,n}^{\geqslant 0}$ does not depend on k as a stratified space. We view this result as a step towards constructing a family of conjectural *shift maps* $\mathrm{Gr}(k,n) \dashrightarrow \mathrm{Gr}(k+1,n)$, which should restrict to homeomorphisms $\mathrm{Crit}_{k,n}^{\geqslant 0} \xrightarrow{\sim} \mathrm{Crit}_{k+1,n}^{\geqslant 0}$. Constructing such shift maps is of great importance in relation to physics and statistical mechanics. For example, it would yield a connection between electrical networks and the Ising model (see [GP20, Question 9.2]) as well as provide insight into the construction of the *BCFW triangulation* [BCFW05] of the *amplituhedron* [AHT14]; see [AHBC+16, LPW20, GL20] and [Gal21a, Section 8] for context and related results.

Recall that the totally nonnegative parts of positroid varieties, while not being isomorphic to polytopes as stratified spaces, have remarkably simple topological structure [Wil07, PSW09, RW08, RW10, GKL17, GKL19]. It remains an open problem to determine whether each totally nonnegative critical variety $\mathrm{Crit}_f^{\geqslant 0}$ is isomorphic to a polytope as a stratified space.

1. Main results

We give a brief overview of some of our results. The full statements and proofs are given in the main body of the paper.

Let G be a planar graph embedded in a disk. We assume that G has n black degree 1 boundary vertices labeled $b_1, b_2, \dots, b_n$ in clockwise order; see Figure 1(a). A *strand* in G is a path that makes a sharp right (resp., left) turn at each black (resp., white) vertex; see Figure 1(b). For each $p \in [n] := \{1, 2, \dots, n\}$, if a strand starts at the boundary vertex b_p, it must terminate at some boundary vertex $b_{f_G(p)}$. The resulting permutation $f_G \in S_n$ is called the *strand permutation* of G. We say that G is *reduced* [Pos06] if it has the minimal number of faces among all graphs with strand permutation f_G.

For $0 \leqslant k \leqslant n$, the *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geqslant 0}(k,n)$ is the subset of the real Grassmannian $\mathrm{Gr}(k,n)$ where all Plücker coordinates have the same sign; see Section 2.1 for further background. To a weight function $\mathrm{wt} : E(G) \to \mathbb{R}_{>0}$ defined on the edges of G, Postnikov [Pos06] associates a point $\mathrm{Meas}_G(\mathrm{wt}) \in \mathrm{Gr}_{\geqslant 0}(k,n)$, where $0 \leqslant k \leqslant n$ depends only on G.

In order to define a *critical cell* $\mathrm{Crit}_G^{>0}$, we restrict to a special family of weight functions coming from the *critical dimer model* of [Ken02]. We will always assume that G is reduced, in which case the critical cell $\mathrm{Crit}_G^{>0}$ depends only on the strand permutation of G and is denoted $\mathrm{Crit}_{f_G}^{>0}$.

For the purposes of this introduction, we consider the most important special case of the *top cell* strand permutation $f_{k,n}$. By definition, $f_{k,n} \in S_n$ sends $p \mapsto p+k$ modulo n, for all $p \in [n]$. Let $\Theta_{k,n}^{>0}$ be the space of n-tuples $\boldsymbol{v} := (v_1, v_2, \dots, v_n) \in \mathbb{C}^n$ of distinct points ordered counterclockwise on the unit circle, considered modulo global rotations of the circle.

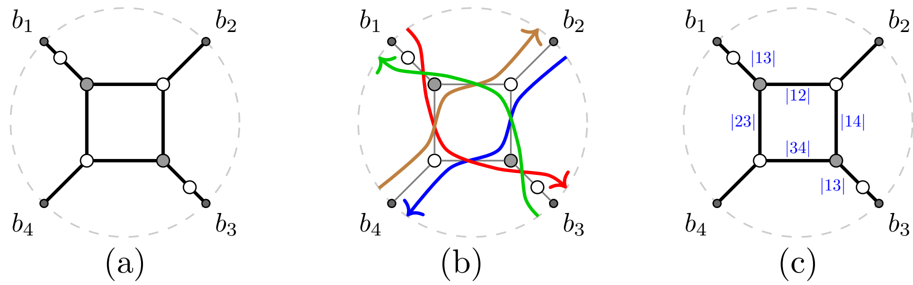


FIGURE 1. (a) A (reduced) planar bipartite graph G; (b) strands in G; (c) edge weights $\mathrm{wt}_{\boldsymbol{v}}$, where the unmarked edges have weight 1 and we abbreviate $|pq| := |v_p - v_q|$. Figure reproduced from [Gal21a, Figure 1].

Remark 1.1. The space $\Theta_{k,n}^{>0}$ is naturally homeomorphic to the interior of an $(n-1)$-dimensional simplex

$$\Theta_{k,n}^{>0} \cong \{\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n \mid 0 = \theta_1 < \theta_2 < \cdots < \theta_n < \pi\},$$

by setting $v_r := \exp(2i\theta_r)$ for all $r \in [n]$. (In particular, $\Theta_{k,n}^{>0}$ does not depend on k.)

Every edge e of G belongs to exactly two strands. Denoting the endpoints of these strands by b_p, b_q for $p, q \in [n]$, we say that e is *labeled by* $\{p, q\}$. In this case, we define its weight by

$$\mathrm{wt}_{\boldsymbol{v}}(e) := \begin{cases} |v_p - v_q|, & \text{if } e \text{ is not incident to a boundary vertex;} \\ 1, & \text{otherwise.} \end{cases} \tag{1.1}$$

We obtain a weight function $\mathrm{wt}_{\boldsymbol{v}} : E(G) \to \mathbb{R}_{>0}$. See Figure 1(c) for an example. It turns out that the resulting point $\mathrm{Meas}_G(\mathrm{wt}_{\boldsymbol{v}}) \in \mathrm{Gr}_{\geqslant 0}(k, n)$ does not depend on the choice of G. We denote $\mathrm{Meas}_{k,n}(\boldsymbol{v}) := \mathrm{Meas}_G(\mathrm{wt}_{\boldsymbol{v}})$. The *critical cell* $\mathrm{Crit}_{k,n}^{>0}$ is defined as

$$\mathrm{Crit}_{k,n}^{>0} := \{\mathrm{Meas}_{k,n}(\boldsymbol{v}) \mid \boldsymbol{v} \in \Theta_{k,n}^{>0}\}.$$

Throughout, we assume that $2 \leqslant k \leqslant n-1$. (For $k = 1$ or $k = n$, $\mathrm{Crit}_{k,n}^{>0}$ is a single point.) According to [Gal21a, Theorem 1.10], the map $\mathrm{Meas}_{k,n}$ restricts to a homeomorphism $\Theta_{k,n}^{>0} \xrightarrow{\sim} \mathrm{Crit}_{k,n}^{>0}$, and thus $\mathrm{Crit}_{k,n}^{>0}$ is homeomorphic to the interior of an $(n-1)$-simplex. Our goal is to study the *closure* $\mathrm{Crit}_{k,n}^{\geqslant 0}$ of $\mathrm{Crit}_{k,n}^{>0}$ inside $\mathrm{Gr}_{\geqslant 0}(k, n)$, and more generally, the closure $\mathrm{Crit}_f^{\geqslant 0}$ of an arbitrary critical cell $\mathrm{Crit}_f^{>0}$, $f \in S_n$.

Informally, since $\mathrm{Crit}_{k,n}^{>0}$ is parameterized by configurations of n distinct points on a circle, its closure $\mathrm{Crit}_{k,n}^{\geqslant 0}$ should be parameterized by n-point configurations where some points are allowed to collide. The map Meas_G is invariant under *gauge transformations*: given a weighted graph (G, wt), for each interior vertex u of G, rescaling the weights of all edges incident to u by the same nonzero scalar does not alter the image of wt under Meas_G. Modulo gauge transformations, $\mathrm{Meas}_{k,n}(\boldsymbol{v})$ depends only on the ratios of the distances between the points $v_1, v_2, \dots, v_n$. For instance, even if all points $v_1, v_2, \dots, v_n$ collide together, it could happen that after we apply gauge transformations at the vertices of G, in the limit *none* of the edge weights tend to zero, as the following example demonstrates.

Example 1.2. Consider the graph G in Figure 1, and suppose that v_1, v_2, v_3, v_4 collide in such a way that

$$(|v_2 - v_1| : |v_3 - v_2| : |v_3 - v_1| : |v_4 - v_3| : |v_4 - v_2| : |v_4 - v_1|) \to (a : b : a+b : c : b+c : a+b+c),$$

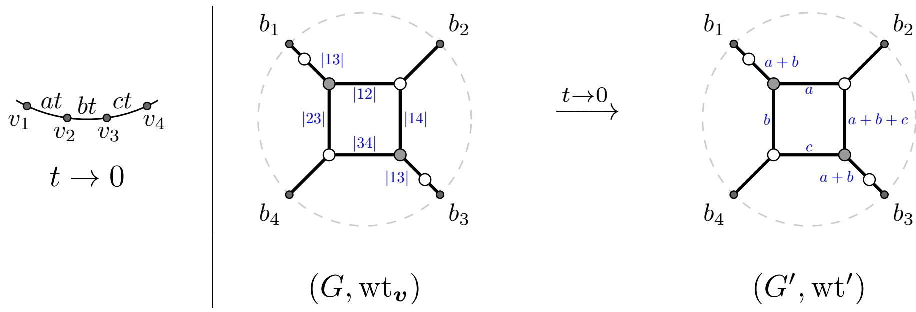


FIGURE 2. Taking a limit where all points in $\boldsymbol{v}$ collide. See Example 1.2.

for some constants $a, b, c > 0$; see Figure 2(left). After applying gauge transformations at the two black interior vertices of G and taking a limit, we obtain a weighted graph (G', wt') shown in Figure 2(right). The point $\mathrm{Meas}_{G'}(\mathrm{wt}')$ belongs to the *totally positive Grassmannian* (i.e., has all Plücker coordinates strictly positive). Yet, $\mathrm{Meas}_{G'}(\mathrm{wt}')$ belongs to the *boundary* of $\mathrm{Crit}_{2,4}^{\geqslant 0}$, i.e., to $\mathrm{Crit}_{2,4}^{\geqslant 0} \setminus \mathrm{Crit}_{2,4}^{>0}$.

A natural compactification of $\Theta_{k,n}^{>0}$ taking into account the ratios of distances between pairs of colliding points in $\boldsymbol{v}$ is the $(n-1)$-dimensional *cyclohedron* $\mathscr{C}_n$ studied in [BT94, Sim03]. See Section 3.2 for background. The cyclohedron $\mathscr{C}_n$ may be obtained as the *Axelrod–Singer* compactification [AS94] of $\Theta_{k,n}^{>0}$. In particular, the interior of $\mathscr{C}_n$ is identified with $\Theta_{k,n}^{>0}$.

Theorem 1.3. *The map* $\mathrm{Meas}_{k,n} : \Theta_{k,n}^{>0} \xrightarrow{\sim} \mathrm{Crit}_{k,n}^{>0}$ *extends to a continuous surjective map*

$$\overline{\mathrm{Meas}}_{k,n} : \mathscr{C}_n \to \mathrm{Crit}_{k,n}^{\geqslant 0}.$$

A similar result (Theorem 4.1) holds for arbitrary critical cells. Here, instead of the cyclohedron, one needs to take an *affine poset cyclohedron* introduced in [Gal21b]. For an arbitrary permutation $f \in S_n$, the critical cell $\mathrm{Crit}_f^{>0}$ is parameterized by a configuration space $\Theta_f^{>0}$ of n points on a circle where some points are allowed to pass through each other. To this data, we associate an *affine poset* $\tilde{P}_f$ such that the corresponding affine poset cyclohedron $\mathscr{C}(\tilde{P}_f)$ gives a suitable compactification of $\Theta_f^{>0}$. This allows us to extend the boundary measurement map $\mathrm{Meas}_f : \Theta_f^{>0} \to \mathrm{Crit}_f^{>0}$ to a surjective continuous map

$$\overline{\mathrm{Meas}}_f : \mathscr{C}(\tilde{P}_f) \to \mathrm{Crit}_f^{\geqslant 0}.$$

By considering images of different faces of $\mathscr{C}(\tilde{P}_f)$, we obtain a stratification of $\mathrm{Crit}_f^{\geqslant 0}$.

It turns out that the map $\overline{\mathrm{Meas}}_{k,n}$ is far from a homeomorphism. Instead, it has the following remarkable property, which we call *independence of infinitesimal ratios*. Suppose that $\boldsymbol{v}^{(t)} \in \mathrm{Crit}_{k,n}^{>0}$ is a sequence of point configurations converging to some $\boldsymbol{v} \in \mathscr{C}_n$ as $t \to 0$. Let $d^{(t)} := \max_{p,q \in [n]} |v_p^{(t)} - v_q^{(t)}|$. It turns out that for all p, q such that $\lim_{t\to 0} \frac{|v_p^{(t)} - v_q^{(t)}|}{d^{(t)}} = 0$, the limit $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{v})$ of $\mathrm{Meas}_{k,n}(\boldsymbol{v}^{(t)})$ does not depend on distance ratios involving $|v_p^{(t)} - v_q^{(t)}|$. This property is surprising since the limiting edge weight function wt' *does* depend on such distance ratios. However, the resulting limiting graph G' is not reduced in general, and after applying *reduction moves* (see Figure 4) to it, all such ratios miraculously cancel each other out.

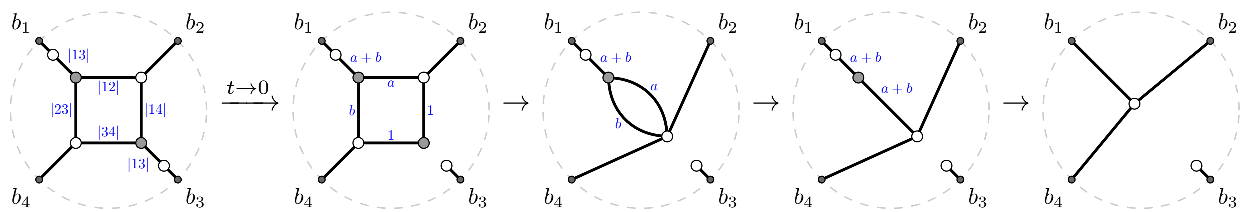


FIGURE 3. Taking a limit where the points v_1, v_2, v_3 collide but are far from v_4. After applying a sequence of reduction moves from Figure 4, the edge weights involving relative distances between v_1, v_2, v_3 cancel out. See Example 1.4 and Theorem 1.5.

Example 1.4. Let G be the graph in Figure 1, and suppose that v_1, v_2, v_3, v_4 collide so that

$$(|v_2 - v_1| : |v_3 - v_2| : |v_3 - v_1| : |v_4 - v_3| : |v_4 - v_2| : |v_4 - v_1|) \to (0 : 0 : 0 : 1 : 1 : 1),$$
$$(|v_2 - v_1| : |v_3 - v_2| : |v_3 - v_1|) \to (a : b : a + b),$$

for some constants $a, b > 0$. After applying gauge transformations and taking a limit, we obtain a weighted graph (G', wt') shown in Figure 3(middle left). Thus the edge weights wt' of G' depend on the ratio $a : b$ in a non-trivial fashion. The graph G' is not reduced, and after applying reduction moves to it, we see that all edge weights involving a and b cancel out; see Figure 3(right). Our result (Theorem 1.5) claims that this phenomenon occurs more generally for arbitrary k and n, and for an arbitrary choice of the limiting ratios of distances between the points in $\boldsymbol{v}$.

To explain independence of infinitesimal ratios formally, consider a map

$$\phi : \Theta^{>0}_{k,n} \to \mathbb{RP}^{n-1}, \quad \boldsymbol{v} \mapsto (|v_2 - v_1| : |v_3 - v_2| : \cdots : |v_n - v_{n-1}| : |v_1 - v_n|).$$

Passing to the closure, ϕ can be extended to a continuous map $\phi : \mathscr{C}_n \to \mathbb{RP}^{n-1}$. The image $\phi(\mathscr{C}_n)$ is essentially described by triangle inequalities, and it is straightforward to check (Proposition 5.6) that it may be identified with the *second hypersimplex*

$$\Delta_{2,n} := \{(x_1, x_2, \dots, x_n) \in [0,1]^n \mid x_1 + x_2 + \cdots + x_n = 2\}.$$

Theorem 1.5 (Independence of infinitesimal ratios)**.** *The map $\overline{\mathrm{Meas}}_{k,n}$ factors through the map ϕ. That is, there exists a continuous map*

$$\psi : \Delta_{2,n} \to \mathrm{Crit}^{\geqslant 0}_{k,n}$$

making the following diagram commutative:

$$\begin{tikzcd} \mathscr{C}_n \arrow[r, two heads, "\phi"'] \arrow[rr, bend left, two heads, "\overline{\mathrm{Meas}}_{k,n}"] & \Delta_{2,n} \arrow[r, dashed, "\psi"'] & \mathrm{Crit}^{\geqslant 0}_{k,n} \end{tikzcd} \tag{1.2}$$

Moreover, the map $\psi : \Delta_{2,n} \to \mathrm{Crit}^{\geqslant 0}_{k,n}$ is a stratification-preserving homeomorphism.

