BRAID VARIETY CLUSTER STRUCTURES, II: GENERAL TYPE

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ABSTRACT. We show that braid varieties for any complex simple algebraic group G are cluster varieties. This includes open Richardson varieties inside the flag variety G/B.

1. INTRODUCTION

This is one of two papers concerned with the construction of cluster structures on braid varieties. In the present paper, we work in the setting of a general simple algebraic group G and construct cluster seeds using algebraic geometry. In the companion paper [GLSBS22], joint with David Speyer, we give an alternative proof in the special case $G=\mathrm{SL}_n$, using the combinatorics of plabic graphs and surfaces. The current work is logically independent of [GLSBS22], which, however, ultimately produces the same cluster structure in the case $G=\mathrm{SL}_n$.

Let G be a complex, simple, simply-connected algebraic group, $B_\pm$ opposing Borel subgroups, $U_\pm$ their unipotent radicals, $H:=B_+\cap B_-$ the torus, I the vertex set of the Dynkin diagram, W the Weyl group with simple generators $s_i, i\in I$, and denote by $\dot{w}$ the lift of $w\in W$ to G as in (2.1). Let $w_\circ\in W$ denote the longest element and $i\mapsto i^*$ the action of $w_\circ$ on I. Let $\alpha_i,\alpha_i^\vee,\omega_i$ for $i\in I$ denote the simple roots, simple coroots, fundamental weights, respectively, and let $A=(a_{ij})_{i,j\in I}$ be the Cartan matrix given by $a_{ij}:=\langle\alpha_i,\alpha_j^\vee\rangle$. Denote $d_i:=2/(\alpha_i,\alpha_i)$ so that $d_ia_{ij}=d_ja_{ji}$.

1.1. **Double braid varieties.** A *double braid word* $\beta=i_1i_2\cdots i_m$ is a word in the alphabet $\pm I$. For $i\in I$, we set $(-i)^*:=-i^*$. For $i\in\pm I$, define

$$s_i^+:=\begin{cases} s_i, & \text{if } i>0,\\ \mathrm{id}, & \text{if } i<0,\end{cases}\quad s_i^-:=\begin{cases} \mathrm{id}, & \text{if } i>0,\\ s_{-i}, & \text{if } i<0.\end{cases} \tag{1.1}$$

A *weighted* (or *framed*) *flag* is an element $F=gU_+\in G/U_+$. Two weighted flags (F,F') are *weakly w-related* (resp., *strictly w-related*) if there exist $g\in G$ and $h\in H$ (resp., $g\in G$) such that $(gF,gF')=(U_+,h\dot{w}U_+)$ (resp., $(gF,gF')=(U_+,\dot{w}U_+)$). We write this as $F\xRightarrow{w}F'$ (resp., $F\xrightarrow{w}F'$).

Suppose that the Demazure product of β is $w_\circ$; see (2.4). We consider the set $\mathring{\mathcal{Y}}_\beta$ of tuples $(X_\bullet,Y_\bullet)$ of weighted flags satisfying the relative position conditions

$$\begin{tikzcd} X_0 & X_1 \arrow[l, "s_{i_1}^+"'] & \cdots \arrow[l, "s_{i_2}^+"'] & X_m \arrow[l, "s_{i_m}^+"'] \\ Y_0 \arrow[u, Rightarrow, "w_\circ"] \arrow[r, "s_{i_1^*}^-"'] & Y_1 \arrow[r, "s_{i_2^*}^-"'] & \cdots \arrow[r, "s_{i_m^*}^-"'] & Y_m. \arrow[u, equal] \end{tikzcd} \tag{1.2}$$

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The group G acts on $\mathring{\mathcal{Y}}_\beta$ by acting on each individual weighted flag, and this action is free. The *double braid variety* $\mathring{R}_\beta := \mathring{\mathcal{Y}}_\beta/G$ is defined as the quotient of $\mathring{\mathcal{Y}}_\beta$ modulo this G-action. It is a smooth, affine, irreducible complex algebraic variety (Proposition 2.5).

Double braid varieties include open Richardson varieties [Rie98, KLS14, Lec16], open positroid varieties [KLS13], double Bott-Samelson cells [SW21], the strata in [WY07], and the braid varieties of [Mel19, CGGS24]; see [GLSBS22] for further discussion. Braid varieties have deep connections to knot theory, as their cohomology recovers part of the Khovanov–Rozansky homology [KR08, Kho07] of the associated link [GL20, Tri21, CGGS21].

For each β, we construct a cluster seed Σ_β. Our main result settles conjectures of [Lec16, CGGS21] and generalizes work of [BFZ05, GL23, Ing19, SW21].

Theorem 1.1. *The coordinate ring of $\mathring{R}_\beta$ is isomorphic to the cluster algebra $\mathcal{A}_\beta = \mathcal{A}(\Sigma_\beta)$.*

It would be interesting to compare our construction and the cluster-categorical approach of [GLS06, BIRS09, Lec16, Mén22, CK22], as was done in type A in [SSB24].

Remark 1.2. At the final stages of completing our construction, we learned that a cluster structure for braid varieties was independently announced in a recent preprint [CGG+22]. We thank the authors of [CGG+22] for updating us on their progress. It would be interesting to understand the relation between our approach and their Legendrian-geometric viewpoint.