We note that currently we have no analog of Theorem 1.5 for other critical cells. First, independence of infinitesimal ratios is very special to the top cell, and does not appear to

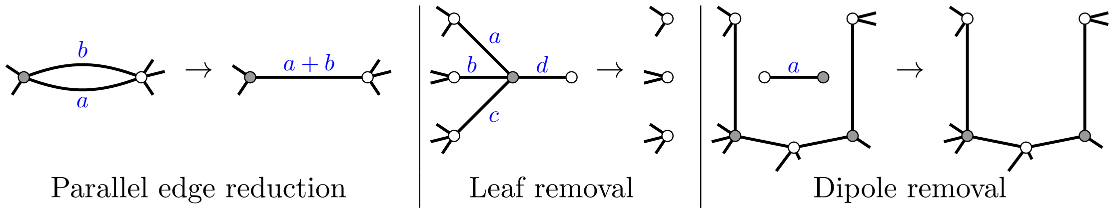


FIGURE 4. Reduction moves for planar bipartite graphs. Each move preserves the boundary measurements.

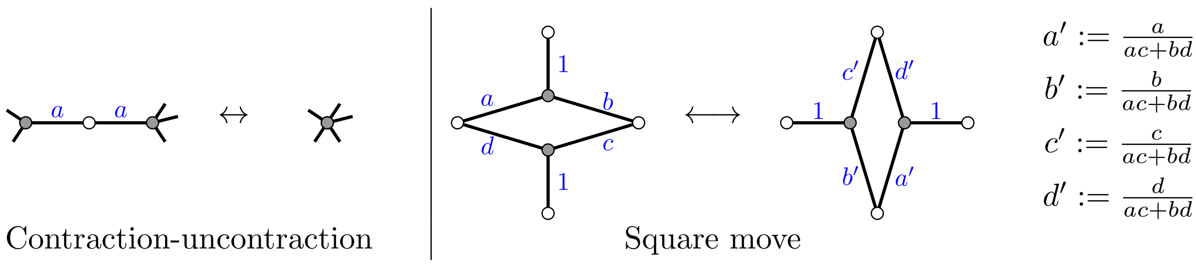


FIGURE 5. Moves for planar bipartite graphs preserving the boundary measurements and the strand permutation.

hold for lower cells. Second, showing that the map ψ is a homeomorphism relies on the *injectivity conjecture* [Gal21a, Conjecture 4.3] being true for a certain family of critical cells; see Section 5.4. This conjecture remains wide open for arbitrary critical cells.

2. Background on critical varieties

We review the background on *positroid cells* inside the *totally nonnegative Grassmannian* [Pos06]; see also [Lam16]. We then recall the construction of *critical cells* introduced in [Gal21a].

2.1. Planar bipartite graphs. Fix a planar graph G as in Section 1. Recall that the n boundary vertices of G are assumed to be black and to have degree 1, and that G is assumed to be reduced. Any non-reduced graph G may be transformed into a reduced one using the moves in Figures 4 and 5.

We switch to denoting strand permutations by $\bar{f}_G$, and reserve the notation f_G for *bounded affine permutations* introduced below.

Definition 2.1. A (k,n)-*bounded affine permutation* is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ such that

- $f(j+n) = f(j) + n$ for all $j \in \mathbb{Z}$,
- $\sum_{j=1}^{n}(f(j) - j) = kn$, and
- $j \leqslant f(j) \leqslant j + n$ for all $j \in \mathbb{Z}$.

We denote the set of (k,n)-bounded affine permutations by $\mathcal{B}(k,n)$. For $f \in \mathcal{B}(k,n)$, we let $\bar{f} \in S_n$ be obtained by reducing f modulo n. In other words, $\bar{f}$ is uniquely determined by the conditions $\bar{f}(j) \in [n]$ and $\bar{f}(j) \equiv f(j)$ modulo n for all $j \in [n]$.

Remark 2.2. We say that $f \in \mathcal{B}(k,n)$ is *loopless* if $f(j) \neq j$ for all $j \in \mathbb{Z}$. Each permutation $\bar{f} \in S_n$ arises via the above procedure from a unique loopless bounded affine permutation $f \in \mathcal{B}(k,n)$: for $j \in [n]$, one sets $f(j) := \bar{f}(j)$ if $\bar{f}(j) > j$ and $f(j) := \bar{f}(j) + n$ otherwise.

The remaining values $f(j+dn) = f(j)+dn$ are automatically determined for all $d \in \mathbb{Z}$. Positroid cells are labeled by arbitrary bounded affine permutations while critical cells are labeled by loopless bounded affine permutations, which is why in the introduction we used permutations in S_n to label critical cells.

In general, the bounded affine permutation f_G is recovered from $\bar{f}_G$ as follows. For $j \in [n]$, if $\bar{f}_G(j) \neq j$ then $f_G(j)$ is uniquely determined by the conditions $j \leqslant f_G(j) \leqslant j+n$ and $f_G(j) \equiv \bar{f}_G(j)$ modulo n. If $\bar{f}_G(j) = j$ then, depending on the structure of G (see Definition 2.3), either j is a *loop* (i.e., $f_G(j) = j$) or j is a *coloop* (i.e., $f_G(j) = j+n$).

An *affine inversion* of $f \in \mathcal{B}(k,n)$ is a pair $(p,q) \in \mathbb{Z}^2$ such that $p<q$ and $f(p) > f(q)$. The *length* $\ell(f)$ of f is the number of affine inversions of f considered modulo n:

$$\ell(f) := \#\{p, q \in \mathbb{Z} \mid p<q,\ f(p)>f(q),\ \text{and } p \in [n]\}.$$

The *(real) Grassmannian* $\mathrm{Gr}(k,n)$ is the set of all linear k-dimensional subspaces of $\mathbb{R}^n$. Choosing a basis of each subspace, $\mathrm{Gr}(k,n)$ may be identified with the space of full rank $k \times n$ matrices M considered modulo row operations. With this identification, one has a collection of *Plücker coordinates* on $\mathrm{Gr}(k,n)$. Let $\binom{[n]}{k}$ denote the set of k-element subsets of $[n]$, and for each $I \in \binom{[n]}{k}$ and a $k \times n$ matrix M we let $\Delta_I(M)$ denote the maximal minor of M with column set I. Letting I vary, we obtain the *Plücker embedding* $\mathrm{Gr}(k,n) \hookrightarrow \mathbb{RP}^{\binom{n}{k}-1}$ sending the row span of M to $(\Delta_I(M))_{I \in \binom{[n]}{k}} \in \mathbb{RP}^{\binom{n}{k}-1}$.

Let $\mathbb{RP}^{r-1}_{>0}$ be the subset of $\mathbb{RP}^{r-1}$ where all coordinates are nonzero and have the same sign, and let $\mathbb{RP}^{r-1}_{\geqslant 0}$ be the closure of $\mathbb{RP}^{r-1}_{>0}$. The *totally nonnegative Grassmannian* $\mathrm{Gr}_{\geqslant 0}(k,n)$ is the subset of $\mathrm{Gr}(k,n)$ where all nonzero Plücker coordinates have the same sign. In other words, $\mathrm{Gr}_{\geqslant 0}(k,n)$ is the preimage of $\mathbb{RP}^{\binom{n}{k}-1}_{\geqslant 0}$ under the Plücker embedding.

Given a planar bipartite graph G as above, the *boundary measurement map* $\mathrm{Meas}_G : \mathbb{R}^{E(G)}_{>0} \to \mathrm{Gr}_{\geqslant 0}(k,n)$ is defined using the *dimer model* on G. An *almost perfect matching* $\mathcal{A}$ of G is a collection of edges of G which uses each interior vertex exactly once. Importantly (cf. Lemma 4.2 below), in order to define the boundary measurement map Meas_G, we assume that G admits at least one almost perfect matching.

Recall that the boundary vertices of G are assumed to be black and have degree 1. For an almost perfect matching $\mathcal{A}$, let $\partial(\mathcal{A}) \subseteq [n]$ denote the set of $p \in [n]$ such that the boundary vertex b_p is used by $\mathcal{A}$. There is an integer $0 \leqslant k \leqslant n$ depending only on G such that $|\partial(\mathcal{A})| = k$ for any almost perfect matching $\mathcal{A}$ of G. Given an edge weight function $\mathrm{wt} : E(G) \to \mathbb{R}_{>0}$, the weight $\mathrm{wt}(\mathcal{A}) := \prod_{e \in \mathcal{A}} \mathrm{wt}(e)$ of $\mathcal{A}$ is the product of the weights of the edges used by $\mathcal{A}$. For $I \in \binom{[n]}{k}$, we set

$$\Delta_I(G, \mathrm{wt}) := \sum_{\mathcal{A}:\ \partial(\mathcal{A}) = I} \mathrm{wt}(\mathcal{A}).$$

We view the resulting boundary measurements

$$\mathrm{Meas}_G(\mathrm{wt}) := (\Delta_I(G, \mathrm{wt}))_{I \in \binom{[n]}{k}} \tag{2.1}$$

up to multiplication by a common scalar, i.e., as an element of $\mathbb{RP}^{\binom{n}{k}-1}$. It was shown in [Pos06, Tal08] (see [Lam18, Theorem 4.1]) that the entries of $\mathrm{Meas}_G(\mathrm{wt})$ are the Plücker coordinates of some point of $\mathrm{Gr}_{\geqslant 0}(k,n)$ which we also denote by $\mathrm{Meas}_G(\mathrm{wt})$.

Definition 2.3. It is known that when $\bar{f}_G(j) = j$, exactly one of the following holds:

- $j \notin \mathcal{A}$ for any almost perfect matching $\mathcal{A}$ of G;
- $j \in \mathcal{A}$ for any almost perfect matching $\mathcal{A}$ of G.

In the former case, we say that j is a *loop* and set $f_G(j) = j$. In the latter case, we say that j is a *coloop* and set $f_G(j) = j + n$. This completes the definition of the bounded affine permutation $f_G \in \mathcal{B}(k, n)$ associated to G. For $f \in \mathcal{B}(k, n)$, we let $\mathcal{G}_{\mathrm{red}}(f)$ denote the set of all reduced planar bipartite graphs G satisfying $f_G = f$. For $G \in \mathcal{G}_{\mathrm{red}}(f)$, the *positroid cell* $\Pi_G^{>0} := \{\mathrm{Meas}_G(\mathrm{wt}) \mid \mathrm{wt} : E(G) \to \mathbb{R}_{>0}\}$ depends only on f and is denoted $\Pi_f^{>0}$. The *top cell* bounded affine permutation $f_{k,n} \in \mathcal{B}(k, n)$ is defined by $f_{k,n}(p) = p + k$ for all $p \in \mathbb{Z}$.

2.2. **Critical cells.** Let $f \in \mathcal{B}(k, n)$ be a loopless bounded affine permutation and let $\bar{f} \in S_n$ be the corresponding permutation. The combinatorics of the critical cell $\mathrm{Crit}_f^{>0}$ associated to f is described by the following objects.

Definition 2.4. Place $2n$ points $b_1^-, b_1^+, \dots, b_n^-, b_n^+$ on the circle in clockwise order. The *reduced strand diagram* of f is obtained by drawing an arrow $b_s^+ \to b_{\bar{f}(s)}^-$ for each $s \in [n]$. We say that $p, q \in [n]$, $p \neq q$, *form an f-crossing* if the arrows $b_s^+ \to b_p^-$ and $b_t^+ \to b_q^-$ cross, where $s := \bar{f}^{-1}(p)$ and $t := \bar{f}^{-1}(q)$. We say that f has a *connected strand diagram* if the resulting union of n arrows is topologically connected. See Figure 6(left) for an example.

Throughout the paper, we assume that f has a connected strand diagram. When the strand diagram of f is not connected, the corresponding critical cell $\mathrm{Crit}_f^{>0}$ (as well as its closure $\mathrm{Crit}_f^{\geqslant 0}$) factorizes as a product over its connected components; see [Gal21a, Section 4.4].

Definition 2.5. A tuple $\boldsymbol{\theta} = (\theta_1, \theta_2, \dots, \theta_n) \in \mathbb{R}^n$ is called *f-admissible* if whenever two indices $1 \leqslant p < q \leqslant n$ form an f-crossing, we have

$$\theta_p < \theta_q < \theta_p + \pi. \tag{2.2}$$

We let

$$\Theta_f^{>0} := \{\boldsymbol{\theta} \in \mathbb{R}^n \mid \theta_1 = 0 \text{ and } \boldsymbol{\theta} \text{ is } f\text{-admissible}\}. \tag{2.3}$$

Letting $v_r := \exp(2i\theta_r)$ for $r \in [n]$, we obtain a configuration $\boldsymbol{v} = (v_1, v_2, \dots, v_n)$ of n points on the unit circle which are not necessarily distinct or ordered counterclockwise. The condition $\theta_1 = 0$ reflects that we consider these points modulo rotations of the circle.

A graph $G \in \mathcal{G}_{\mathrm{red}}(f)$ is called *contracted* if it has no degree 2 vertices that are not adjacent to the boundary. Any graph $G \in \mathcal{G}_{\mathrm{red}}(f)$ may be transformed into a contracted one using *contraction-uncontraction moves* (Figure 5(left)) which do not affect the boundary measurements of G.

Given a contracted graph $G \in \mathcal{G}_{\mathrm{red}}(f)$ and an f-admissible tuple $\boldsymbol{\theta} \in \Theta_f^{>0}$, we define a weight function $\mathrm{wt}_{\boldsymbol{\theta}} : E(G) \to \mathbb{R}_{>0}$ similarly to (1.1): if $e \in E(G)$ is labeled by $\{p, q\}$ with $1 \leqslant p < q \leqslant n$ then we set

$$\mathrm{wt}_{\boldsymbol{\theta}}(e) := \begin{cases} \sin(\theta_q - \theta_p), & \text{if } e \text{ is not incident to a boundary vertex;} \\ 1, & \text{otherwise.} \end{cases} \tag{2.4}$$

By [Gal21a, Proposition 4.2], we indeed get $\mathrm{wt}_{\boldsymbol{\theta}}(e) > 0$ for all $e \in E(G)$. Setting $v_r := \exp(2i\theta_r)$ for $r \in [n]$, we get $\sin(\theta_q - \theta_p) = \frac{1}{2}|v_q - v_p|$. Thus $\mathrm{wt}_{\boldsymbol{\theta}}$ differs from $\mathrm{wt}_{\boldsymbol{v}}$ defined in (1.1) by applying gauge transformations at all black interior vertices.

The crucial property of this assignment of edge weights is that the resulting boundary measurements are invariant under *square moves* (Figure 5(right)). Thus it follows from the results of [Pos06] that the point $\mathrm{Meas}_G(\mathrm{wt}_{\boldsymbol{\theta}})$ does not depend on the choice of G. We denote $\mathrm{Meas}_f(\boldsymbol{\theta}) := \mathrm{Meas}_G(\mathrm{wt}_{\boldsymbol{\theta}})$. The *critical cell* is given by

$$\mathrm{Crit}_f^{>0} := \{\mathrm{Meas}_f(\boldsymbol{\theta}) \mid \boldsymbol{\theta} \in \mathbb{R}^n \text{ is } f\text{-admissible}\}.$$

3. Affine poset cyclohedra

We review some definitions and properties of affine posets and the associated polytopes; see [Gal21b] for further details.

3.1. **Order polytopes and tubings.** We start with ordinary posets. Let $(P, \preceq_P)$ be a connected (i.e., having a connected Hasse diagram) poset with $|P| \geqslant 2$. Let $\alpha_P : \mathbb{R}^P \to \mathbb{R}$ be a linear function given by

$$\alpha_P(\boldsymbol{x}) := \sum_{p \prec_P q} x_q - x_p,$$

where the sum is taken over all covering relations $p \prec_P q$ in P. Let $\mathbb{R}^P_{\Sigma=0}$ denote the linear subspace of $\mathbb{R}^P$ consisting of vectors whose sum of coordinates is zero. Consider a $(|P|-2)$-dimensional polytope

$$\mathscr{O}(P) := \{\boldsymbol{x} \in \mathbb{R}^P_{\Sigma=0} \mid \alpha_P(\boldsymbol{x}) = 1 \text{ and } x_p \leqslant x_q \text{ for all } p \preceq_P q\}.$$

When P has a maximal and a minimal element, $\mathscr{O}(P)$ is projectively equivalent to the *order polytope* [Sta86] of P; see [Gal21b, Remark 2.5].

For a subset $\tau \subseteq P$, we say that τ is *convex* if for any three elements $p \preceq_P q \preceq_P r$ such that $p, r \in \tau$, we have $q \in \tau$. We say that τ is *connected* if the restriction of $\preceq_P$ to τ is a connected poset. A *P-tube* is a convex connected nonempty subset $\tau \subseteq P$. A *tubing partition* of P is a set partition $\mathbf{T}$ of P into disjoint P-tubes such that the directed graph $D_{\mathbf{T}}$ with vertex set $V(D_{\mathbf{T}}) := \mathbf{T}$ and edge set

$$E(D_{\mathbf{T}}) := \{(\tau, \tau') \mid \tau \cap \tau' = \emptyset \text{ and } p \prec_P q \text{ for some } p \in \tau,\ q \in \tau'\} \tag{3.1}$$

is acyclic. The faces of $\mathscr{O}(P)$ are in bijection with tubing partitions of P. Explicitly, given a point $\boldsymbol{x} \in \mathscr{O}(P)$, consider a maximal by inclusion set $I \subseteq P$ such that all coordinates in $\{x_p\}_{p \in I}$ coincide. Then I is a disjoint union of P-tubes, which are the connected components of the induced subgraph of the Hasse diagram of P with vertex set I. Collecting these P-tubes for all such sets I, we obtain a tubing partition of P denoted $\mathbf{B}(\boldsymbol{x})$.

Definition 3.1. An *affine poset (of order $n \geqslant 1$)* is a poset $\tilde{P} = (\mathbb{Z}, \preceq_{\tilde{P}})$ such that:

- for all $p \in \mathbb{Z}$, $p \prec_{\tilde{P}} p + n$;
- for all $p, q \in \mathbb{Z}$, $p \preceq_{\tilde{P}} q$ if and only if $p + n \preceq_{\tilde{P}} q + n$;
- for all $p, q \in \mathbb{Z}$, we have $p \preceq_{\tilde{P}} q + dn$ for some $d \geqslant 0$.

We denote $|\tilde{P}| := n$.

We identify points $\boldsymbol{\theta} \in \mathbb{R}^{|\tilde{P}|}$ with infinite sequences $\tilde{\boldsymbol{\theta}} = (\tilde{\theta}_p)_{p \in \mathbb{Z}}$ satisfying $\tilde{\theta}_p = \theta_p$ for $p \in [n]$ and $\tilde{\theta}_{p+n} = \tilde{\theta}_p + \pi$ for $p \in \mathbb{Z}$. Consider the $(n-1)$-dimensional *affine order polytope* $\mathscr{O}(\tilde{P})$ and its interior $\mathscr{O}^\circ(\tilde{P})$ defined by

$$\mathscr{O}(\tilde{P}) := \{\boldsymbol{\theta} \in \mathbb{R}^{|\tilde{P}|} \mid \theta_1 = 0 \text{ and } \tilde{\theta}_p \leqslant \tilde{\theta}_q \text{ for all } p \preceq_{\tilde{P}} q\}, \tag{3.2}$$

$$\mathscr{O}^\circ(\tilde{P}) := \{\boldsymbol{\theta} \in \mathbb{R}^{|\tilde{P}|} \mid \theta_1 = 0 \text{ and } \tilde{\theta}_p < \tilde{\theta}_q \text{ for all } p \prec_{\tilde{P}} q\}. \tag{3.3}$$

A $\tilde{P}$*-tube* (or simply a *tube*) is a convex connected nonempty subset $\tau \subseteq \tilde{P}$ such that either $\tau = \tilde{P}$ or τ contains at most one element in each residue class modulo n. For each tube τ, we denote by $\overline{\tau} := \{\tau + dn \mid d \in \mathbb{Z}\}$ its *equivalence class*, where $\tau + dn := \{p + dn \mid p \in \tau\}$. A collection $\mathbf{T}$ of tubes is called n*-periodic* if it is a union of such equivalence classes.

We say that two sets A, B are *nested* if either $A \subseteq B$ or $B \subseteq A$.

Definition 3.2. A $\tilde{P}$*-tubing* (or simply a *tubing*) is an n-periodic collection $\mathbf{T}$ of tubes such that any two tubes in $\mathbf{T}$ are either nested or disjoint, and such that the directed graph $D_{\mathbf{T}}$ given by (3.1) is acyclic. A tube τ is called *proper* if $\tau \neq \tilde{P}$ and $|\tau| > 1$. A tubing $\mathbf{T}$ is called *proper* if it consists of proper tubes. A *tubing partition* of $\tilde{P}$ is a tubing $\mathbf{T}$ which is simultaneously a set partition of $\mathbb{Z}$.

The face poset of $\mathscr{O}(\tilde{P})$ is isomorphic to the poset of tubing partitions of $\tilde{P}$ ordered by refinement. For example, the vertices of $\mathscr{O}(\tilde{P})$ are in bijection with equivalence classes of *maximal proper tubes* which are tubes $\tau \neq \tilde{P}$ satisfying $|\tau| = n$. For a point $\boldsymbol{x} \in \mathscr{O}(\tilde{P})$, we let $\mathbf{B}(\boldsymbol{x})$ denote the corresponding tubing partition of $\tilde{P}$.

3.2. **Affine poset cyclohedra and compactifications.** We showed in [Gal21b] that there is an $(n-1)$-dimensional polytope $\mathscr{C}(\tilde{P})$, called an *affine poset cyclohedron*, whose face poset is the poset of proper tubings ordered by reverse inclusion. For example, the vertices of $\mathscr{C}(\tilde{P})$ are in bijection with proper tubings $\mathbf{T}$ satisfying $|\overline{\mathbf{T}}| = n-1$, where $\overline{\mathbf{T}} := \{\overline{\tau} \mid \tau \in \mathbf{T}\}$ is the set of equivalence classes of tubes in $\mathbf{T}$. We refer the reader to [Gal21b, Section 1.3] for examples of affine poset cyclohedra. In addition, we showed that $\mathscr{C}(\tilde{P})$ arises as the compactification of the space $\mathscr{O}^\circ(\tilde{P})$, which may be identified with a certain configuration space of n points on a circle; see [Gal21b, Remark 1.12]. We now review the construction of this compactification.

Let $\tau \subsetneq \tilde{P}$ be a proper tube. We treat τ as a finite subposet $(\tau, \preceq_{\tilde{P}})$ of $\tilde{P}$, thus we may consider the order polytope $\mathscr{O}(\tau)$. The projection $\mathbb{R}^{|\tilde{P}|} \to \mathbb{R}^\tau$ sending $(\tilde{\theta}_p)_{p\in\mathbb{Z}} \mapsto (\tilde{\theta}_p)_{p\in\tau}$ gives rise to a map $\rho_\tau : \mathscr{O}^\circ(\tilde{P}) \to \mathscr{O}^\circ(\tau)$. More precisely, given any set $A \supseteq \tau$, define the following maps:

$$\operatorname{avg}_\tau : \mathbb{R}^A \to \mathbb{R}, \quad \boldsymbol{x} \mapsto \frac{1}{|\tau|}\sum_{p\in\tau} x_p; \qquad \pi^\tau_{\Sigma=0} : \mathbb{R}^A \to \mathbb{R}^\tau_{\Sigma=0}, \quad \boldsymbol{x} \mapsto (x_p - \operatorname{avg}_\tau(\boldsymbol{x}))_{p\in\tau};$$

$$\alpha_\tau : \mathbb{R}^A \to \mathbb{R}, \quad \boldsymbol{x} \mapsto \sum_{p,q\in\tau:\ p \prec_{\tilde{P}} q} x_q - x_p; \qquad \rho_\tau : \mathbb{R}^A \dashrightarrow \mathbb{R}^\tau, \quad \boldsymbol{x} \mapsto \frac{1}{\alpha_\tau(\boldsymbol{x})}\pi^\tau_{\Sigma=0}(\boldsymbol{x}).$$

Here ρ_τ is a rational map defined on the subset of $\mathbb{R}^A$ where $\alpha_\tau(\boldsymbol{x}) \neq 0$. Applying this construction to the case $A = \mathbb{Z}$, we obtain a map $\rho_\tau : \mathscr{O}^\circ(\tilde{P}) \to \mathscr{O}^\circ(\tau)$. Notice that α_τ takes strictly positive values on $\mathscr{O}^\circ(\tilde{P})$. By convention, for $\boldsymbol{\theta} \in \mathscr{O}^\circ(\tilde{P})$, we set $\rho_{\tilde{P}}(\boldsymbol{\theta}) := \boldsymbol{\theta}$. Let

$$\tilde{\rho} : \mathscr{O}^\circ(\tilde{P}) \to \overline{\prod_{|\tau|>1}} \mathscr{O}(\tau), \quad \boldsymbol{\theta} \mapsto (\rho_\tau(\boldsymbol{\theta}))_{|\tau|>1}.$$

Here $\overline{\prod}_{|\tau|>1} \mathscr{O}(\tau)$ is the set of points $(\boldsymbol{\theta}[\tau])_{|\tau|>1} \in \prod_{|\tau|>1} \mathscr{O}(\tau)$ satisfying $\boldsymbol{\theta}[\tau] = \boldsymbol{\theta}[\tau']$ whenever two tubes τ, τ' are equivalent. The product is taken over all non-singleton tubes τ, including the case $\tau = \tilde{P}$. We let

$$\operatorname{Comp}(\tilde{P}) := \overline{\tilde{\rho}(\mathscr{O}^\circ(\tilde{P}))} \tag{3.4}$$

denote the closure of the image of $\tilde{\rho}$.

By definition, each point $\boldsymbol{\theta} \in \mathrm{Comp}(\tilde{P})$ is an element $(\boldsymbol{\theta}[\tau])_{|\tau|>1}$ of the product $\bar{\prod}_{|\tau|>1} \mathscr{O}(\tau)$. We refer to its coordinates as $(\tilde{\theta}_i[\tau])_{i\in\tau}$ for each non-singleton tube τ. We showed in [Gal21b, Proposition 3.9] that $\mathrm{Comp}(\tilde{P})$ may be alternatively described as the subset of $\bar{\prod}_{|\tau|>1} \mathscr{O}(\tau)$ consisting of all points satisfying the following *coherence condition*:

$$\text{for any } \tau \subsetneq \tau_+ \text{ with } |\tau| > 1, \text{ there exists } \lambda \in \mathbb{R}_{\geqslant 0} \text{ such that } \pi^{\tau}_{\Sigma=0}(\boldsymbol{\theta}[\tau_+]) = \lambda \boldsymbol{\theta}[\tau]. \tag{3.5}$$

Definition 3.3. For $\boldsymbol{\theta} \in \mathrm{Comp}(\tilde{P})$, let $\widehat{\mathbf{T}}(\boldsymbol{\theta})$ be the smallest collection of tubes such that

- $\widehat{\mathbf{T}}(\boldsymbol{\theta})$ contains $\tilde{P}$;
- for each non-singleton $\tau \in \widehat{\mathbf{T}}(\boldsymbol{\theta})$, $\widehat{\mathbf{T}}(\boldsymbol{\theta})$ also contains all tubes in $\mathbf{B}(\boldsymbol{\theta}[\tau])$.

We let $\mathbf{T}(\boldsymbol{\theta})$ be obtained from $\widehat{\mathbf{T}}(\boldsymbol{\theta})$ by removing $\tilde{P}$ and all singleton tubes. More generally, for a proper tubing $\mathbf{T}$, we let $\widehat{\mathbf{T}}$ be obtained from $\mathbf{T}$ by adding $\tilde{P}$ and all singleton tubes, and vice versa.

The space $\mathrm{Comp}(\tilde{P})$ is naturally subdivided into cells labeled by proper tubings: for a proper tubing $\mathbf{T}$, the corresponding cell is given by

$$\mathrm{Comp}_{\mathbf{T}}(\tilde{P}) := \{\boldsymbol{\theta} \in \mathrm{Comp}(\tilde{P}) \mid \mathbf{T}(\boldsymbol{\theta}) = \mathbf{T}\}.$$

Cell closure relations are given by reverse inclusion of tubings:

$$\overline{\mathrm{Comp}_{\mathbf{T}}(\tilde{P})} = \bigsqcup_{\mathbf{T}' \supseteq \mathbf{T}} \mathrm{Comp}_{\mathbf{T}'}(\tilde{P}),$$

Theorem 3.4 ([Gal21b, Theorem 1.11])**.** *There exists a stratification-preserving homeomorphism $\mathscr{C}(\tilde{P}) \xrightarrow{\sim} \mathrm{Comp}(\tilde{P})$.*

Remark 3.5. In what follows, we always identify $\mathscr{C}(\tilde{P})$ with $\mathrm{Comp}(\tilde{P})$. The map $\tilde{\rho}$ gives a homeomorphism between $\mathscr{O}^{\circ}(\tilde{P})$ and the unique open dense cell $\mathrm{Comp}_{\emptyset}(\tilde{P})$ of $\mathrm{Comp}(\tilde{P})$, and we identify each of these spaces with the interior of the affine poset cyclohedron:

$$\mathscr{O}^{\circ}(\tilde{P}) \cong \mathrm{Comp}_{\emptyset}(\tilde{P}) \cong \mathscr{C}^{\circ}(\tilde{P}).$$

3.3. Circular chains.

Let $\tilde{P}$ be an affine poset. Our goal is to construct a particular family of continuous functions on $\mathscr{C}(\tilde{P})$ indexed by *circular $\tilde{P}$-chains*.

Definition 3.6. We say that a tuple $\boldsymbol{p} := (p_1, p_2, \dots, p_r)$ of integers is a *circular $\tilde{P}$-chain* if

$$p_1 \prec_{\tilde{P}} p_2 \prec_{\tilde{P}} \cdots \prec_{\tilde{P}} p_r \prec_{\tilde{P}} p_1 + n. \tag{3.6}$$

Thus $\boldsymbol{p}$ is a circular $\tilde{P}$-chain if and only if $\sigma(\boldsymbol{p}) := (p_2, \dots, p_r, p_1 + n)$ is a circular $\tilde{P}$-chain. We say that two such tuples differ by *cyclic relabeling*. We say that a tube τ *contains the residues of $\boldsymbol{p}$ modulo n* if for each $j \in [r]$, we have $p_j + d_j n \in \tau$ for some $d_j \in \mathbb{Z}$. Equivalently, since each tube τ is convex, it follows that τ contains the residues of $\boldsymbol{p}$ modulo n if and only if τ contains all elements of a circular $\tilde{P}$-chain $\sigma^s(\boldsymbol{p})$ for some $s \in \mathbb{Z}$.

Given a circular $\tilde{P}$-chain $\boldsymbol{p}$ and a point $\boldsymbol{\theta} \in \mathscr{C}(\tilde{P})$, the point $\boldsymbol{\theta}[\tilde{P}] \in \mathscr{O}(\tilde{P})$ satisfies

$$\tilde{\theta}_{p_1}[\tilde{P}] \leqslant \tilde{\theta}_{p_2}[\tilde{P}] \leqslant \cdots \leqslant \tilde{\theta}_{p_r}[\tilde{P}] \leqslant \tilde{\theta}_{p_1+n}[\tilde{P}] = \tilde{\theta}_{p_1}[\tilde{P}] + \pi. \tag{3.7}$$

For any tube $\tau \subsetneq \tilde{P}$ satisfying $p_1, p_2, \dots, p_r \in \tau$, the vector $\boldsymbol{\theta}[\tau] \in \mathscr{O}(\tau)$ satisfies

$$\tilde{\theta}_{p_1}[\tau] \leqslant \tilde{\theta}_{p_2}[\tau] \leqslant \cdots \leqslant \tilde{\theta}_{p_r}[\tau]. \tag{3.8}$$

Lemma 3.7. *Let $\tilde{P}$ be an affine poset, and suppose that $\boldsymbol{p} = (p_1, p_2, \dots, p_r)$ is a circular $\tilde{P}$-chain. Then the map*

$$\zeta^\circ_{\boldsymbol{p}} : \mathscr{C}^\circ(\tilde{P}) \to \mathbb{RP}^{r-1}_{>0}, \quad \boldsymbol{\theta} \mapsto \Big(\sin(\tilde{\theta}_{p_2} - \tilde{\theta}_{p_1}) : \cdots : \sin(\tilde{\theta}_{p_r} - \tilde{\theta}_{p_{r-1}}) : \sin(\tilde{\theta}_{p_1+n} - \tilde{\theta}_{p_r})\Big) \tag{3.9}$$

extends to a continuous map

$$\zeta_{\boldsymbol{p}} : \mathscr{C}(\tilde{P}) \to \mathbb{RP}^{r-1}_{\geqslant 0}.$$

Proof. Let $\boldsymbol{\theta} \in \mathrm{Comp}(\tilde{P}) \cong \mathscr{C}(\tilde{P})$ and let $\mathbf{T} := \mathbf{T}(\boldsymbol{\theta})$ be the associated tubing. Let $\tau \in \widehat{\mathbf{T}}$ be a minimal by inclusion tube containing the residues of $\boldsymbol{p}$ modulo n.

If $\tau = \tilde{P}$ then we set

$$\zeta_{\boldsymbol{p}}(\boldsymbol{\theta}) := \Big(\sin(\tilde{\theta}_{p_2}[\tilde{P}] - \tilde{\theta}_{p_1}[\tilde{P}]) : \cdots : \sin(\tilde{\theta}_{p_r}[\tilde{P}] - \tilde{\theta}_{p_{r-1}}[\tilde{P}]) : \sin(\tilde{\theta}_{p_1+n}[\tilde{P}] - \tilde{\theta}_{p_r}[\tilde{P}])\Big). \tag{3.10}$$

We would like to show that the vector on the right hand side is nonzero. Otherwise, by (3.7), we would have $\tilde{\theta}_{p_s}[\tilde{P}] = \cdots = \tilde{\theta}_{p_r}[\tilde{P}] = \tilde{\theta}_{p_1+n}[\tilde{P}] = \cdots = \tilde{\theta}_{p_{s-1}+n}[\tilde{P}]$ for some $s \in [r]$. Let

$$S := \{p \in \mathbb{Z} \mid \tilde{\theta}_p[\tilde{P}] = \tilde{\theta}_{p_s}[\tilde{P}]\}.$$

Thus S is a convex subset of $\tilde{P}$ containing all elements in $\sigma^{s-1}(\boldsymbol{p}) = (p_s, \dots, p_r, p_1 + n, \dots, p_{s-1} + n)$. It follows that S splits as a disjoint union of tubes, all of which belong to $\widehat{\mathbf{T}} \setminus \{\tilde{P}\}$. Because $\sigma^{s-1}(\boldsymbol{p})$ is a circular $\tilde{P}$-chain, there exists a path in the Hasse diagram of $\tilde{P}$ which starts at p_s, ends at $p_{s-1} + n$, and passes through all elements of $\sigma^{s-1}(\boldsymbol{p})$. For each vertex p on this path, we see that $p \in S$ since S is convex. Thus all elements of $\sigma^{s-1}(\boldsymbol{p})$ belong to the same proper tube $\tau' \in \mathbf{T}$. This contradicts the minimality of τ. We have shown that the vector on the right hand side of (3.10) is nonzero, thus $\zeta_{\boldsymbol{p}}(\boldsymbol{\theta})$ is a well defined element of $\mathbb{RP}^{r-1}_{\geqslant 0}$ when $\tau = \tilde{P}$.