One application of Theorem 1.1 is that a *curious Lefschetz theorem* (see [HRV08, LS22], [GLSBS22, Theorem 10.1], and [GL20, Theorem 1.5]) holds for double braid varieties; see Theorem 6.8. In the case of open Richardson varieties, this implies that the doubly-graded extension group $\mathrm{Ext}_{\mathcal{O}}(M_w, M_v)$ of two Verma modules in Category $\mathcal{O}$ satisfies curious Lefschetz; cf. [GL20, Section 1.11].

1.2. **Cluster variables and Deodhar geometry.** To construct a cluster structure on $\mathring{R}_\beta$, we need to identify certain regular functions on $\mathring{R}_\beta$ as (initial) cluster variables, and then construct the exchange matrix $\tilde{B}$, or quiver, of the initial seed. In previous works [Sco06, GL23, BFZ05, SW21] establishing cluster structures on the Grassmannian, positroid varieties, double Bruhat cells and double Bott Samelson cells, the cluster variables are *(generalized) minors* of some kind. Determinantal identities satisfied by the minors become exchange relations in the cluster algebra. A fundamental obstacle, already pointed out by Leclerc [Lec16], to extending these constructions to open Richardson varieties or braid varieties, is that the (generalized) minors are no longer irreducible elements of the coordinate ring $\mathbb{C}[\mathring{R}_\beta]$.

In the present work, we construct cluster variables using *Deodhar geometry*; see Section 2 for details and [GLSBS22, Gal23] for examples.[1] We introduce an open dense algebraic torus $T_\beta \subset \mathring{R}_\beta$ called the *Deodhar torus*, so named for its relation to the Deodhar decomposition of Richardson varieties [Deo85, MR04]. It is defined by requiring the weighted flags X_c, Y_c to be weakly w_c-related, where $w_c \in W$ is maximal possible subject to (1.2) (for each $c = 0, 1, \dots, m$). The complement $\mathring{R}_\beta \setminus T_\beta$ is a union of irreducible *mutable Deodhar hypersurfaces* $\{V_c \mid c \in J_\beta^{\mathrm{mut}}\}$. We define a partial compactification of $\mathring{R}_\beta$ so that the complement of T_β in it also includes *frozen Deodhar hypersurfaces* $\{V_c \mid c \in J_\beta^{\mathrm{fro}}\}$. We let $J_\beta := J_\beta^{\mathrm{fro}} \sqcup J_\beta^{\mathrm{mut}}$. The following definition, suggested by David Speyer, is key to our approach.

Proposition-Definition 1.3. For $c \in J_\beta$, define the *cluster variable* x_c to be the unique character of T_β that vanishes to order one on V_c and has neither a pole nor a zero on V_e for $e \in J_\beta \setminus \{c\}$. We denote the cluster by $\mathbf{x}_\beta = \{x_c\}_{c \in J_\beta}$.

We show that the cluster variables thus defined form a basis of the character lattice of T_β, and that they extend to regular functions on the braid variety $\mathring{R}_\beta$. A particular set of generalized minors also

[1] To compare our quivers to the quivers in [GLSBS22, Gal23], all arrows need to be reversed.

form a basis of the character lattice of T_β; the two sets of functions are related by an upper-triangular monomial transformation. Since Deodhar hypersurfaces are irreducible, cluster variables are the irreducible factors of these generalized minors.

1.3. **Exchange matrix.** In earlier works on the construction of cluster structures, the exchange matrix $\tilde{B}$ is obtained directly from the combinatorics of planar bipartite graphs [Pos06, Sco06, GL23] or double wiring diagrams [BFZ05, SW21], and can be defined using "local contributions" from each edge or each crossing, respectively. Our construction of $\tilde{B}$ uses similar combinatorics, but in addition we must take into account the monomial transformation between cluster variables and generalized minors.

In the spirit of Fock and Goncharov [FG06], the exchange matrix $\tilde{B}$ is equivalent to the datum of a 2-form ω_β on $\mathring{R}_\beta$. We define ω_β in terms of generalized minors as a sum of local contributions for each letter of the double braid word β. We introduce integers $\mathbf{d}_\beta=(d_c)_{c\in J_\beta}$ and then expand ω_β in the basis of cluster variables:

$$\omega_\beta = \sum_{c,e\in J_\beta:\ c\le e} d_e\tilde{B}_{ce}\mathrm{dlog}x_c\wedge\mathrm{dlog}x_e = \sum_{c,e\in J_\beta:\ c\le e} d_c\tilde{B}_{ec}\mathrm{dlog}x_e\wedge\mathrm{dlog}x_c. \tag{1.3}$$

The coefficients $\tilde{B}_{ce}$ define a $J_\beta\times J_\beta^{\mathrm{mut}}$ integer matrix $\tilde{B}:=(\tilde{B}_{ce})$. The principal $J_\beta^{\mathrm{mut}}\times J_\beta^{\mathrm{mut}}$ part of the matrix $\tilde{B}$ is skew-symmetrizable, with symmetrizer $\mathrm{diag}(d_c\,|\,c\in J_\beta^{\mathrm{mut}})$. Therefore $\Sigma_\beta:=(\mathbf{x}_\beta,\tilde{B})$ is a seed of a cluster algebra $\mathcal{A}(\Sigma_\beta)$. The content of Theorem 1.1 is that $\mathcal{A}(\Sigma_\beta)=\mathbb{C}[\mathring{R}_\beta]$.