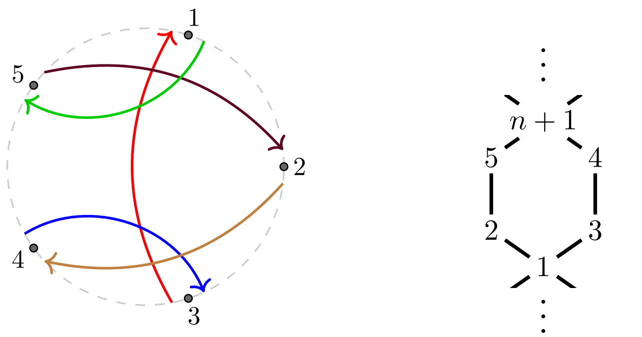
Assume now that $\tau \subsetneq \tilde{P}$. Since τ is convex, we may assume after some cyclic relabeling[1] that $p_1, p_2, \dots, p_r \in \tau$, in which case we set

$$\zeta_{\boldsymbol{p}}(\boldsymbol{\theta}) := \Big((\tilde{\theta}_{p_2}[\tau] - \tilde{\theta}_{p_1}[\tau]) : \cdots : (\tilde{\theta}_{p_r}[\tau] - \tilde{\theta}_{p_{r-1}}[\tau]) : (\tilde{\theta}_{p_r}[\tau] - \tilde{\theta}_{p_1}[\tau])\Big). \tag{3.11}$$

The entries on the right hand side are nonnegative by (3.8). Similarly to the above, we see that they cannot all be zero because that would imply $\tilde{\theta}_{p_1}[\tau] = \tilde{\theta}_{p_2}[\tau] = \cdots = \tilde{\theta}_{p_r}[\tau]$, contradicting the minimality of τ.

It remains to show that $\zeta_{\boldsymbol{p}}$ is continuous. Let $\boldsymbol{\theta}^{(m)}$ be a sequence of elements of $\mathscr{C}(\tilde{P})$ converging to $\boldsymbol{\theta}$ as $m \to \infty$. By definition, this means that $\boldsymbol{\theta}^{(m)}[\tau']$ converges to $\boldsymbol{\theta}[\tau']$ inside $\mathscr{O}(\tau')$ for each non-singleton tube τ'. Without loss of generality, we may assume that all points $\boldsymbol{\theta}^{(m)}$ belong to $\mathrm{Comp}_{\mathbf{T}'}(\tilde{P})$ for some fixed $\mathbf{T}' \subseteq \mathbf{T}$. Let $\tau' \in \widehat{\mathbf{T}}'$ be a minimal by inclusion tube containing the residues of $\boldsymbol{p}$ modulo n. Then $\tau \subseteq \tau'$. If $\tau = \tau'$ then clearly $\zeta_{\boldsymbol{p}}(\boldsymbol{\theta}^{(m)}) \to \zeta_{\boldsymbol{p}}(\boldsymbol{\theta})$ as $m \to \infty$. If $\tau \subsetneq \tau' \subsetneq \tilde{P}$ then the result follows from (3.5). Finally, if $\tau \subsetneq \tau' = \tilde{P}$, we see that because $\tau \in \mathbf{T} = \mathbf{T}(\boldsymbol{\theta})$, all coordinates of the vector on the right hand side of (3.10) tend to zero. But since this vector is treated as an element of $\mathbb{RP}^{r-1}$, we may replace the sines by their arguments. For the last coordinate, we replace

[1] Observe that the maps $\zeta^\circ_{\boldsymbol{p}}$ and $\zeta^\circ_{\sigma(\boldsymbol{p})}$ are related by a cyclic shift on $\mathbb{RP}^{r-1}$.



Strand diagram of $f \in S_n$ Affine poset $\tilde{P}_f$

FIGURE 6. Associating an affine poset $\tilde{P}_f$ (right) to a strand diagram of a permutation $f \in S_n$ (left). Figure reproduced from [Gal21b].

$\sin(\tilde{\theta}_{p_1+n}[\tilde{P}] - \tilde{\theta}_{p_r}[\tilde{P}]) = \sin(\tilde{\theta}_{p_r}[\tilde{P}] - \tilde{\theta}_{p_1}[\tilde{P}])$ with $\tilde{\theta}_{p_r}[\tilde{P}] - \tilde{\theta}_{p_1}[\tilde{P}]$. Therefore the limit of $\zeta_{\boldsymbol{p}}(\boldsymbol{\theta}^{(m)})$ coincides with the limit of

$$\left((\tilde{\theta}^{(m)}_{p_2}[\tilde{P}] - \tilde{\theta}^{(m)}_{p_1}[\tilde{P}]) : \cdots : (\tilde{\theta}^{(m)}_{p_r}[\tilde{P}] - \tilde{\theta}^{(m)}_{p_{r-1}}[\tilde{P}]) : (\tilde{\theta}^{(m)}_{p_r}[\tilde{P}] - \tilde{\theta}^{(m)}_{p_1}[\tilde{P}])\right) \tag{3.12}$$

as $m \to \infty$. By the coherence condition (3.5) applied to $\tau_+ := \tilde{P}$, the vector in (3.12) equals

$$\left((\tilde{\theta}^{(m)}_{p_2}[\tau] - \tilde{\theta}^{(m)}_{p_1}[\tau]) : \cdots : (\tilde{\theta}^{(m)}_{p_r}[\tau] - \tilde{\theta}^{(m)}_{p_{r-1}}[\tau]) : (\tilde{\theta}^{(m)}_{p_r}[\tau] - \tilde{\theta}^{(m)}_{p_1}[\tau])\right). \tag{3.13}$$

Since $\boldsymbol{\theta}^{(m)}[\tau] \to \boldsymbol{\theta}[\tau]$ as $m \to \infty$, the vector in (3.13) converges to $\zeta_{\boldsymbol{p}}(\boldsymbol{\theta})$. □

3.4. **From bounded affine permutations to affine posets.** Suppose that $f \in \mathcal{B}(k,n)$ is loopless and has a connected strand diagram. Let $\tilde{P}_f$ be the n-periodic transitive closure of the relations $p \prec_{\tilde{P}_f} q \prec_{\tilde{P}_f} p+n$ whenever $1 \leqslant p < q \leqslant n$ form an f-crossing. (Explicitly, $\prec_{\tilde{P}_f}$ is the transitive closure of the relations $p + dn \prec_{\tilde{P}_f} q + dn \prec_{\tilde{P}_f} p + (d+1)n$ for all $d \in \mathbb{Z}$.) It follows that $\tilde{P}_f$ is an affine poset. See Figure 6 for an example.

Comparing (3.3) to (2.3), we see that the sets

$$\mathscr{O}^\circ(\tilde{P}_f) = \Theta_f^{>0}$$

coincide as subsets of $\mathbb{R}^n$. As explained in Remark 3.5, these spaces are identified with the interior $\mathscr{C}^\circ(\tilde{P}_f)$ of the corresponding affine poset cyclohedron.

4. Taking the closure

Suppose that $f \in \mathcal{B}(k,n)$ is loopless and has a connected strand diagram. Recall from Section 3.4 that $\Theta_f^{>0}$ is naturally identified with the interior $\mathscr{C}^\circ(\tilde{P}_f)$. Thus we have a map

$$\mathrm{Meas}_f : \mathscr{C}^\circ(\tilde{P}_f) \to \mathrm{Crit}_f^{>0}.$$

Our goal is to show the following result.

Theorem 4.1. *For any loopless $f \in \mathcal{B}(k,n)$, the map Meas_f extends to a surjective continuous map between the closures*

$$\overline{\mathrm{Meas}}_f : \mathscr{C}(\tilde{P}_f) \to \mathrm{Crit}_f^{\geqslant 0}. \tag{4.1}$$

First, we describe a simple way to take a limit of a family of boundary measurements. See [PSW09, Lemma 3.1] for a closely related result.

Lemma 4.2. *Let $G \in \mathcal{G}_{\mathrm{red}}(f)$. Suppose that we are given a sequence $\mathrm{wt}^{(m)} \in \mathbb{R}_{>0}^{E(G)}$, $m = 1, 2, \dots$, such that for each $e \in E(G)$, there exists a finite limit*

$$\mathrm{wt}(e) := \lim_{m\to\infty} \mathrm{wt}^{(m)}(e) \quad \in [0, \infty).$$

Let G' be given by

$$V(G') := V(G), \quad E(G') := \{e \in E(G) \mid \mathrm{wt}(e) > 0\},$$

and let $\mathrm{wt}' \in \mathbb{R}_{>0}^{E(G')}$ be the restriction of wt to $E(G')$. Then we have

$$\lim_{m\to\infty} \mathrm{Meas}_G(\mathrm{wt}^{(m)}) = \mathrm{Meas}_{G'}(\mathrm{wt}') \quad \textit{inside } \mathrm{Gr}_{\geqslant 0}(k, n), \tag{4.2}$$

provided that G' admits at least one almost perfect matching.

Proof. Clearly, we have

$$\lim_{m\to\infty} (\Delta_I(G, \mathrm{wt}^{(m)}))_{I\in\binom{[n]}{k}} = (\Delta_I(G', \mathrm{wt}'))_{I\in\binom{[n]}{k}} \quad \text{inside } \mathbb{R}^{\binom{n}{k}}. \tag{4.3}$$

By construction, any almost perfect matching of G' is an almost perfect matching of G. Since the set of such almost perfect matchings is nonempty, the right hand side of (4.3) is nonzero. Thus (4.3) also holds inside $\mathbb{RP}^{\binom{n}{k}-1}$. This implies (4.2). □

Remark 4.3. We caution that if G' admits no almost perfect matchings, the limit on the left hand side of (4.2) may still exist, since applying a gauge transformation to each $\mathrm{wt}^{(m)}$ may give rise to a different graph G' in the limit.

Our next goal is to define the map $\overline{\mathrm{Meas}}_f$ in (4.1). We identify $\mathscr{C}(\tilde{P}_f)$ with $\mathrm{Comp}(\tilde{P}_f)$ via Theorem 3.4. Fix $\boldsymbol{\theta} \in \mathscr{C}(\tilde{P}_f)$ and let $\mathbf{T} := \mathbf{T}(\boldsymbol{\theta})$ be the corresponding proper tubing. Choose a contracted graph $G \in \mathcal{G}_{\mathrm{red}}(f)$.

Lemma 4.4. *Let $v \in V(G)$ be an interior vertex of G of degree r, and let $1 \leqslant p_1 < p_2 < \dots < p_r \leqslant n$ be the endpoints of the strands emanating from v. Then $(p_1, p_2, \dots, p_r)$ is a circular $\tilde{P}_f$-chain.*

Proof. It is easy to see from the "no bad double crossings" condition on the strands [Pos06, Theorem 13.2] that the edges incident to v are labeled by $\{p_1, p_2\}, \dots, \{p_{r-1}, p_r\}, \{p_r, p_1\}$ in clockwise order. The result follows by [Gal21a, Proposition 4.2]. □

In the setting of the above lemma, we denote $\boldsymbol{p}_G(v) := (p_1, p_2, \dots, p_r)$. Observe that the entries of $\zeta_{\boldsymbol{p}_G(v)}(\boldsymbol{\theta})$ are naturally labeled by $\{p_1, p_2\}, \dots, \{p_{r-1}, p_r\}, \{p_r, p_1\}$; see (3.9). Thus we may treat the entries of $\zeta_{\boldsymbol{p}_G(v)}(\boldsymbol{\theta})$ as nonnegative real edge weights assigned to the edges incident to v. They form an element of $\mathbb{RP}_{\geqslant 0}^{r-1}$ since rescaling them by a common positive scalar corresponds to a gauge transformation at v.

Definition 4.5. Let $\boldsymbol{\theta} \in \mathrm{Comp}(\tilde{P}_f)$. We define a weight function $\mathrm{wt}_{\boldsymbol{\theta}} \in \mathbb{R}_{\geqslant 0}^{E(G)}$ as follows. For each boundary edge e, set $\mathrm{wt}_{\boldsymbol{\theta}}(e) := 1$. For each black interior vertex $b \in V(G)$, set the weights of the edges incident to b to be proportional to the entries of $\zeta_{\boldsymbol{p}_G(b)}(\boldsymbol{\theta})$. Let G' be given by

$$V(G') := V(G), \quad E(G') := \{e \in E(G) \mid \mathrm{wt}_{\boldsymbol{\theta}}(e) > 0\}, \tag{4.4}$$

and let $\mathrm{wt}'_{\boldsymbol{\theta}}$ be the restriction of $\mathrm{wt}_{\boldsymbol{\theta}}$ to $E(G')$. Define

$$\overline{\mathrm{Meas}}_f(\boldsymbol{\theta}) := \mathrm{Meas}_{G'}(\mathrm{wt}'_{\boldsymbol{\theta}}). \tag{4.5}$$

See Figures 2 and 3 for examples of weighted graphs (G', wt').

Remark 4.6. For $\boldsymbol{\theta} \in \Theta_f^{>0} \cong \mathscr{O}^\circ(\tilde{P}_f)$, we have $\overline{\mathrm{Meas}}_f(\boldsymbol{\theta}) = \mathrm{Meas}_f(\boldsymbol{\theta})$ in view of (2.4) and Lemma 3.7.

Remark 4.7. The construction of $\overline{\mathrm{Meas}}_f$ in Definition 4.5 formally depends on the choice of $G \in \mathcal{G}_{\mathrm{red}}(f)$. However, we will see later that the choice of G is immaterial: we will show that $\overline{\mathrm{Meas}}_f$ is a *continuous* extension of Meas_f to $\mathscr{C}(\tilde{P}_f)$. If such a continuous extension exists, it must be unique, and thus any other choice of G would give rise to the same map $\overline{\mathrm{Meas}}_f$.

While the graph G in Definition 4.5 was assumed to be reduced and contracted, these properties need not hold for G'. But first, in order for (4.5) to give a well-defined element of the Grassmannian, we must show that not all coordinates of the vector $\mathrm{Meas}_{G'}(\mathrm{wt}'_{\boldsymbol{\theta}})$ are zero, which is equivalent to the following statement.

Proposition 4.8. *The graph G' given by* (4.4) *admits at least one almost perfect matching.*

Proof. Recall that we have set $\mathbf{T} := \mathbf{T}(\boldsymbol{\theta})$. Our first goal is to show that there exists a maximal proper tube τ' such that $\mathbf{T} \cup \{\tau'\}$ is a tubing.

The tubing $\mathbf{T}$ corresponds to a face $\mathrm{Comp}_{\mathbf{T}}(\tilde{P}_f)$ of $\mathscr{C}(\tilde{P}_f)$. Let $\mathrm{Comp}_{\mathbf{T}'}(\tilde{P}_f)$ be any vertex of the closed face $\overline{\mathrm{Comp}_{\mathbf{T}}(\tilde{P}_f)}$. Thus $\mathbf{T} \subseteq \mathbf{T}'$ and $\mathrm{Comp}_{\mathbf{T}'}(\tilde{P}_f)$ is a zero-dimensional face, which means $|\overline{\mathbf{T}'}| = n-1$. We claim that any proper tubing $\mathbf{T}'$ satisfying $|\overline{\mathbf{T}'}| = n-1$ contains a maximal proper tube.

To see this, consider a rooted tree T' (cf. [Gal21b, Definition 3.5]) with vertex set $\{\tilde{P}_f\} \sqcup \overline{\mathbf{T}'} \sqcup \mathbb{Z}/n\mathbb{Z}$, where $\mathbb{Z}/n\mathbb{Z}$ is identified with the set of equivalence classes of singleton tubes. (We identify the set of singleton tubes with $\mathbb{Z}$.) The root of T' is $\tilde{P}_f$, while $\mathbb{Z}/n\mathbb{Z}$ is the set of leaves of T'. The children of each $\overline{\tau} \in \{\tilde{P}_f\} \sqcup \overline{\mathbf{T}'}$ are of the form $\overline{\tau_-}$ where τ_- is a maximal by inclusion element of $\mathbf{T}' \sqcup \mathbb{Z}$ satisfying $\tau_- \subsetneq \tau$. We find that T' has $2n$ vertices, including n leaves. Moreover, each non-leaf vertex of T' other than $\tilde{P}_f$ has at least two children. Since a binary tree on n leaves contains $2n-1$ vertices, it follows that the root $\tilde{P}_f$ has exactly one child in T'. In other words, $\mathbf{T}'$ contains a maximal proper tube τ'. Thus $\mathbf{T} \cup \{\tau'\}$ is contained in a tubing $\mathbf{T}'$, and therefore is itself a tubing.

We now construct an almost perfect matching $\mathcal{A}$ of G. Let v be a (black or white) interior vertex of G. Since τ' is a maximal proper tube, it contains the residues of $\boldsymbol{p}_G(v)$ modulo n, and we let $p_1, p_2, \dots, p_r \in \tau'$ be such that $(p_1, p_2, \dots, p_r) = \sigma^s(\boldsymbol{p}_G(v))$ for some $s \in \mathbb{Z}$. Thus the strands emanating from v are labeled by $p_1, p_2, \dots, p_r$ in clockwise order, where we consider their labels modulo n.

We see that v is incident to an edge e_v labeled by $\{p_1, p_r\}$. Set

$$\mathcal{A} := \{e_v \mid v \text{ is an interior vertex of } G\}.$$

Thus $\mathcal{A}$ is a collection of edges of G covering each interior vertex at least once.