The factorization of generalized minors into cluster variables, and thus the exchange matrix $\tilde{B}$ itself, is difficult to describe directly in a combinatorial manner. In the case $G=\mathrm{SL}_n$, we described the factorization in [GLSBS22] using the combinatorics of *3D plabic graphs*. In this work, we approach it geometrically, via orders of vanishing of the minors on Deodhar hypersurfaces. In Section 7, we give an algorithm, implemented in [Gal23], for computing these orders of vanishing — and thus the entire cluster seed — using only root-system combinatorics.

1.4. **Deletion-contraction induction.** Our proof of Theorem 1.1 is inductive. We introduce in Section 3.2 a *deletion-contraction recurrence* in the context of cluster varieties, which serves as the main ingredient of the inductive step. Let X be an algebraic variety and let Σ be a cluster seed on X. Consider a sink in the mutable part of the quiver, with corresponding cluster variable x. We consider two subvarieties $W:=\{x\neq 0\}$ and $V:=\{x=0\}$ of X. We show in Theorem 3.13 that if both V and W are cluster varieties, and if some technical assumptions on X and Σ are satisfied, then X is a cluster variety. We expect that deletion-contraction can be applied in situations beyond braid varieties.

We apply deletion-contraction to $\mathring{R}_\beta$ when $\beta=ii\beta'$ starts with a repeated letter; the braids $i\beta'$ and β' correspond to deletion and contraction, respectively. To transform an arbitrary braid word β to one of this form, we utilize *double braid moves* $\beta\sim\beta'$ on double braid words that induce natural isomorphisms $\mathring{R}_\beta\cong\mathring{R}_{\beta'}$. The full flexibility of these moves is the reason we consider double braid words and varieties here. Every double braid variety is isomorphic to some braid variety of [Mel19, CGGS24], but using double braid words rather than usual braid words gives us access to more seeds.

In Theorem 4.2, we prove the key feature of double braid moves $\beta\sim\beta'$: under the isomorphism $\mathring{R}_\beta\cong\mathring{R}_{\beta'}$ the corresponding seeds Σ_β and $\Sigma_{\beta'}$ are related by mutation. In Section 4, we prove Theorem 4.2 in the simply-laced case (i.e., for G of type A, D, E); the seeds Σ_β and $\Sigma_{\beta'}$ either coincide or are related by a single mutation. The proof of Theorem 4.2 in the multiply-laced case is achieved via folding in Sections 5 and 6; the seeds Σ_β and $\Sigma_{\beta'}$ are related by a sequence of mutations.

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2. Deodhar geometry

We discuss the geometry of the double braid variety $\mathring{R}_\beta$ with the goal of defining a cluster seed on it. The ingredients of a cluster seed were outlined in Section 1.2. In Section 2.4, we construct a *Deodhar torus* $T_\beta \subset \mathring{R}_\beta$. In Section 2.8, we introduce a family $\mathbf{x}_\beta = \{x_c\}_{c\in J_\beta}$ of *cluster variables* and show that they are regular functions on $\mathring{R}_\beta$. Finally, in Section 2.9, we introduce a 2-form ω_β on T_β from which the $\tilde{B}$-matrix can be extracted via (1.3).

2.1. Background.

For each $i\in I$, we fix a group homomorphism

$$\phi_i : \mathrm{SL}_2 \to G, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_i(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto y_i(t),$$

where $x_i(t), y_i(t)$ are the exponentiated Chevalley generators. The data $(H, B_+, B_-, x_i, y_i; i \in I)$ is a *pinning* of G; see [Lus94, Section 1.1].

Let Φ be the root system of G, with positive roots Φ^+ corresponding to B_+. Let $X^*(H) := \mathrm{Hom}(H, \mathbb{C}^\times)$ be the *character lattice* of H and $X_*(H) := \mathrm{Hom}(\mathbb{C}^\times, H)$ be the *cocharacter lattice* of H. Let $\{\alpha_i\}_{i\in I} \subset X^*(H)$ (resp., $\{\alpha_i^\vee\}_{i\in I} \subset X_*(H)$, $\{\omega_i\}_{i\in I} \subset X^*(H)$) be the simple roots (resp., simple coroots, fundamental weights) of Φ^+. We have a natural pairing $\langle \cdot,\cdot \rangle : X^*(H) \times X_*(H) \to \mathbb{Z}$ satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ and $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$, where $A = (a_{ij})_{i,j\in I}$ is the Cartan matrix of G.

Let the Weyl group W have simple generators $\{s_i\}_{i\in I}$, length function $\ell(\cdot)$, and identity $\mathrm{id} \in W$. For $i \in I$, we set

$$\dot{s}_i = \overline{s_i} := \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \dot{s}_i^{-1} = \overline{\overline{s_i}} := \phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$, where $l = \ell(w)$, we set

$$\dot{w} = \overline{w} := \overline{s_{i_1}} \cdot \overline{s_{i_2}} \cdots \overline{s_{i_l}}, \quad \overline{\overline{w}} := \overline{\overline{s_{i_1}}} \cdot \overline{\overline{s_{i_2}}} \cdots \overline{\overline{s_{i_l}}}. \tag{2.1}$$

The resulting product does not depend on the choice of the reduced expression. For $u \in W$ and $h \in H$, we set $u \cdot h := \dot{u} h \dot{u}^{-1} = \overline{u} h \overline{u}^{-1} = \overline{\overline{u}} h \overline{\overline{u}}^{-1}$. We also consider elements

$$z_i(t) := \phi_i \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} = x_i(t) \dot{s}_i, \quad \bar{z}_i(t) := \phi_i \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} = x_i(-t) \dot{s}_i^{-1}. \tag{2.2}$$

For each $w \in W$, it is well known [Hum75, Proposition 28.1] that the multiplication map gives rise to an isomorphism

$$(\dot{w}^{-1} U_+ \dot{w} \cap U_-) \times (\dot{w}^{-1} U_+ \dot{w} \cap U_+) \xrightarrow{\sim} \dot{w}^{-1} U_+ \dot{w}. \tag{2.3}$$

2.2. Weighted flags.

Recall from Section 1.1 that a *weighted flag* is an element $F = gU_+ \in G/U_+$. Associated to a weighted flag F is the flag $\pi(F) = gB_+$, the image of F in G/B_+.

The following elementary facts can be found in e.g. [SW21, Appendix]; see also [GLSBS22, Section 6.2].

Lemma 2.1. *Let F, F', F'' be weighted flags. Suppose $v, w \in W$ with $\ell(vw) = \ell(v) + \ell(w)$.*

(1) $F \xrightarrow{\mathrm{id}} F'$ if and only if $F = F'$.
(2) If $F \xrightarrow{v} F' \xrightarrow{w} F''$, then $F \xrightarrow{vw} F''$.
(3) If $F \xrightarrow{vw} F''$, then there exists a unique F' such that $F \xrightarrow{v} F' \xrightarrow{w} F''$. If $F \xRightarrow{vw} F''$ then there exist unique F_1', F_2' such that $F \xrightarrow{v} F_1' \xRightarrow{w} F''$ and $F \xRightarrow{v} F_2' \xrightarrow{w} F''$.

Lemma 2.2. *Suppose $F \xrightarrow{s_i} F'$ and say $F = gU_+$. Then there exists a unique $t \in \mathbb{C}$ such that $F' = g z_i(t) U_+$. Similarly, if $F' = g' U_+$, there exists a unique $t' \in \mathbb{C}$ such that $F = g' \bar{z}_i(t') U_+$. The maps $(g, F') \mapsto t$ and $(g', F) \mapsto t'$ are regular on the appropriate subvarieties of $G \times G/U_+$.*

Lemma 2.3. *Suppose $F \xRightarrow{v} gU_+ \xrightarrow{s_i} g z_i(t) U_+$ and $F \xRightarrow{w} g z_i(t) U_+$. If $vs_i > v$, then $w = vs_i$ for all $t \in \mathbb{C}$. If $vs_i < v$, then there exists $t^* \in \mathbb{C}$ such that $w = vs_i$ for $t = t^*$ and $w = v$ for $t \in \mathbb{C} \setminus \{t^*\}$.*

2.3. **Distinguished subexpressions.** To define the Deodhar torus of $\mathring{R}_\beta$, we need the following combinatorics.

Fix $\beta = i_1 i_2 \dots i_m \in (\pm I)^m$. We write $[m] := \{1,2,\dots,m\}$ and $[0,m] := \{0,1,\dots,m\}$. Recall the notation s_i^+ and s_i^- from (1.1).

Given two elements $u, v \in W$ with $u \le v$ in the Bruhat order, we write $\max(u,v) := v$ and $\min(u,v) := u$. We also define $u * s_i := \max(u, us_i)$. The *Demazure product* of β is defined by

$$\delta(\beta) := s_{i_m}^- * s_{i_{m-1}}^- * \cdots * s_{i_1}^- * s_{i_1}^+ * s_{i_2}^+ * \cdots * s_{i_m}^+ \quad \in W. \tag{2.4}$$

From now on, we assume that $\delta(\beta) = w_\circ$.

Definition 2.4. A $w_\circ$*-subexpression* of β is a sequence $\mathbf{u} = (u_0, u_1, \dots, u_m) \in W^{m+1}$ such that $u_0 = \mathrm{id}$, $u_m = w_\circ$, and such that for each $c \in [m]$, we have either $u_{c-1} = u_c$ or $u_{c-1} = s_{i_c}^- u_c s_{i_c}^+$. Since $\delta(\beta) = w_\circ$ there exists a unique "rightmost" subexpression, called the *positive distinguished subexpression*; see [MR04, Lemma 3.5]. It is given by $u_m := w_\circ$ and $u_{c-1} := \min(u_c, s_{i_c}^- u_c s_{i_c}^+)$ for all $c = m, m-1, \dots, 1$.

From now on, we fix $\mathbf{u} = (u_0, u_1, \dots, u_m)$ to be the positive distinguished $w_\circ$-subexpression of β. We also define $w_c := w_\circ u_c$ for $c \in [0,m]$ and $\mathbf{w} = w_\circ \mathbf{u} := (w_0, w_1, \dots, w_m)$. Note that $w_0 = w_\circ$. We set $J_\beta := \{c \in [m] \mid u_c = u_{c-1}\}$. We refer to the indices in J_β as *solid crossings* and to the indices in $[m] \setminus J_\beta$ as *hollow crossings*. We denote $d(\beta) := m - \ell(w_\circ) = |J_\beta|$.

2.4. **Double braid varieties and the Deodhar torus.** Recall that

$$\mathring{\mathcal{Y}}_\beta := \{(X_\bullet, Y_\bullet) \in (G/U_+)^{[0,m]} \times (G/U_+)^{[0,m]} \mid (X_\bullet, Y_\bullet) \text{ satisfy (1.2)}\}$$

and that G acts on $\mathring{\mathcal{Y}}_\beta$ by acting on each weighted flag.