Let b (resp., w) be a black (resp., white) interior vertex of G. We claim that

$$e_b \text{ connects } b \text{ to } w \iff e_w \text{ connects } b \text{ to } w. \tag{4.6}$$

Suppose that e_b connects b to w. Label the strands emanating from b (resp., from w) by $p_1, p_2, \dots, p_r \in \tau'$ (resp., $q_1, q_2, \dots, q_s \in \tau'$) in clockwise order. Thus e_b is labeled by

$\{p_1, p_r\}$ while e_w is labeled by $\{q_1, q_s\}$. Since w is also incident to the edge e_b labeled by $\{p_1, p_r\}$, we see that $p_1, p_r \in \{q_1, q_2, \dots, q_s\}$, and moreover, p_r appears right before p_1 in the sequence $(q_1, q_2, \dots, q_s, q_1)$. It follows that $q_1 = p_1$ and $q_s = p_r$, therefore $e_w = e_b$. The converse direction is handled similarly, except that for a white interior vertex w, e_w may be a boundary edge (in which case there is no black interior vertex b satisfying $e_b = e_w$).

It follows from (4.6) that $\mathcal{A}$ is an almost perfect matching of G. It remains to show that $\mathcal{A}$ is an almost perfect matching of G'. Recall that $V(G') = V(G)$. Let b be a black interior vertex of G with outgoing strands labeled by $p_1, p_2, \dots, p_r \in \tau'$. Thus the edge $e_b \in \mathcal{A}$ is labeled by $\{p_1, p_r\}$. Our goal is to show that the r-th entry y_r of $\boldsymbol{y} := \zeta_{(p_1,p_2,\dots,p_r)}(\boldsymbol{\theta})$ is nonzero. By Lemma 3.7, the entries of $\boldsymbol{y}$ are not all zero.

Let $\tau \in \hat{\mathbf{T}}$ be a minimal by inclusion tube containing the residues of $(p_1, p_2, \dots, p_r)$ modulo n. We first consider the case $\tau = \tilde{P}_f$. By (3.10), we have $y_r = \sin(\tilde{\theta}_{p_1+n}[\tau] - \tilde{\theta}_{p_r}[\tau])$. Thus by (3.7), $y_r = 0$ implies that either $\tilde{\theta}_{p_r}[\tau] = \tilde{\theta}_{p_1}[\tau]$ or $\tilde{\theta}_{p_r}[\tau] = \tilde{\theta}_{p_1}[\tau] + \pi$. In the former case, the vector $\boldsymbol{y}$ would be zero, a contradiction. Thus assume $\tilde{\theta}_{p_r}[\tau] = \tilde{\theta}_{p_1}[\tau] + \pi$. Let

$$S := \{p \in \mathbb{Z} \mid \tilde{\theta}_p[\tau] = \tilde{\theta}_{p_r}[\tau]\}.$$

We see that S is a convex subset of $\tilde{P}_f$ containing both p_r and $p_1 + n$. Recall that S is a disjoint union of tubes. Since $p_r \prec_{\tilde{P}_f} p_1 + n$, these two elements belong to the same connected component τ_- of S. By Definition 3.3, we must have $\tau_- \in \mathbf{T}$. This is a contradiction: $\mathbf{T} \cup \{\tau'\}$ is a tubing, however, the tubes $\tau_-, \tau' \in \mathbf{T} \cup \{\tau'\}$ are neither nested nor disjoint. We have shown that $y_r \neq 0$ when $\tau = \tilde{P}_f$.

Assume now that $\tau \subsetneq \tilde{P}_f$ is a proper tube. By choosing a particular representative in $\overline{\tau}$, we may assume that $p_1 \in \tau$. Since any two tubes in $\mathbf{T} \cup \{\tau'\}$ are either nested or disjoint, and since $p_1 \in \tau \cap \tau'$, we must have $\tau \subseteq \tau'$. (Because $|\tau'| = n$, we cannot have $\tau' \subsetneq \tau$.) It follows that $p_1, p_2, \dots, p_r \in \tau$. Since $\tau \subsetneq \tilde{P}_f$, $\boldsymbol{y}$ is given by (3.11). In particular, $y_r = \tilde{\theta}_{p_r}[\tau] - \tilde{\theta}_{p_1}[\tau]$. By (3.7), $y_r = 0$ implies $\boldsymbol{y} = 0$, a contradiction. □

Proof of Theorem 4.1. By Proposition 4.8, the map $\overline{\mathrm{Meas}}_f$ lands inside $\mathrm{Gr}(k, n)$. By Remark 4.6, it extends the map Meas_f to $\mathscr{C}(\tilde{P}_f)$. Next, we show that it is continuous.

Let $(\boldsymbol{\theta}^{(m)})_{m \geqslant 1}$ be a sequence of points in $\mathscr{C}(\tilde{P}_f)$ converging to $\boldsymbol{\theta} \in \mathscr{C}(\tilde{P}_f)$ as $m \to \infty$. Let b be a black interior vertex of G of degree r. By Lemma 3.7, the map $\zeta_{\boldsymbol{p}_G(b)}$ is continuous on $\mathscr{C}(\tilde{P}_f)$:

$$\lim_{m \to \infty} \zeta_{\boldsymbol{p}_G(b)}(\boldsymbol{\theta}^{(m)}) = \zeta_{\boldsymbol{p}_G(b)}(\boldsymbol{\theta}) \quad \text{inside } \mathbb{RP}^{r-1}.$$

Thus, after applying gauge transformations to each $\mathrm{wt}_{\boldsymbol{\theta}^{(m)}}$ at black interior vertices, we get

$$\lim_{m \to \infty} \mathrm{wt}_{\boldsymbol{\theta}^{(m)}}(e) = \mathrm{wt}_{\boldsymbol{\theta}}(e) \quad \text{for all } e \in E(G).$$

(Recall that the weight of each boundary edge e is not affected by gauge transformations at black interior vertices, and satisfies $\mathrm{wt}_{\boldsymbol{\theta}^{(m)}}(e) = \mathrm{wt}_{\boldsymbol{\theta}}(e) = 1$ for all m.) By Proposition 4.8, G' admits an almost perfect matching, therefore $\overline{\mathrm{Meas}}_f$ is continuous by Lemma 4.2.

It remains to show that $\overline{\mathrm{Meas}}_f(\mathscr{C}(\tilde{P}_f)) = \mathrm{Crit}_f^{\geqslant 0}$. We see that the image of $\overline{\mathrm{Meas}}_f$ is compact (since $\mathscr{C}(\tilde{P}_f)$ is compact) and thus closed. Since the image contains $\mathrm{Crit}_f^{>0} = \mathrm{Meas}_f(\mathscr{C}^\circ(\tilde{P}_f))$, it contains the closure $\mathrm{Crit}_f^{\geqslant 0}$ of $\mathrm{Crit}_f^{>0}$. On the other hand, $\overline{\mathrm{Meas}}_f(\mathscr{C}(\tilde{P}_f))$ must be contained inside $\mathrm{Crit}_f^{\geqslant 0}$ because $\mathscr{C}(\tilde{P}_f)$ is the closure of $\mathscr{C}^\circ(\tilde{P}_f)$. □

Definition 4.9. We endow $\mathrm{Crit}_f^{\geqslant 0}$ with a stratification obtained by taking the common refinement of the images of all open faces of $\mathscr{C}(\tilde{P}_f)$.

Conjecture 4.10. For any two open faces $\mathrm{Comp}_{\mathbf{T}}(\tilde{P}_f), \mathrm{Comp}_{\mathbf{T}'}(\tilde{P}_f)$ of $\mathscr{C}(\tilde{P}_f)$, their images under $\overline{\mathrm{Meas}}_f$ either coincide or are disjoint.

Below we prove this conjecture for $f = f_{k,n}$.

5. Top cell and the second hypersimplex

We concentrate on the case of the top cell ($f = f_{k,n}$), where $2 \leqslant k \leqslant n-1$. We denote $\mathrm{Crit}_{k,n}^{\geqslant 0} := \mathrm{Crit}_{f_{k,n}}^{\geqslant 0}$, $\tilde{P}_{k,n} := \tilde{P}_{f_{k,n}}$, etc. Note that $\mathscr{C}(\tilde{P}_{k,n}) \cong \mathscr{C}_n$ is just the standard $(n-1)$-dimensional cyclohedron of [BT94, Sim03]. Our goal is to prove Theorem 1.5.

5.1. **From $\mathscr{C}(\tilde{P}_{k,n})$ to $\Delta_{2,n}$.** Recall from Theorem 4.1 that $\mathrm{Crit}_{k,n}^{\geqslant 0}$ is the image of the cyclohedron $\mathscr{C}(\tilde{P}_{k,n})$ under the map $\overline{\mathrm{Meas}}_{k,n} : \mathscr{C}(\tilde{P}_{k,n}) \to \mathrm{Crit}_{k,n}^{\geqslant 0}$. Our first goal is to introduce a map $\phi : \mathscr{C}(\tilde{P}_{k,n}) \to \Delta_{2,n}$ to the second hypersimplex and to show that $\overline{\mathrm{Meas}}_{k,n}$ factors through ϕ.

We start with a few preliminary observations and definitions.

Notation 5.1. For $a, b \in \mathbb{Z}$ with $a \leqslant b$, we set $[a, b) := \{a, a+1, \dots, b-1\}$. For $a, b \in [n]$, we introduce a *cyclic interval* $[a, b) := \{a, a+1, \dots, b-1\}$ if $a \leqslant b$ and $[a, b) := \{a, a+1, \dots, n, 1, \dots, b-1\}$ if $a > b$. The intervals $(a, b], [a, b] \subseteq \mathbb{Z}$ (for $a \leqslant b$) and cyclic intervals $(a, b], [a, b] \subseteq [n]$ (for $a, b \in [n]$) are defined analogously.

Definition 5.2. An *inscribed polygon* (resp., *degenerate inscribed polygon*) is a polygon all of whose vertices lie on a single circle (resp., on a single line).

We view (degenerate) inscribed polygons modulo transformations that preserve the ratios of the distances between their vertices. We write $R = (v_1, v_2, \dots, v_m)$ for a polygon with vertices $v_1, v_2, \dots, v_m$ given in cyclic order. The following result is well known.

Lemma 5.3. *Let $(a_1, a_2, \dots, a_m) \in \mathbb{R}_{\geqslant 0}^m$ be such that $a_p \leqslant \sum_{q \neq p} a_q$ for all $p \in [m]$. Then there exists a unique possibly degenerate inscribed polygon $R = (v_1, v_2, \dots, v_m)$ such that*

$$\frac{|v_{p+1} - v_p|}{|v_{q+1} - v_q|} = \frac{a_p}{a_q} \quad \text{for all } p, q \in [m],$$

where we set $v_{m+1} := v_1$. □

Thus, up to a common scalar, the diagonals of a possibly degenerate inscribed polygon may be reconstructed from its sides.

Next, observe that the order $\preceq_{\tilde{P}_{k,n}}$ coincides with the usual total order $\leqslant$ on $\mathbb{Z}$. In particular, $\boldsymbol{p}_{(n)} := (1, 2, \dots, n)$ is a circular $\tilde{P}_{k,n}$-chain. By Lemma 3.7, we therefore have a continuous map

$$\zeta_{\boldsymbol{p}_{(n)}} : \mathscr{C}(\tilde{P}_{k,n}) \to \mathbb{RP}_{\geqslant 0}^{n-1}.$$

Definition 5.4. Let $\boldsymbol{\theta} \in \mathscr{C}(\tilde{P}_{k,n})$. We denote by $\mathbf{B}_{\boldsymbol{\theta}}$ the partition of $\mathbb{Z}$ into intervals consisting of the tubes in $\mathbf{B}(\boldsymbol{\theta}[\tau])$ for each minimal by inclusion $\tau \in \widehat{\mathbf{T}}(\boldsymbol{\theta})$ satisfying $|\tau| = n$.

In the above definition, either $\tau = \tilde{P}_{k,n}$ or τ is a maximal proper tube, which in the case of $\tilde{P}_{k,n}$ is just an interval of the form $[p, p+n) \subseteq \mathbb{Z}$ for some $p \in \mathbb{Z}$. Thus $\mathbf{B}(\boldsymbol{\theta}[\tau])$ forms a partition of τ into intervals. Considering $\mathbf{B}_{\boldsymbol{\theta}}$ modulo n, we get a partition $\bar{\mathbf{B}}_{\boldsymbol{\theta}} = (\bar{B}_1, \bar{B}_2, \dots, \bar{B}_m)$ of $[n]$ into $m \geqslant 2$ nonempty cyclic intervals.

Remark 5.5. Recall from Remark 1.1 that for $\boldsymbol{\theta} \in \mathscr{C}^\circ(\tilde{P}_{k,n})$, setting $v_r := \exp(2i\theta_r)$ for $r \in [n]$ gives n distinct points $v_1, v_2, \dots, v_n$ on the unit circle ordered counterclockwise. The map $\zeta^\circ_{\boldsymbol{p}_{(n)}}$ in this case records the side length ratios of the n-gon $R = (v_1, v_2, \dots, v_n)$. When we pass to the boundary ($\boldsymbol{\theta} \in \mathscr{C}(\tilde{P}_{k,n})$), some of these points will collide. If not all points collide then the cyclic intervals in $\bar{\mathbf{B}}_{\boldsymbol{\theta}} = (\bar{B}_1, \bar{B}_2, \dots, \bar{B}_m)$ record precisely the groups of collided points, and $\zeta_{\boldsymbol{p}_{(n)}}(\boldsymbol{\theta})$ records the side length ratios of the corresponding m-gon. If all points collide then $\mathbf{T}(\boldsymbol{\theta})$ contains a maximal proper tube τ. In this case, $\boldsymbol{\theta}[\tau]$ records the positions of n points on a line, the cyclic intervals in $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$ record which groups of those points collided together, and $\zeta_{\boldsymbol{p}}(\boldsymbol{\theta})$ records the side length ratios of the corresponding degenerate inscribed m-gon.

Consider a map

$$\xi : \mathbb{RP}^{n-1}_{\geqslant 0} \to \mathbb{R}^n_{\geqslant 0}, \quad (x_1 : x_2 : \dots : x_n) \mapsto \frac{2}{x_1 + x_2 + \dots + x_n}(x_1, x_2, \dots, x_n).$$

We note that the entries of an element of $\mathbb{RP}^{n-1}_{\geqslant 0}$ are nonnegative and at least one of them is nonzero, thus their sum is strictly positive. The image of ξ belongs to the subspace of $\mathbb{R}^n_{\geqslant 0}$ where the sum of coordinates is equal to 2. Let

$$\phi : \mathscr{C}(\tilde{P}_{k,n}) \to \mathbb{R}^n_{\geqslant 0}, \quad \phi := \xi \circ \zeta_{\boldsymbol{p}_{(n)}}.$$

Proposition 5.6. *The image of the map ϕ equals*

$$\Delta_{2,n} = \{(y_1, y_2, \dots, y_n) \in [0,1]^n \mid y_1 + y_2 + \dots + y_n = 2\}.$$

Proof. By Remark 5.5, the map $\zeta^\circ_{\boldsymbol{p}_{(n)}}$ records the side length ratios of an inscribed n-gon, and thus its image is described by triangle inequalities:

$$\zeta^\circ_{\boldsymbol{p}_{(n)}}(\mathscr{C}^\circ(\tilde{P}_{k,n})) = \{(x_1 : x_2 : \dots : x_n) \in \mathbb{RP}^{n-1}_{\geqslant 0} \mid 0 < x_p < \sum_{q \neq p} x_q \text{ for each } p \in [n]\}.$$

Observe that $0 < x_p < \sum_{q \neq p} x_q$ is equivalent to $0 < 2x_p < \sum_{q=1}^n x_q$. Substituting $y_p := \frac{2x_p}{x_1 + x_2 + \dots + x_n}$, we get

$$\phi(\mathscr{C}^\circ(\tilde{P}_{k,n})) = \{(y_1, y_2, \dots, y_n) \in \mathbb{R}^n \mid 0 < y_p < 1 \text{ for each } p \in [n] \text{ and } y_1 + y_2 + \dots + y_n = 2\}.$$

The result follows by taking the closure. □

5.2. **From $\Delta_{2,n}$ to $\mathrm{Crit}^{\geqslant 0}_{k,n}$.** The goal of this section is to prove the following result.

Theorem 5.7. *There exists a continuous map*

$$\psi : \Delta_{2,n} \to \mathrm{Crit}^{\geqslant 0}_{k,n}$$

making the diagram (1.2) *commutative.*

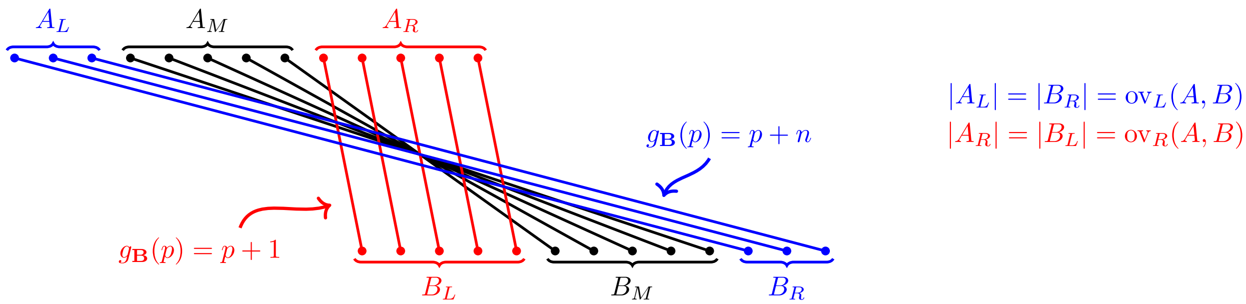


FIGURE 7. The permutation $g_{\mathbf{B}}$ described in Lemma 5.8.