Proposition 2.5. *The G-action on $\mathring{\mathcal{Y}}_\beta$ is free. The quotient $\mathring{R}_\beta := \mathring{\mathcal{Y}}_\beta / G$ is a smooth, affine, irreducible complex algebraic variety of dimension $d(\beta)$.*

Proof. We repeat the argument from [GLSBS22, Proposition 6.13]. Consider the space of tuples of weighted flags satisfying

$$\begin{tikzcd} X_0 = U_+ & X_1 \arrow[l, "s_{i_1}^+"'] & \cdots \arrow[l, "s_{i_2}^+"'] & X_m \arrow[l, "s_{i_m}^+"'] \\ Y_0 \arrow[r, "s_{i_1^*}^-"'] & Y_1 \arrow[r, "s_{i_2^*}^-"'] & \cdots \arrow[r, "s_{i_m^*}^-"'] & Y_m \arrow[u, "\mathrm{id}"'] \end{tikzcd} \tag{2.5}$$

This space is an iterated $\mathbb{C}$-bundle and is thus affine. Imposing the condition that U_+ and Y_0 are weakly $w_\circ$-related (that is, $Y_0 \in B_+ w_\circ B_+ = U_+ w_\circ B_+$) cuts out a nonempty smooth affine open subset V of the iterated $\mathbb{C}$-bundle. The braid variety $\mathring{R}_\beta$ is the quotient of V by the diagonal action of $U_+ = \mathrm{Stab}_G(U_+)$. The group U_+ acts freely on $U_+ \dot{w}_0 B_+$ and thus acts freely on V. It follows that the quotient $\mathring{R}_\beta$ is also smooth and affine; it is also clearly irreducible. Explicitly, one may fix the U_+-action by identifying $\mathring{R}_\beta$ with the subvariety of V where $Y_m \in w_\circ B_+$, which is the viewpoint of [CGGS24].

For the dimension, note that

$$\dim(\mathring{\mathcal{Y}}_\beta) = \dim(G/U_+) + m = \dim(G) - \ell(w_\circ) + m,$$

so $\dim(\mathring{R}_\beta) = \dim(\mathring{\mathcal{Y}}_\beta) - \dim(G) = m - \ell(w_\circ) = d(\beta)$. □

Let $\mathcal{Y}_\beta$ be a partial compactification of $\mathring{\mathcal{Y}}_\beta$ obtained by removing the condition $X_0 \overset{w_\circ}{\Longleftarrow} Y_0$ from (1.2).

Definition 2.6. Let $\tilde{T}_\beta \subset \mathcal{Y}_\beta$ be the set of tuples $(X_\bullet, Y_\bullet) \in \mathcal{Y}_\beta$ satisfying

$$X_c \overset{w_c}{\Longleftarrow} Y_c \quad \text{for } c \in [0,m]. \tag{2.6}$$

Since $w_0 = w_\circ$, we have $\tilde{T}_\beta \subset \mathring{\mathcal{Y}}_\beta$ and thus G acts freely on $\tilde{T}_\beta$. Define the *Deodhar torus* $T_\beta \subset \mathring{R}_\beta$ to be the quotient $T_\beta := \tilde{T}_\beta / G$.

We will show in Corollary 2.10 below that T_β is indeed a $d(\beta)$-dimensional algebraic torus.

Lemma 2.7. *The subsets $\tilde{T}_\beta \subset \mathring{\mathcal{Y}}_\beta$ are open dense in $\mathcal{Y}_\beta$.*

Proof. We can parameterize the variety $\mathcal{Y}_\beta$ as follows. We choose an arbitrary weighted flag $X_m = Y_m$, and then for $c = m, m-1, \dots, 1$, assuming $(X_c, Y_c) = (g_c U_+, g'_c U_+)$, we set (cf. Lemma 2.2)

$$(X_{c-1}, Y_{c-1}) := \begin{cases} (g_c z_{i_c}(t'_c) U_+, g'_c U_+), & \text{if } i_c > 0, \\ (g_c U_+, g'_c \bar{z}_{|i_c|^*}(t'_c) U_+), & \text{if } i_c < 0, \end{cases} \tag{2.7}$$

for arbitrary parameters $\mathbf{t}' := (t'_1, t'_2, \dots, t'_m) \in \mathbb{C}^m$.

For $(X_\bullet, Y_\bullet)$ to be a point in $\tilde{T}_\beta$, (2.6) needs to be satisfied for each $c \in [0, m]$. It is clearly satisfied for $c = m$. If it is satisfied for some solid index $c \in [m]$, then it will be satisfied for $c-1$ if and only if the parameter t'_c is not equal to the value t^* from Lemma 2.3. If the index $c \in [m]$ is instead hollow and (2.6) is satisfied for c it will be satisfied for $c-1$ regardless of the value of t'_c. Thus, $\tilde{T}_\beta$ is indeed an open dense subset of $\mathcal{Y}_\beta$. Since $\tilde{T}_\beta \subset \mathring{\mathcal{Y}}_\beta$, $\mathring{\mathcal{Y}}_\beta$ is dense in $\mathcal{Y}_\beta$. Since $\mathring{\mathcal{Y}}_\beta$ is cut out of $\mathcal{Y}_\beta$ by an open condition $X_0 \xLeftarrow{w_\circ} Y_0$, it is open in $\mathcal{Y}_\beta$. □

2.5. **H-valued functions.** Recall that $H = B_+ \cap B_-$ is the Cartan torus of G. Over the next sections, we will discuss various functions on T_β and $\mathring{R}_\beta$. To do so, it is convenient to introduce a regular map $T_\beta \to H$. We also use this map in this section to show that T_β is an algebraic torus.

Given $(X_\bullet, Y_\bullet) \in \mathring{R}_\beta$, let $Z_c := Y_c^{-1} X_c \in U_+ \backslash G / U_+$. Abusing notation, we use double cosets $Z_c \in U_+ \backslash G / U_+$ interchangeably with their representatives in G. For $(X_\bullet, Y_\bullet) \in T_\beta$, Z_c belongs to the Bruhat cell $\mathring{\mathcal{X}}_{w_c} := B_+ w_c B_+ = U_+ w_c H U_+$ of G.[2] There exist unique elements $h_c^+, h_c^- \in H$ satisfying

$$Z_c \in U_+ \dot{w}_c h_c^+ U_+, \quad Z_c \in U_+ \overline{w_\circ} h_c^- \overline{\overline{u_c}} U_+, \quad \text{thus,} \quad h_c^- = u_c \cdot h_c^+. \tag{2.8}$$

The third statement follows from the first two since $\dot{w}_c = \overline{w_\circ} \cdot \overline{\overline{u_c}}$ and $u_c \cdot h_c^+ = \overline{\overline{u_c}} h_c^+ \overline{\overline{u_c}}^{-1}$. The map $(X_\bullet, Y_\bullet) \mapsto h_c^\pm$ is a rational H-valued function on $\mathring{R}_\beta$ (resp., on $\mathcal{Y}_\beta$), regular on T_β (resp., on $\tilde{T}_\beta$).

Lemma 2.8. *There exist rational functions $(t_c)_{c \in J_\beta}$ on $\mathring{R}_\beta$ such that for $c \in [m]$,*

$$h_{c-1}^+ = \begin{cases} s_{i_c} \cdot h_c^+, & \text{if } c \text{ is hollow, } i_c \in I; \\ \alpha_{i_c}^\vee(t_c) h_c^+, & \text{if } c \text{ is solid, } i_c \in I; \end{cases} \qquad h_{c-1}^- = \begin{cases} s_{|i_c|} \cdot h_c^-, & \text{if } c \text{ is hollow, } i_c \in -I; \\ \alpha_{|i_c|}^\vee(t_c) h_c^-, & \text{if } c \text{ is solid, } i_c \in -I. \end{cases} \tag{2.9}$$

Proof. Fix $(X_\bullet, Y_\bullet) \in T_\beta$. For $c \in J_\beta$, define t_c to be such that if $Z_c = \dot{w}_c h_c^+ = \overline{w_\circ} h_c^- \overline{\overline{u_c}}$ then $Z_{c-1} = Z_c z_{i_c}(t_c)$ if $i_c \in I$ and $Z_{c-1} = \bar{z}_{|i_c|^*}(t_c)^{-1} Z_c$ if $i_c \in -I$; see Lemma 2.2. The following identities in G can be checked inside SL_2:

$$x_i(t) \dot{s}_i = y_i(1/t) \alpha_i^\vee(t) x_i(-1/t) \quad \text{and} \quad \dot{s}_i x_i(t) = x_i(-1/t) \alpha_i^\vee(1/t) y_i(1/t). \tag{2.10}$$

Suppose that $i_c \in I$. We have $Z_{c-1} = \dot{w}_c h_c^+ x_{i_c}(t_c) \dot{s}_{i_c}$. If c is hollow then $\dot{w}_c h_c^+ x_{i_c}(t_c) \in U_+ \dot{w}_c h_c^+$, and thus $Z_{c-1} \in U_+ \dot{w}_c \dot{s}_{i_c} (s_{i_c} \cdot h_c^+)$. This implies that $h_{c-1}^+ = s_{i_c} \cdot h_c^+$. If c is solid then we use the first identity in (2.10) to write $Z_{c-1} = \dot{w}_c h_c^+ y_{i_c}(1/t_c) \alpha_{i_c}^\vee(t_c) x_{i_c}(-1/t_c)$. Since $\dot{w}_c h_c^+ y_{i_c}(t) \in U_+ \dot{w}_c h_c^+$, we see that $Z_{c-1} \in U_+ \dot{w}_c h_c^+ \alpha_{i_c}^\vee(t_c) U_+$, which implies that $h_{c-1}^+ = h_c^+ \alpha_{i_c}^\vee(t_c)$.