Thus, Theorem 1.5 consists of Theorem 5.7 together with the statement that the map ψ is a homeomorphism, which we prove in Section 5.4.

Let $\boldsymbol{\theta} \in \mathscr{C}(\tilde{P}_{k,n})$. Since $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta}) \in \mathrm{Gr}_{\geqslant 0}(k, n)$, it must belong to some positroid cell $\Pi_g^{>0}$, $g \in \mathcal{B}(k, n)$. We will see later (Proposition 5.14) that the bounded affine permutation g has the following description. For a subset $A \subseteq \mathbb{Z}$ and $p \in \mathbb{Z}$, we let $A + p := \{a + p \mid a \in A\}$. By an *$n$-periodic interval partition of $\mathbb{Z}$* we mean a collection $\mathbf{B}$ of disjoint nonempty intervals in $\mathbb{Z}$ of size strictly less than n such that their union is $\mathbb{Z}$ and for each interval $B \in \mathbf{B}$, we have $B + dn \in \mathbf{B}$ for all $d \in \mathbb{Z}$.

Lemma 5.8. *For any n-periodic interval partition $\mathbf{B}$ of $\mathbb{Z}$, there exists a unique loopless $g_{\mathbf{B}} \in \mathcal{B}(k, n)$ of maximal length such that $g_{\mathbf{B}}(B - k) = B$ for all $B \in \mathbf{B}$.*

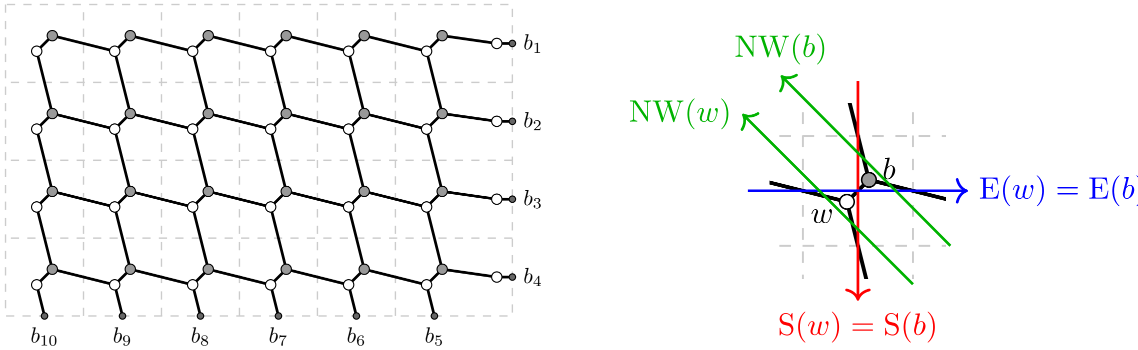
Proof. We describe $g_{\mathbf{B}}$ explicitly; see Figure 7. Let $B \in \mathbf{B}$ and denote $A := B - k$. Let

$$\mathrm{ov}_L(A, B) := |(A + n) \cap B| \quad \text{and} \quad \mathrm{ov}_R(A, B) := |(A + 1) \cap B|. \tag{5.1}$$

We have $\mathrm{ov}_L(A, B) + \mathrm{ov}_R(A, B) \leqslant |A| = |B| < n$. Let A_L consist of the smallest $\mathrm{ov}_L(A, B)$ elements of A, let A_R consist of the largest $\mathrm{ov}_R(A, B)$ elements of A, and let A_M consist of the remaining elements of A. Thus we have a partition $A = A_L \sqcup A_M \sqcup A_R$ into intervals. Next, we partition $B = B_L \sqcup B_M \sqcup B_R$ into intervals given by $B_L = A_R + 1$ and $B_R := A_L + n$. For $p \in A_R$, we let $g_{\mathbf{B}}(p) := p + 1 \in B_L$, and for $p \in A_L$, we let $g_{\mathbf{B}}(p) := p + n \in B_R$. The restriction of $g_{\mathbf{B}}$ to A_M is an order reversing bijection $A_M \to B_M$. This ensures that $g_{\mathbf{B}}$ has maximal possible length among all loopless bounded affine permutations sending A to B. It is also clear that $g_{\mathbf{B}} \in \mathcal{B}(k, n)$ (as opposed to $\mathcal{B}(k', n)$ for some $k' \neq k$) since it can be obtained from $f_{k,n}$ by applying (affine) simple transpositions. $\square$

Recall from Remark 4.7 that any choice of a graph $G \in \mathcal{G}_{\mathrm{red}}(f_{k,n})$ gives rise to the same map $\overline{\mathrm{Meas}}_{k,n}$. We will take advantage of this observation by using a particular graph $G_{k,n} \in \mathcal{G}_{\mathrm{red}}(f_{k,n})$ called the *Le-diagram graph*; see Figure 8(a) for an example and [Pos06, Section 20] for background.

Notation 5.9. All interior vertices of $G_{k,n}$ have degree either 2 or 3. Each interior vertex v belongs to one horizontal strand directed east, one vertical strand directed south, and one diagonal strand directed northwest; see Figure 8(b). We denote the endpoints of these strands by $\mathrm{E}(v), \mathrm{S}(v), \mathrm{NW}(v) \in [n]$, respectively. If a black vertex b has degree 2 then we have $\mathrm{NW}(b) = \mathrm{S}(b)$. If a white vertex w has degree 2 then we have $\mathrm{NW}(w) = \mathrm{E}(w)$. We denote by $V_\bullet(G_{k,n})$ the set of black interior vertices of $G_{k,n}$.



(a) Le-diagram graph $G_{k,n}$ for $k = 4$, $n = 10$ (b) Strands in $G_{k,n}$; see Notation 5.9

FIGURE 8. A Le-diagram graph and its strands.

Thus each $b \in V_\bullet(G_{k,n})$ is uniquely determined by $\mathrm{E}(b)$ and $\mathrm{S}(b)$, which are its vertical and horizontal coordinates in the plane.

After a cyclic shift, we may assume that

$$k \text{ and } k+1 \text{ belong to different intervals in } \mathbf{B}_{\boldsymbol{\theta}}. \tag{5.2}$$

Definition 5.10. An interval $B \in \mathbf{B}_{\boldsymbol{\theta}}$ is called *special* if it contains both n and $n+1$. We also refer to the corresponding cyclic interval $\bar{B} \in \bar{\mathbf{B}}_{\boldsymbol{\theta}}$ as *special*.

Clearly, $\mathbf{B}_{\boldsymbol{\theta}}$ contains at most one special interval.

Next, we consider the weighted graph (G', wt') obtained from $G_{k,n}$ via Definition 4.5.

Definition 5.11. We say that $b \in V_\bullet(G_{k,n})$ is of *type* (1) (resp., *type* (2) or *type* (3)) if the endpoints of the strands emanating from b belong to exactly one (resp., two or three) distinct cyclic intervals in $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$.

Remark 5.12. If b is of type (3), all three edges of b are present in G'. Their weights coincide with their weights in G, and can be computed from $\phi(\boldsymbol{\theta})$; cf. Remark 5.5 and Lemma 5.3. If b is of type (2), only two edges of b are present in G'. Their weights are equal, and after a gauge transformation at b, can be made equal to 1. Finally, if b is of type (1), either two or three edges of b are present in G', and their weights cannot in general be computed from $\phi(\boldsymbol{\theta})$. See e.g. Figures 3 and 9.

Lemma 5.13. *If $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$ does not contain a special cyclic interval (in the sense of Definition 5.10) then $V_\bullet(G_{k,n})$ contains no vertices of type (1). If $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$ contains a special cyclic interval $\bar{B}$ then for each $b \in V_\bullet(G_{k,n})$, b is of type (1) if and only if $\mathrm{S}(b), \mathrm{E}(b) \in \bar{B}$.*

Proof. In order for $b \in V_\bullet(G_{k,n})$ to be of type (1), $\mathrm{S}(b), \mathrm{E}(b), \mathrm{NW}(b)$ must belong to some cyclic interval $\bar{B} \in \bar{\mathbf{B}}_{\boldsymbol{\theta}}$. But since $\mathrm{S}(b) \in [k+1, n]$ and $\mathrm{E}(b) \in [k]$, $\bar{B}$ must be special in view of (5.2). Conversely, suppose that $\bar{B} \in \mathbf{B}_{\boldsymbol{\theta}}$ is special and $\mathrm{S}(b), \mathrm{E}(b) \in \bar{B}$. Since $\bar{B}$ is of the form $[n-h+1, n] \sqcup [v]$ for some $v \in [k]$ and $h \in [n-k]$, $\mathrm{S}(b), \mathrm{E}(b) \in \bar{B}$ implies $\mathrm{NW}(b) \in \bar{B}$. $\square$

Thus the set of type (1) vertices forms a top left justified $h \times v$ rectangle in $G_{k,n}$; see Figure 9 for an example.

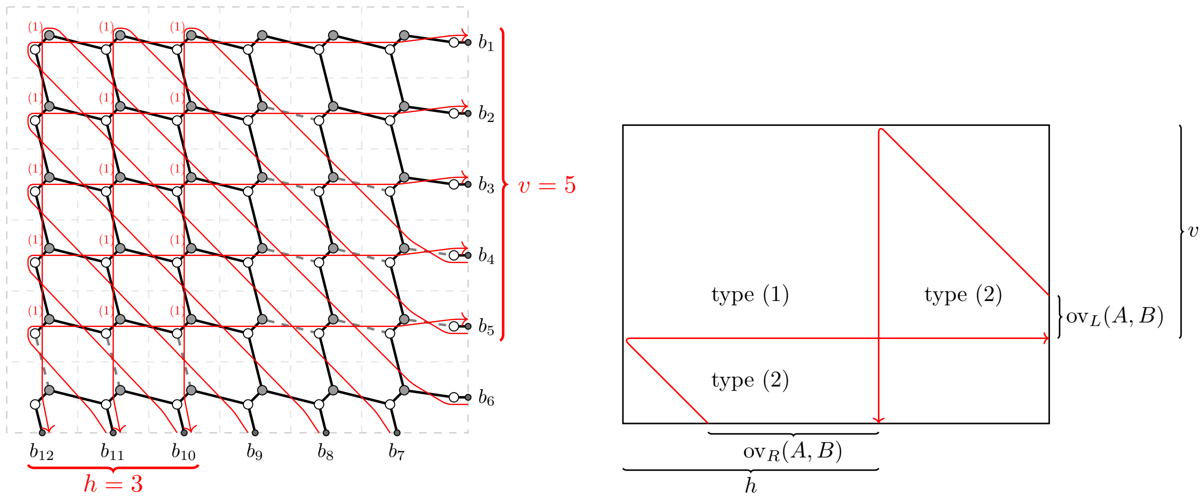


FIGURE 9. Left: an example for Lemma 5.13. Here $k = 6$, $n = 12$, and $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$ contains a special cyclic interval $\bar{B} = [10, 5] = [10, 12] \sqcup [1, 5]$. The strands terminating in $\bar{B}$ are shown in red. Type (1) black vertices are marked by (1). Dashed edges are present in G but not in G'. Each of them is incident to a black vertex of type (2). Right: regions of $G_{k,n}$ containing vertices of types (1) and (2) with respect to the special region $\bar{B}$.

For the next result, we need to refer explicitly to the edges of $G_{k,n}$. Each black vertex $b \in V_\bullet(G_{k,n})$ of degree 3 is incident to a *northern*, *eastern*, and *southwestern* edge labeled by $\{\mathrm{S}(b), \mathrm{NW}(b)\}$, $\{\mathrm{E}(b), \mathrm{NW}(b)\}$, and $\{\mathrm{S}(b), \mathrm{E}(b)\}$, respectively; see Figure 8(b). Recall from (5.2) that k and $k+1$ cannot both belong to the special interval in $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$. We let $g_{\boldsymbol{\theta}} := g_{\mathbf{B}_{\boldsymbol{\theta}}}$ be given by Lemma 5.8. Let us say that a *self-loop* is an edge of a graph connecting a vertex to itself.

Proposition 5.14. *If $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$ does not contain a special cyclic interval then set $G'' := G'$. Otherwise, let $\bar{B}$ be the special cyclic interval of $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$, and let G'' be obtained from G' in one of the following two ways:*

- *(if $k \notin \bar{B}$) remove all black vertices of type* (1) *and their southwestern white neighbors;*
- *(if $k+1 \notin \bar{B}$) contract all edges incident to black vertices of types* (1) *and* (2) *and remove all self-loops in the resulting graph.*

Let $\mathrm{wt}''_{\boldsymbol{\theta}}$ be the restriction of $\mathrm{wt}'_{\boldsymbol{\theta}}$ to the edges of G''. Then

$$G'' \in \mathcal{G}_{\mathrm{red}}(g_{\boldsymbol{\theta}}) \quad \textit{and} \quad \mathrm{Meas}_{G''}(\mathrm{wt}''_{\boldsymbol{\theta}}) = \mathrm{Meas}_{G'}(\mathrm{wt}'_{\boldsymbol{\theta}}). \tag{5.3}$$

For the example in Figure 9(left), we have $k, k+1 \notin \bar{B}$, so either of the two above procedures yields a reduced graph G'' satisfying the conditions in (5.3).

Proof. Consider a vertex $b \in V_\bullet(G_{k,n})$ of type (1) and let e be its southwestern edge. We claim that $\mathrm{wt}_{\boldsymbol{\theta}}(e)$ equals the sum of $\mathrm{wt}_{\boldsymbol{\theta}}(e')$ over all other edges e' of b. (In particular, $\mathrm{wt}_{\boldsymbol{\theta}}(e) > 0$ so e is present in G'.) Indeed, this is clear if b has degree 2. If b has degree 3 then $\mathrm{NW}(b)$ belongs to the cyclic interval $[\mathrm{S}(b), \mathrm{E}(b)]$. Thus there exists a $\tilde{P}_{k,n}$-circular chain (p, q, r) such that p, q, r are equal respectively to $\mathrm{S}(b)$, $\mathrm{NW}(b)$, $\mathrm{E}(b)$ modulo n, and such that

$p, q, r \in \tau$ for some proper tube $\tau \in \mathbf{T}(\boldsymbol{\theta})$. This implies that $\mathrm{wt}_{\boldsymbol{\theta}}(e) = \mathrm{wt}_{\boldsymbol{\theta}}(e') + \mathrm{wt}_{\boldsymbol{\theta}}(e'')$, where e, e', e'' are labeled by $\{p, r\}$, $\{p, q\}$, and $\{q, r\}$, respectively.

Let $\bar{B} := [n-h+1, n] \sqcup [v]$ for $v \in [k]$ and $h \in [n-k]$; see Figure 9. Assume first that $k \notin \bar{B}$, thus $v < k$. Let $b \in V_\bullet(G_{k,n})$ be a vertex satisfying $\mathrm{E}(b) = v+1$ and $\mathrm{S}(b) \in \bar{B}$. Then b is of type (2) with $\mathrm{S}(b), \mathrm{NW}(b) \in \bar{B}$, and its northern edge labeled by $\{\mathrm{S}(b), \mathrm{NW}(b)\}$ is not present in G'. Thus the bottom left black vertex b of type (1) (defined by $\mathrm{S}(b) = n$, $\mathrm{E}(b) = v$) is adjacent to a white vertex of degree 1 in G'. Applying a sequence of leaf removals (Figure 4(middle)) starting with b and proceeding up and to the right, we remove all black vertices of type (1) and their southwestern white neighbors.

Assume now that $k+1 \notin \bar{B}$, thus $h < n-k$. Let $b' \in V_\bullet(G_{k,n})$ be a vertex satisfying $\mathrm{S}(b') = n-h$ and $\mathrm{E}(b') \in [2, v]$. Then b' is of type (2) with $\mathrm{E}(b), \mathrm{NW}(b) \in \bar{B}$, and its eastern edge labeled by $\{\mathrm{E}(b), \mathrm{NW}(b)\}$ is not present in G'. For a connected subgraph H of G', let G'/H be obtained from G' by contracting all edges in H and removing all self-loops in the resulting graph. Initialize H to consist of all edges incident to vertices $b \in V_\bullet(G_{k,n})$ of type (2). This includes the edges incident to black vertices at the top ($\mathrm{E}(b) = 1$, $\mathrm{S}(b) \in \bar{B}$) and the right ($\mathrm{S}(b) = n-h$, $\mathrm{E}(b) \in \bar{B}$) boundaries of the $(h+1) \times v$ rectangle. Choose the top right type (1) black vertex that is not a vertex of H. Its northern and eastern white neighbors are in H. Let e, e', e'' be the edges of G incident to b as above (where one of e', e'' may not be present in G'), so that e is the southwestern edge. If both e', e'' are present then their images in G'/H form a double edge. Applying a parallel edge reduction move (Figure 4(left)), we transform this double edge into a single edge of weight $\mathrm{wt}_{\boldsymbol{\theta}}(e') + \mathrm{wt}_{\boldsymbol{\theta}}(e'')$, which, as we have shown above, equals $\mathrm{wt}_{\boldsymbol{\theta}}(e)$. Thus the image of b in G/H is a vertex of degree 2, and the two edges incident to it have the same weight. These two edges may be contracted using a contraction-uncontraction move (Figure 5(left)). This corresponds to adding e, e', e'' and their endpoints to H, and constitutes the induction step. Once all type (1) vertices have been added to H, we arrive at $G'/H = G''$.