The case when $i_c \in -I$ is handled similarly. When c is solid, we use the second identity in (2.10) together with $\alpha_{|i_c|^*}^\vee(1/t_c) \overline{w_\circ} = \overline{w_\circ} \alpha_{|i_c|}^\vee(t_c)$; see [FZ99, Equation (1.2)]. □

Corollary 2.9. *Suppose c is hollow. If $i_c \in I$ then $h_{c-1}^- = h_c^-$, and if $i_c \in -I$ then $h_{c-1}^+ = h_c^+$.*

Corollary 2.10. *The Deodhar torus $T_\beta \subset \mathring{R}_\beta$ is isomorphic to an algebraic torus of dimension $d(\beta)$, and the functions $(t_c)_{c \in J_\beta}$ form a basis of the character lattice of T_β.*

Proof. For $c \in J_\beta$, the function t_c is regular on T_β by Lemma 2.2. With notation as in the proof of Lemma 2.8, we have $Z_c = \dot{w}_c h_c^+ = \overline{w_\circ} h_c^- \overline{\overline{u_c}}$ and $Z_{c-1} = Z_c z_{i_c}(t_c)$ if $i_c \in I$ and $Z_{c-1} = \bar{z}_{|i_c|^*}(t_c)^{-1} Z_c$

[2] Here and below, we omit the dot over $\dot{w}_c$ in products such as $B_+ w_c B_+$ that involve multiplying $\dot{w}_c$ by a subgroup of G containing H.

if $i_c \in -I$. It follows that $t_c \neq 0$ if and only if $Z_{c-1} \in \mathring{\mathcal{X}}_{w_{c-1}} = \mathring{\mathcal{X}}_{w_c}$ (thus, $t^* = 0$ in the notation of Lemma 2.3). Thus, we get a regular map $T_\beta \to (\mathbb{C}^\times)^{J_\beta}$, $(X_\bullet, Y_\bullet) \mapsto (t_c)_{c \in J_\beta}$.

To show that this map is an isomorphism, we construct the inverse: given $(t_c)_{c\in J_\beta}$, we explain how to recover $(X_\bullet,Y_\bullet) \in T_\beta$. We have $h_m^+ = h_m^- = 1 \in H$, and for $c = m-1, m-2, \dots, 0$, we recover h_c^+, h_c^- using (2.9). Next, since $(X_\bullet,Y_\bullet)$ is defined up to G-action, we may assume that $Y_0 = U_+$ and $X_0 = \dot{w}_\circ h_0^+$. Suppose that for $c = 1,2,\dots,m$, (X_{c-1},Y_{c-1}) has been recovered. If c is hollow then (X_c,Y_c) is recovered uniquely using part (3) of Lemma 2.1. If c is solid then the parameter t'_c in (2.7) is recovered uniquely from t_c by an argument similar to the proof of Lemma 2.8. (Alternatively, one can recover t'_c as a ratio of grid minors discussed in the next subsection.) We have constructed the inverse map, and it is clearly regular on the torus $(\mathbb{C}^\times)^{c\in J_\beta}$. □

2.6. **Grid and chamber minors.** In this section, we introduce a basis of characters of T_β consisting of certain generalized minors called *grid minors*. This basis will be crucial in our definition of the exchange matrix. We will show later in Corollary 2.24 that grid minors are related to cluster variables by an invertible monomial transformation.

Recall that $A = (a_{ij})_{i,j\in I}$, $a_{ij} := \langle \alpha_i, \alpha_j^\vee \rangle$ is the Cartan matrix, and that $d_i a_{ij} = d_j a_{ji}$. For $i,j \in \pm I$, we define $a_{ij} = 0$ if i,j have different signs, and $a_{ij} = a_{(-i)(-j)}$ otherwise. Also set $d_{-i} := d_i$ for $i \in I$. Abusing notation, given a double braid word β, we let

$$\mathbf{d}_\beta = (d_c)_{c\in J_\beta}, \quad \text{where} \quad d_c := d_{i_c} \quad \text{for } c \in J_\beta. \tag{2.11}$$

Following [FZ99, Definition 1.4], for $i \in I$ and $v,w \in W$, we have a *generalized minor* $\Delta_{v\omega_i,w\omega_i} : G \to \mathbb{C}$. It is a regular function satisfying

$$\Delta_{\omega_i,\omega_i}(y_- x y_+) = \Delta_{\omega_i,\omega_i}(x) \quad \text{for all } (y_-,x,y_+) \in U_- \times G \times U_+; \tag{2.12}$$

$$\Delta_{v\omega_i,w\omega_i}(x) = \Delta_{\omega_i,\omega_i}\left(\overline{\overline{v^{-1}}} x \overline{w}\right) = \Delta_{\omega_i,\omega_i}\left(\dot{v}^{-1} x \dot{w}\right); \tag{2.13}$$

see [FZ99, Section 1.4]. For $h \in H$, we have $\Delta_{\omega_i,\omega_i}(h) = h^{\omega_i}$. For $x \in G$, we also have [FZ99, Equation (2.14)]

$$\Delta_{\omega_k,\omega_k}(xh) = \Delta_{\omega_k,\omega_k}(hx) = h^{\omega_k} \Delta_{\omega_k,\omega_k}(x). \tag{2.14}$$

Definition 2.11. For $c \in [0,m]$ and $k \in I$, we define the *grid minors*

$$\Delta_{c,k}(X_\bullet,Y_\bullet) = \Delta_{w_c\omega_k,\omega_k}(Z_c) \quad \text{and} \quad \Delta_{c,-k}(X_\bullet,Y_\bullet) = \Delta_{w_\circ\omega_k, u_c^{-1}\omega_k}(Z_c). \tag{2.15}$$

The *chamber minors* are defined as $\Delta_c := \Delta_{c-1,i_c}$, for $c \in [m]$.