A straightforward consequence of the above construction is that G'' has strand permutation $f_{G''} = g_{\boldsymbol{\theta}}$ and satisfies $\mathrm{Meas}_{G''}(\mathrm{wt}''_{\boldsymbol{\theta}}) = \mathrm{Meas}_{G'}(\mathrm{wt}'_{\boldsymbol{\theta}})$. Indeed, we have $\mathrm{Meas}_{G''}(\mathrm{wt}''_{\boldsymbol{\theta}}) = \mathrm{Meas}_{G'}(\mathrm{wt}'_{\boldsymbol{\theta}})$ since $(G'', \mathrm{wt}''_{\boldsymbol{\theta}})$ was obtained from $(G', \mathrm{wt}'_{\boldsymbol{\theta}})$ via a sequence of moves in Figures 4 and 5. To see that $f_{G''} = g_{\boldsymbol{\theta}}$, we first observe directly that $f_{G''}(p) = p+n$ (resp., $f_{G''}(p) = p+1$) if and only if $g_{\boldsymbol{\theta}}(p) = p+n$ (resp., $g_{\boldsymbol{\theta}}(p) = p+1$). Next, since our edge removals taking G to G' and reduction moves taking G' to G'' only involved edges labeled by $\{p, q\}$ where p, q belong to a single interval $\bar{B}'$ of $\bar{\mathbf{B}}$, we have $f^{-1}_{G''}(B') = B' - k = g^{-1}_{\boldsymbol{\theta}}(B')$ for any interval $B' \in \mathbf{B}$. Finally, it is clear that the contracted version of G'' contains no edge labeled $\{p, q\}$ where p, q belong to the same cyclic interval in $\bar{\mathbf{B}}$. Thus no two strands terminating at any given B' form a crossing, so $f_{G''}$ coincides with $g_{\boldsymbol{\theta}}$.

We further note that for any black interior vertex $b \in V(G'')$,

$$\text{the weights } \mathrm{wt}''_{\boldsymbol{\theta}}(e) \text{ of the edges } e \text{ of } G'' \text{ incident to } b \text{ are proportional to } \zeta_{\boldsymbol{p}_{G''(b)}}(\boldsymbol{\theta}). \tag{5.4}$$

Morally, the last property is close to the statement $\mathrm{Meas}_{G''}(\mathrm{wt}''_{\boldsymbol{\theta}}) = \overline{\mathrm{Meas}}_{g_{\boldsymbol{\theta}}}(\boldsymbol{\theta})$, except that we have not yet shown that G'' is reduced, and we also have not defined the map $\overline{\mathrm{Meas}}_g$ for the case when g does not have a connected strand diagram (cf. Definition 2.4).

In order to complete the proof of the proposition, we need to show that G'' is reduced. For that, we will use the following well-known characterization [Pos06] of reduced graphs: G'' is reduced if and only if it has no isolated connected components and has exactly $k(n-k) + 1 - \ell(g_{\boldsymbol{\theta}})$ faces. It is not hard to check that G'' has no isolated connected components. Since

$G_{k,n}$ has $k(n-k)+1$ faces, we need to show that our process above decreases the number of faces precisely by $\ell(g_{\boldsymbol{\theta}})$. Since each affine inversion of $g_{\boldsymbol{\theta}}$ involves two strands with endpoints in the same interval of $\mathbf{B}_{\boldsymbol{\theta}}$, it suffices to show, for each interval B of $\mathbf{B}_{\boldsymbol{\theta}}$, that the number of affine inversions involving indices from B matches the number of faces removed from G due to deleting/contracting edges labeled by $\{p,q\}$ for $p,q \in \bar{B}$.

Let $B \in \mathbf{B}_{\boldsymbol{\theta}}$, and let $A := B - k$. It follows from the proof of Lemma 5.8 that the number of affine inversions of the restriction of $g_{\boldsymbol{\theta}}$ to A equals

$$\binom{|B|}{2} - \binom{\mathrm{ov}_R(A,B)}{2} - \binom{\mathrm{ov}_L(A,B)}{2}. \tag{5.5}$$

If B is not special then we see that (5.5) also describes the number of type (2) vertices involving two indices in $\bar{B}$. Indeed, if B is not special then either $\bar{B} \subseteq [k+1,n]$ or $\bar{B} \subseteq [k]$. In the former case, we have $\mathrm{ov}_L(A,B) = 0$ and the number of type (2) vertices involving two indices in B equals $\binom{|B|}{2} - \binom{\mathrm{ov}_R(A,B)}{2}$. In the latter case, we have $\mathrm{ov}_R(A,B) = 0$ and the number of type (2) vertices involving two indices in B equals $\binom{|B|}{2} - \binom{\mathrm{ov}_L(A,B)}{2}$. Each such type (2) vertex is incident to an edge of G which is not present in G'. We therefore see that in both cases, the number of faces decreases by the quantity given in (5.5).

We concentrate on the case where B is special, so assume $\bar{B} = [n-h+1,n] \sqcup [v]$. Either of the two ways to reduce G' to G'' removes exactly $h(v-1)$ faces contained in the rectangular region. (When $k+1 \in \bar{B}$, this includes joining the $v-1$ boundary faces contained between the boundary vertices b_p for $p \in [v]$ into a single boundary face.) Next, we count the number of edges removed when passing from G to G'. All of them are adjacent to type (2) black vertices, and are contained in two trapezoidal regions shown in Figure 9(right). The lower left (resp., upper right) region is a trapezoid if $\mathrm{ov}_R(A,B) > 0$ (resp., $\mathrm{ov}_L(A,B) > 0$) and a triangle if $\mathrm{ov}_R(A,B) = 0$ (resp., $\mathrm{ov}_L(A,B) = 0$). It contains $\binom{h+1}{2} - \binom{\mathrm{ov}_R(A,B)}{2}$ (resp., $\binom{v}{2} - \binom{\mathrm{ov}_L(A,B)}{2}$) vertices of type (2) involving two indices in B. The result follows since

$$h(v-1) + \binom{h+1}{2} + \binom{v}{2} = \binom{h+v}{2} = \binom{|B|}{2}. \qquad \square$$

Proof of Theorem 5.7. By Definition 4.5, we have $\overline{\mathrm{Meas}}_f(\boldsymbol{\theta}) = \mathrm{Meas}_{G'}(\mathrm{wt}'_{\boldsymbol{\theta}})$, which equals $\mathrm{Meas}_{G''}(\mathrm{wt}''_{\boldsymbol{\theta}})$ by Proposition 5.14. By Remark 5.12, the edge weights of G'' may be computed purely in terms of the side length ratios encoded in $\phi(\boldsymbol{\theta})$. Thus $\overline{\mathrm{Meas}}_f$ factors through ϕ. Since ϕ is surjective, there exists a unique map $\psi : \Delta_{2,n} \to \mathrm{Crit}_{k,n}^{\geqslant 0}$ making the diagram (1.2) commutative. It remains to show that ψ is continuous. Letting $X := \mathscr{C}(\tilde{P}_{k,n})$, $Y := \Delta_{2,n}$, and $Z := \mathrm{Crit}_{k,n}^{\geqslant 0}$, we have maps $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ such that the composition $\psi \circ \phi$ is continuous. Choose a closed subset $Z' \subseteq Z$. Then $X' := (\psi \circ \phi)^{-1}(Z')$ is a closed subset of X. Observe that X is compact while Y is Hausdorff, thus ϕ is closed. Therefore $Y' := \phi(X')$ is a closed subset of Y. It follows from the surjectivity of ϕ that $Y' = \psi^{-1}(Z')$. Thus ψ is continuous. $\square$

5.3. **Positroids and weak separation.** Before we proceed with the final step of the proof, we need to introduce some constructions related to *positroids*; see [Pos06, OPS15] for background.

Let $g \in \mathcal{B}(k,n)$ be a bounded affine permutation. For $q \in \mathbb{Z}$, let

$$\tilde{I}_q := \{g(p) \mid p \in \mathbb{Z} \text{ is such that } p < q \leqslant g(p)\}. \tag{5.6}$$

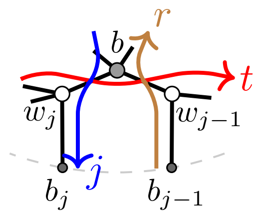


FIGURE 10. An I_j-arch; see Definition 5.15.

We set $\tilde{\mathcal{I}}_g = (\tilde{I}_q)_{q\in\mathbb{Z}}$. For $q \in [n]$, let $I_q \in \binom{[n]}{k}$ be obtained from $\tilde{I}_q$ by reducing all elements modulo n. The *Grassmann necklace* of g is the sequence $\mathcal{I}_g = (I_1, I_2, \dots, I_n)$. For each $q \in [n]$, consider a total order $\leqslant_q$ on $[n]$ given by $q \leqslant_q q+1 \leqslant_q \cdots \leqslant_q q-1$. For two sets $I = \{i_1 <_q i_2 <_q \cdots <_q i_k\}$ and $J = \{j_1 <_q j_2 <_q \cdots <_q j_k\}$, we write $I \leqslant_q J$ if $i_r \leqslant_q j_r$ for all $r \in [k]$. The *positroid* $\mathcal{M}_g$ of g is defined as the collection of all $J \in \binom{[n]}{k}$ satisfying $I_q \leqslant_q J$ for each $q \in [n]$.

We say that $I, J \in \binom{[n]}{k}$ are *weakly separated* [LZ98] if there do not exist indices $1 \leqslant a < b < c < d \leqslant n$ such that $a, c \in I \setminus J$ and $b, d \in J \setminus I$ or vice versa.

For $G \in \mathcal{G}_{\mathrm{red}}(g)$ and $j \in [n]$, we let w_j denote the unique neighbor of the degree 1 boundary vertex b_j.

Definition 5.15. Let $g \in \mathcal{B}(k,n)$ and $j, t \in [n]$. Let $r := \bar{g}(j-1) \in [n]$. (Here and below the index $j-1$ is taken modulo n.) Assume that $t \neq j \neq r \neq t$. We say that t *touches an I_j-arch* (with respect to g) if there exists a contracted graph $G \in \mathcal{G}_{\mathrm{red}}(g)$ such that the boundary face of G between b_j and b_{j-1} is a pentagon with vertices $(b_j, w_j, b, w_{j-1}, b_{j-1})$ for some black interior vertex b, and such that the strand labeled t passes through the edges connecting w_j to b and b to w_{j-1}. See Figure 10.

Our notion of an I_j*-arch* is closely related to the notion of a *BCFW bridge*; see [BCFW05, AHBC+16, Lam16]. In fact, a bridge is a special case of an arch when either w_j or w_{j-1} has degree 2; compare Figure 10 to e.g. [Gal21a, Figure 7]. We now establish a useful criterion for the existence of an I_j-arch.

Lemma 5.16. *Let $g \in \mathcal{B}(k,n)$, $j, t \in [n]$, $r := \bar{g}(j-1)$ be such that $t \neq j \neq r \neq t$. Then t touches an I_j-arch if and only if the sets*

$$J := I_j \cup \{t\} \setminus \{j\} \quad \textit{and} \quad R := I_j \cup \{t\} \setminus \{r\} \tag{5.7}$$

belong to $\mathcal{M}_g$ and are weakly separated from all sets in $\mathcal{I}_g$.

Proof. We start with the *if* direction. Since $I_j, J, R \in \mathcal{M}_g$, they are all of size k, and since $t \neq j \neq r \neq t$, we have $I_j \neq J \neq R \neq I_j$. In particular, $j, j-1$ are neither loops nor coloops. (Otherwise, either j or r would appear in either none or all of the three sets I_j, J, R.) Clearly, J and R are weakly separated from each other. Since they are also weakly separated from all sets in $\mathcal{I}_g$, by [OPS15, Theorem 1.5], there exists a contracted graph $G \in \mathcal{G}_{\mathrm{red}}(g)$ such that J, R appear as *face labels* of G. Here we label the faces of $G \in \mathcal{G}_{\mathrm{red}}(g)$ by k-element sets as follows: for each face F of G, the label of F contains $s \in [n]$ if and only if F is to the left of the strand terminating at b_s.

Observe that $|J \cup I_j \cup R| = |I_j \cup \{t\}| = k+1$. Thus J, I_j, R belong to a *non-trivial black clique* in the sense of [OPS15, Section 9]. In particular, the faces of G labeled J, I_j, R share a black vertex $b \in V(G)$.

Since $j, j-1$ are neither loops nor coloops, we have $I_{j-1} \neq I_j \neq I_{j+1}$. Suppose that $J \neq I_{j+1}$. Then I_j, J, I_{j+1} belong to a *non-trivial white clique*, and thus the corresponding faces of G share a common white vertex, which, since G is contracted, equals w_j. If $J = I_{j+1}$ then the two faces labeled by J and I_j still share the degree 2 vertex w_j. Similarly, the faces labeled by R and I_j share w_{j-1}. By [OPS15, Lemma 9.2], I_j and J share an edge connecting w_j to b while I_j and R share an edge connecting w_{j-1} to b. The strand labeled t therefore must pass through both of these edges, so t touches an I_j-arch.

The *only if* direction is a trivial consequence of the results of [OPS15]: if t touches an I_j-arch then J, R appear as labels of the faces of G containing b, and therefore J, R belong to $\mathcal{M}_g$ and are weakly separated from all sets in $\mathcal{I}_g$ by [OPS15, Theorem 1.5]. □

Next, we apply the above lemma to a particular class of permutations $g_{\mathbf{B}}$ constructed in Lemma 5.8.

Definition 5.17. An n-periodic interval partition $\mathbf{B}$ of $\mathbb{Z}$ is called *generic* if we have

$$|B| \leqslant \min(k-1, n-k) \quad \text{for all } B \in \mathbf{B}.$$

In other words, $\mathbf{B}$ is generic if and only if $\mathrm{ov}_L(A, B) = \mathrm{ov}_R(A, B) = 0$ for all $B \in \mathbf{B}$ and $A := B - k$. Consequently, $g_{\mathbf{B}}$ restricts to an order-reversing map $A \to B$ for each such pair (A, B). For the rest of this subsection, we fix some generic $\mathbf{B}$. Recall from Notation 5.1 that for $p, q \in [n]$, $[p, q)$ denotes the corresponding cyclic interval.

Lemma 5.18. *Let $[p, q) \in \bar{\mathbf{B}}$ and $r \in [p, q)$. Let $j \in [n]$ be equal to $p + q - k - r$ modulo n. Then the corresponding element of the Grassmann necklace $\mathcal{I}_{g_{\mathbf{B}}}$ is given by*

$$I_j := [j, p) \sqcup [r, q).$$

Moreover, every element of $\mathcal{I}_{g_{\mathbf{B}}}$ appears in this way for a unique triple (p, q, r).

We note that such Grassmann necklaces have been previously studied in [FG18, Section 4.4].

Proof. Follows from (5.6) by direct observation. □

Definition 5.19. A set $J \in \binom{[n]}{k}$ is called *right-aligned* if for each $[p, q) \in \bar{\mathbf{B}}$, we have

$$J \cap [p, q) = [r, q) \quad \text{for some } r \in [p, q].$$

Lemma 5.20. *Let $J \in \binom{[n]}{k}$ be right-aligned. Then $J \in \mathcal{M}_{g_{\mathbf{B}}}$ and J is weakly separated from all sets in $\mathcal{I}_{g_{\mathbf{B}}}$.*

Proof. Let $I_j \in \mathcal{I}_{g_{\mathbf{B}}}$. After a cyclic shift, we may assume $j = 1$, thus $I_1 = [1, p) \sqcup [r, q)$ for some $[p, q) \in \bar{\mathbf{B}}$ with $r \in [p, q)$. Our goal is to show that $I_1 \leqslant_1 J$ and that J is weakly separated from I_1. If J does not contain any elements in $[p, r)$ then both claims are clear. Otherwise, let $I_1' := [1, p)$, $J' := J \cap [1, p)$, $I_1'' := [r, q)$, and $J'' := J \cap [p, q)$, thus J'' contains an element $s \in [p, r)$. However, since $[p, q) \in \bar{\mathbf{B}}$ and J is right-aligned, we must have $J'' \supsetneq I_1''$. On the other hand, $J' \subseteq I_1'$, so I_1 and J are weakly separated. Moreover, because $|I_1| = |J|$ and J contains the last $q - r$ elements of I_1, we get $I_1 \leqslant_1 J$. □

Corollary 5.21. *Let $j \in [n]$ and $[s, s') \in \bar{\mathbf{B}}$ be such that $I_j \cap [s, s') = \emptyset$. Then $t := s' - 1$ touches an I_j-arch with respect to $g_{\mathbf{B}}$.*

Proof. Let $I_j = [j, p) \sqcup [r, q)$ with $r \in [p, q) \in \bar{\mathbf{B}}$ be as in Lemma 5.18. Observe that $r = \bar{g}_{\mathbf{B}}(j-1)$. The sets J, R given by (5.7) are clearly right-aligned. By Lemma 5.20, they satisfy the conditions of Lemma 5.16. □

Corollary 5.22. *Assume that $k \leqslant n-2$. Let $[p_1, q_1), [p_2, q_2), [p_3, q_3), [p_4, q_4) \in \bar{\mathbf{B}}$ be four distinct intervals, listed in clockwise order. Then there exists a contracted graph $G \in \mathcal{G}_{\mathrm{red}}(g_{\mathbf{B}})$ containing a square face whose edges are labeled by $\{t_1, t_2\}$, $\{t_2, t_3\}$, $\{t_3, t_4\}$, $\{t_1, t_4\}$ with $t_j \in [p_j, q_j)$ for each $j = 1, 2, 3, 4$.*

Proof. Consider all right-aligned subsets of $[n]$ whose intersection with $[p_j, q_j)$ is nonempty for each $j = 1, 2, 3, 4$. Clearly, such subsets can have any size between 4 and n. Let I be such a set of size $k+2$, and for $j = 1, 2, 3, 4$, let $I \cap [p_j, q_j) = [t_j, q_j)$, where $t_j \in [p_j, q_j)$. The sets $I \setminus \{t_i, t_j\}$ for $1 \leqslant i < j \leqslant 4$ are all right-aligned. Thus they belong to $\mathcal{M}_{g_{\mathbf{B}}}$ and are weakly separated from all elements of $\mathcal{I}_{g_{\mathbf{B}}}$ by Lemma 5.20. The result follows by combining [OPS15, Proposition 3.2] with [OPS15, Theorem 1.3]. □

5.4. **Injectivity.** Our final goal is to show that the map $\psi : \Delta_{2,n} \to \mathrm{Crit}_{k,n}^{\geqslant 0}$ constructed in Theorem 4.1 is injective, which is closely related to the *injectivity conjecture* for critical cells; see [Gal21a, Conjecture 4.3]. It was proved for $\mathrm{Crit}_{k,n}^{>0}$ in [Gal21a, Theorem 4.4]. What we need is an extension of that result to the closure $\mathrm{Crit}_{k,n}^{\geqslant 0}$ of $\mathrm{Crit}_{k,n}^{>0}$ which turns out to be more subtle.

Theorem 5.23. *The map $\psi : \Delta_{2,n} \to \mathrm{Crit}_{k,n}^{\geqslant 0}$ is a homeomorphism.*

Proof. Since $\Delta_{2,n}$ is compact, $\mathrm{Crit}_{k,n}^{\geqslant 0}$ is Hausdorff, and ψ is a continuous surjection, it remains to show that ψ is an injection. Thus for $\boldsymbol{\theta} \in \mathscr{C}(\tilde{P}_{k,n})$, our goal is to show that the point $\phi(\boldsymbol{\theta}) \in \Delta_{2,n}$ can be uniquely reconstructed from $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta}) \in \mathrm{Crit}_{k,n}^{\geqslant 0}$. Let $\mathbf{B}_{\boldsymbol{\theta}}$, $g_{\boldsymbol{\theta}}$, $G'' \in \mathcal{G}_{\mathrm{red}}(g_{\boldsymbol{\theta}})$, and $\mathrm{wt}''_{\boldsymbol{\theta}}$ be as in Section 5.2 and Proposition 5.14.

First, observe that $\mathbf{B}_{\boldsymbol{\theta}}$ need not be generic in the sense of Definition 5.17 since $g_{\boldsymbol{\theta}}$ may have some coloops and some indices $j \in \mathbb{Z}$ satisfying $g_{\boldsymbol{\theta}}(j) = j+1$. The corresponding strands form isolated connected components of the reduced strand diagram of $g_{\boldsymbol{\theta}}$. We remove these components using the *factorization* procedure from [Gal21a, Section 4.4]. Thus the problem reduces to the case where $\mathbf{B}_{\boldsymbol{\theta}}$ is generic, which allows us to apply the results of Section 5.3.

Let $\bar{\mathbf{B}} = (\bar{B}_1, \bar{B}_2, \dots, \bar{B}_m)$. The point $\phi(\boldsymbol{\theta})$ records the side length ratios of a (possibly degenerate) inscribed m-gon $R_{\boldsymbol{\theta}}$. For $p, q \in [m]$, let $\mathrm{d}_{\boldsymbol{\theta}}(p, q)$ denote the distance between the corresponding vertices of $R_{\boldsymbol{\theta}}$. The ratio of any two such distances can be computed from $\phi(\boldsymbol{\theta})$; see Remark 5.5. Recall from (5.4) that the edge weights of the graph G'' are proportional to the distances between the vertices of $R_{\boldsymbol{\theta}}$. More precisely, if an interior (that is, not incident to a boundary vertex) edge e of G'' is labeled by $\{s, t\}$ then $s \in \bar{B}_p$ and $t \in \bar{B}_q$ belong to different cyclic intervals in $\bar{\mathbf{B}}_{\boldsymbol{\theta}}$, and the weight $\mathrm{wt}''_{\boldsymbol{\theta}}(e)$ is proportional (compared to the other edges sharing a black vertex with e) to $\mathrm{d}_{\boldsymbol{\theta}}(p, q)$.

As explained in [Gal21a, Section 9], for any face F of G'', the alternating ratio of the edge weights that appear on the boundary of F may be reconstructed from $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta})$ using the *left twist* of Muller–Speyer; see [MS17, Corollary 5.11]. We will be interested in two kinds of faces of G'': I_j-arches as in Definition 5.15 and interior square faces as in Corollary 5.22.

Let $s, t \in [m]$ and $j \in \bar{B}_s$ be such that $I_j \cap \bar{B}_t = \emptyset$. By Corollary 5.21, some element in $\bar{B}_t$ touches an I_j-arch. Then for $r \in [m]$ such that $\bar{g}_{\boldsymbol{\theta}}(j-1) \in \bar{B}_r$, we find that the ratio

$$\frac{\mathrm{d}_{\boldsymbol{\theta}}(s, t)}{\mathrm{d}_{\boldsymbol{\theta}}(r, t)} \tag{5.8}$$

may be recovered from $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta})$.

Similarly, assume $k \leqslant n-2$ and let $\bar{B}_p, \bar{B}_q, \bar{B}_t, \bar{B}_s \in \bar{\mathbf{B}}$ be four distinct intervals listed in clockwise order. Then by Corollary 5.22, the *cross-ratio*

$$\frac{\mathrm{d}_{\boldsymbol{\theta}}(p,q) \cdot \mathrm{d}_{\boldsymbol{\theta}}(t,s)}{\mathrm{d}_{\boldsymbol{\theta}}(q,t) \cdot \mathrm{d}_{\boldsymbol{\theta}}(p,s)} \tag{5.9}$$

may be recovered from $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta})$. In fact, since the four corresponding vertices of $R_{\boldsymbol{\theta}}$ lie on a circle or on a line, the ratios

$$(\mathrm{d}_{\boldsymbol{\theta}}(p,q) \cdot \mathrm{d}_{\boldsymbol{\theta}}(t,s)) : (\mathrm{d}_{\boldsymbol{\theta}}(p,t) \cdot \mathrm{d}_{\boldsymbol{\theta}}(q,s)) : (\mathrm{d}_{\boldsymbol{\theta}}(p,s) \cdot \mathrm{d}_{\boldsymbol{\theta}}(q,t)) \tag{5.10}$$

can all be recovered from $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta})$ using standard relations for cross-ratios.

Recall that we have $\bar{\mathbf{B}} = (\bar{B}_1, \bar{B}_2, \dots, \bar{B}_m)$. Consider a directed graph D on $[m]$ with edges $s \to r$ whenever there exists $j \in \bar{B}_s$ such that $\bar{g}_{\boldsymbol{\theta}}(j-1) \in \bar{B}_r$. Thus the ratio in (5.8) may be recovered from $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta})$ for all $t \in [r+1, s-1]$. Clearly, each vertex of D has at least one outgoing arrow. Moreover, since $\bar{\mathbf{B}}$ is generic, we see that each vertex s of D has an outgoing arrow $s \to r$ for $r \neq s, s-1$ (modulo m). Finally, by comparing $\bar{g}_{\boldsymbol{\theta}}(j)$ to $\bar{g}_{\boldsymbol{\theta}}(j+1)$, we see that if D has an arrow $s \to r$ then D also has at least one of the following arrows: $s \to r+1$, $s+1 \to r$, $s+1 \to r+1$.

By Lemma 5.3, it suffices to recover the ratio $\mathrm{d}_{\boldsymbol{\theta}}(s, s-1) : \mathrm{d}_{\boldsymbol{\theta}}(s-1, s-2)$ from $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta})$ for each $s \in [m]$. This task is trivial when $k = n-1$, thus let us assume that $k \leqslant n-2$. As shown above, there exists $r \neq s, s-1$ such that D contains an arrow $s \to r$. If $r = s-2$ then we are done, thus assume $r \neq s, s-1, s-2$ and let $t \in [r+2, s-1]$. We know that D contains another arrow $s' \to r'$ for $s' \in \{s, s+1\}$, $r' \in \{r, r+1\}$. From (5.8), we recover the ratios

$$\mathrm{d}_{\boldsymbol{\theta}}(s,t) : \mathrm{d}_{\boldsymbol{\theta}}(r,t), \quad \mathrm{d}_{\boldsymbol{\theta}}(s,r') : \mathrm{d}_{\boldsymbol{\theta}}(r,r'), \quad \mathrm{d}_{\boldsymbol{\theta}}(s',t) : \mathrm{d}_{\boldsymbol{\theta}}(r',t), \quad \mathrm{d}_{\boldsymbol{\theta}}(s',s) : \mathrm{d}_{\boldsymbol{\theta}}(r',s), \tag{5.11}$$

some of which may coincide or be equal to 1 if $s = s'$ or $r = r'$.

Suppose first that $s' = s+1$ and $r' = r+1$. Using (5.10), we recover the ratios

$$\begin{aligned} &(\mathrm{d}_{\boldsymbol{\theta}}(s,s+1) \cdot \mathrm{d}_{\boldsymbol{\theta}}(r,r+1)) : (\mathrm{d}_{\boldsymbol{\theta}}(s,r) \cdot \mathrm{d}_{\boldsymbol{\theta}}(s+1,r+1)) : (\mathrm{d}_{\boldsymbol{\theta}}(s,r+1) \cdot \mathrm{d}_{\boldsymbol{\theta}}(s+1,r)), \\ &(\mathrm{d}_{\boldsymbol{\theta}}(s+1,r) \cdot \mathrm{d}_{\boldsymbol{\theta}}(r+1,t)) : (\mathrm{d}_{\boldsymbol{\theta}}(s+1,r+1) \cdot \mathrm{d}_{\boldsymbol{\theta}}(r,t)) : (\mathrm{d}_{\boldsymbol{\theta}}(s+1,t) \cdot \mathrm{d}_{\boldsymbol{\theta}}(r,r+1)). \end{aligned} \tag{5.12}$$

Combining (5.11) with (5.12), we recover

$$\mathrm{d}_{\boldsymbol{\theta}}(s,s+1) : \mathrm{d}_{\boldsymbol{\theta}}(s+1,r) : \mathrm{d}_{\boldsymbol{\theta}}(r,r+1) : \mathrm{d}_{\boldsymbol{\theta}}(s,r+1).$$

By Lemma 5.3, we recover the (possibly degenerate) inscribed quadrilateral with vertices $s, s+1, r, r+1$. The cases $s' = s$, $r' = r+1$ and $s = s+1$, $r' = r$ are handled similarly. In the former case, we recover the inscribed triangle with vertices $s, r, r+1$, and in the latter case, we recover the inscribed triangle with vertices $s, s+1, r$. (When we say "we recover a polygon" we mean that the ratio of any two of its side lengths may be recovered from $\overline{\mathrm{Meas}}_{k,n}(\boldsymbol{\theta})$.) Thus we have recovered a possibly degenerate inscribed polygon R whose vertex set $\mathrm{Vert}(R)$ contains s and r. By (5.8), for each $t' \in [r+1, s-1]$, we recover the ratio $\mathrm{d}_{\boldsymbol{\theta}}(s,t) : \mathrm{d}_{\boldsymbol{\theta}}(r,t)$, and thus the possibly degenerate inscribed polygon with vertex set $\mathrm{Vert}(R) \cup [r,s]$ is recovered. In particular, the ratio $\mathrm{d}_{\boldsymbol{\theta}}(s,s-1) : \mathrm{d}_{\boldsymbol{\theta}}(s-1,s-2)$ is recovered. $\square$

References

[AHBC+16] Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, and Jaroslav Trnka. *Grassmannian Geometry of Scattering Amplitudes.* Cambridge University Press, Cambridge, 2016.

[AHT14] Nima Arkani-Hamed and Jaroslav Trnka. The amplituhedron. *Journal of High Energy Physics*, 2014(10):30, Oct 2014.

[AS94] Scott Axelrod and I. M. Singer. Chern-Simons perturbation theory. II. *J. Differential Geom.*, 39(1):173–213, 1994.

[BCFW05] Ruth Britto, Freddy Cachazo, Bo Feng, and Edward Witten. Direct proof of the tree-level scattering amplitude recursion relation in Yang-Mills theory. *Phys. Rev. Lett.*, 94(18):181602, 4, 2005.

[BT94] Raoul Bott and Clifford Taubes. On the self-linking of knots. *J. Math. Phys.*, 35(10):5247–5287, 1994.

[CW11] Sylvie Corteel and Lauren K. Williams. Tableaux combinatorics for the asymmetric exclusion process and Askey-Wilson polynomials. *Duke Math. J.*, 159(3):385–415, 2011.

[FG18] Miriam Farber and Pavel Galashin. Weak separation, pure domains and cluster distance. *Selecta Math. (N.S.)*, 24(3):2093–2127, 2018.

[Gal21a] Pavel Galashin. Critical varieties in the Grassmannian. `arXiv:2102.13339v1`, 2021.

[Gal21b] Pavel Galashin. Poset associahedra. `arXiv:2110.07257v1`, 2021.

[GKL17] Pavel Galashin, Steven N. Karp, and Thomas Lam. The totally nonnegative Grassmannian is a ball. `arXiv:1707.02010`, 2017.

[GKL19] Pavel Galashin, Steven N. Karp, and Thomas Lam. Regularity theorem for totally nonnegative flag varieties. *J. Amer. Math. Soc., to appear.* `arXiv:1904.00527v3`, 2019.

[GL20] Pavel Galashin and Thomas Lam. Parity duality for the amplituhedron. *Compos. Math.*, 156(11):2207–2262, 2020.

[GP20] Pavel Galashin and Pavlo Pylyavskyy. Ising model and the positive orthogonal Grassmannian. *Duke Math. J.*, 169(10):1877–1942, 2020.

[Hai84] Mark Haiman. Constructing the associahedron. Preprint, 1984.

[Ken02] R. Kenyon. The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.*, 150(2):409–439, 2002.

[KLS13] Allen Knutson, Thomas Lam, and David E. Speyer. Positroid varieties: juggling and geometry. *Compos. Math.*, 149(10):1710–1752, 2013.

[Lam16] Thomas Lam. Totally nonnegative Grassmannian and Grassmann polytopes. In *Current developments in mathematics 2014*, pages 51–152. Int. Press, Somerville, MA, 2016.

[Lam18] Thomas Lam. Electroid varieties and a compactification of the space of electrical networks. *Adv. Math.*, 338:549–600, 2018.

[Lee89] Carl W. Lee. The associahedron and triangulations of the n-gon. *European J. Combin.*, 10(6):551–560, 1989.

[LPW20] Tomasz Lukowski, Matteo Parisi, and Lauren K. Williams. The positive tropical Grassmannian, the hypersimplex, and the m=2 amplituhedron. `arXiv:2002.06164v2`, 2020.

[LTV10] Pascal Lambrechts, Victor Turchin, and Ismar Volić. Associahedron, cyclohedron and permutohedron as compactifications of configuration spaces. *Bull. Belg. Math. Soc. Simon Stevin*, 17(2):303–332, 2010.

[Lus94] G. Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pages 531–568. Birkhäuser Boston, Boston, MA, 1994.

[Lus98] George Lusztig. Introduction to total positivity. In *Positivity in Lie theory: open problems*, volume 26 of *De Gruyter Exp. Math.*, pages 133–145. de Gruyter, Berlin, 1998.

[LZ98] Bernard Leclerc and Andrei Zelevinsky. Quasicommuting families of quantum Plücker coordinates. In *Kirillov's seminar on representation theory*, volume 181 of *Amer. Math. Soc. Transl. Ser. 2*, pages 85–108. Amer. Math. Soc., Providence, RI, 1998.

[MS17] Greg Muller and David E. Speyer. The twist for positroid varieties. *Proc. Lond. Math. Soc. (3)*, 115(5):1014–1071, 2017.

[OPS15] Suho Oh, Alexander Postnikov, and David E. Speyer. Weak separation and plabic graphs. *Proc. Lond. Math. Soc. (3)*, 110(3):721–754, 2015.

[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, `http://math.mit.edu/~apost/papers/tpgrass.pdf`, 2006.

[PSW09] Alexander Postnikov, David Speyer, and Lauren Williams. Matching polytopes, toric geometry, and the totally non-negative Grassmannian. *J. Algebraic Combin.*, 30(2):173–191, 2009.

[RW08] Konstanze Rietsch and Lauren Williams. The totally nonnegative part of G/P is a CW complex. *Transform. Groups*, 13(3-4):839–853, 2008.

[RW10] Konstanze Rietsch and Lauren Williams. Discrete Morse theory for totally non-negative flag varieties. *Adv. Math.*, 223(6):1855–1884, 2010.

[Sim03] Rodica Simion. A type-B associahedron. volume 30, pages 2–25. 2003. Formal power series and algebraic combinatorics (Scottsdale, AZ, 2001).

[Sin04] Dev P. Sinha. Manifold-theoretic compactifications of configuration spaces. *Selecta Math. (N.S.)*, 10(3):391–428, 2004.

[Sta63] James Dillon Stasheff. Homotopy associativity of H-spaces. I, II. *Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid.*, 108:293–312, 1963.

[Sta86] Richard P. Stanley. Two poset polytopes. *Discrete Comput. Geom.*, 1(1):9–23, 1986.

[Tal08] Kelli Talaska. A formula for Plücker coordinates associated with a planar network. *Int. Math. Res. Not. IMRN*, pages Art. ID rnn 081, 19, 2008.

[Tam51] Dov Tamari. *Monoïdes préordonnés et chaînes de Malcev*. Thèse, Université de Paris, 1951.

[Wil07] Lauren K. Williams. Shelling totally nonnegative flag varieties. *J. Reine Angew. Math.*, 609:1–21, 2007.

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