Lemma 2.12. *For $c \in [0,m]$ and $k \in I$, the grid minors $\Delta_{c,k}$ and $\Delta_{c,-k}$ are well-defined regular functions on $\tilde{T}_\beta$. For $(X_\bullet,Y_\bullet) \in \tilde{T}_\beta$, we have*

$$\Delta_{c,k}(X_\bullet,Y_\bullet) = (h_c^+)^{\omega_k} \quad \text{and} \quad \Delta_{c,-k}(X_\bullet,Y_\bullet) = (h_c^-)^{\omega_k}. \tag{2.16}$$

Proof. Recall that we view $Z_c \in U_+\backslash G/U_+$ as an element of G, and that for $(X_\bullet,Y_\bullet) \in \tilde{T}_\beta$, we have $Z_c \in \mathring{\mathcal{X}}_{w_c} = U_+ w_c H U_+$ for all $c \in [0,m]$. Write $Z_c = y'_+ \dot{w}_c h_c^+ y''_+$ for $y'_+, y''_+ \in U_+$. We have

$$\Delta_{c,k}(X_\bullet,Y_\bullet) = \Delta_{w_c\omega_k,\omega_k}(Z_c) = \Delta_{\omega_k,\omega_k}\left(\overline{\overline{w_c^{-1}}} y'_+ \dot{w}_c h_c^+ y''_+\right) = \Delta_{\omega_k,\omega_k}(\dot{w}_c^{-1} y'_+ \dot{w}_c h_c^+).$$

Factorizing $\dot{w}_c^{-1} y'_+ \dot{w}_c = b_- b_+$ for $(b_-,b_+) \in U_- \times U_+$ using (2.3), using $(h_c^+)^{-1} b_+ h_c^+ \in U_+$, and applying (2.14), we get the first identity in (2.16). In particular, since the result does not depend on $y'_+, y''_+ \in U_+$, we see that $\Delta_{c,k}(Z_c)$ is invariant under the $U_+ \times U_+$-action on Z_c, and thus descends to a well-defined function on $U_+\backslash \mathring{\mathcal{X}}_{w_c}/U_+$. The proof for $\Delta_{c,-k}$ is similar. □

Corollary 2.13. *For each $c \in [0,m]$ and $k \in I$, the grid minors $\Delta_{c,k}$ and $\Delta_{c,-k}$ are well-defined regular functions on $\mathcal{Y}_\beta$, $\mathring{\mathcal{Y}}_\beta$, and $\mathring{R}_\beta$. These regular functions commute with the quotient map $\mathring{\mathcal{Y}}_\beta \to \mathring{R}_\beta$ and the inclusion map $\mathring{\mathcal{Y}}_\beta \hookrightarrow \mathcal{Y}_\beta$.*

Proof. For $(X_\bullet,Y_\bullet)\in\tilde{T}_\beta$, we have $Z_c\in\mathring{\mathcal{X}}_{w_c}$, and the proof of Lemma 2.12 implies that $\Delta_{w_c\omega_k,\omega_k}$ is well defined on $U_+\backslash\mathring{\mathcal{X}}_{w_c}/U_+$. Since $\tilde{T}_\beta$ is dense in $\mathcal{Y}_\beta$ by Lemma 2.7, it follows that for $(X_\bullet,Y_\bullet)\in\mathcal{Y}_\beta$, we have $Z_c\in\mathcal{X}_{w_c}$. By continuity, it follows that $\Delta_{w_c\omega_k,\omega_k}$ is well defined on $U_+\backslash\mathcal{X}_{w_c}/U_+$. Thus, $\Delta_{c,k}$ (and similarly $\Delta_{c,-k}$) is well defined on $\mathcal{Y}_\beta$. The fact that it commutes with the quotient map $\mathring{\mathcal{Y}}_\beta\to\mathring{R}_\beta$ follows since the map $(X_c,Y_c)\mapsto Z_c=Y_c^{-1}X_c$ commutes with the G-action $(X_c,Y_c)\mapsto(gX_c,gY_c)$, $g\in G$, on $\mathring{\mathcal{Y}}_\beta$. It also trivially commutes with the inclusion map $\mathring{\mathcal{Y}}_\beta\hookrightarrow\mathcal{Y}_\beta$. ☐

Combining Lemma 2.12 with (2.9), we get the following.

Corollary 2.14. *If c is solid and $k\in\pm I$ has the same sign as i_c then*

$$\Delta_{c-1,k}=\begin{cases}t_c\Delta_{c,k}, & \text{if } k=i_c;\\ \Delta_{c,k}, & \text{if } k\neq i_c.\end{cases}\tag{2.17}$$

Proposition 2.15.

(1) The grid minors are characters of T_β.
(2) The solid chamber minors $(\Delta_c)_{c\in J_\beta}$ form a basis of the character lattice of T_β.

Proof. We relate the parameters $(t_c)_{c\in J_\beta}$ from Corollary 2.10 to the grid and chamber minors by combining (2.9) with (2.16). Suppose that $i_c\in I$ and let $k\in I$. If $k\neq i_c$ then $\Delta_{c-1,k}=\Delta_{c,k}$ by Corollary 2.14. If c is hollow then

$$\Delta_{c-1,i_c}=(h_{c-1}^+)^{\omega_{i_c}}=(s_{i_c}\cdot h_c^+)^{\omega_{i_c}}=(h_c^+)^{s_{i_c}\omega_{i_c}}.$$

Expand $s_{i_c}\omega_{i_c}=\omega_{i_c}-\alpha_{i_c}$ in the basis of fundamental weights using $\alpha_{i_c}=\sum_{j\in I}a_{i_cj}\omega_j$. This gives

$$\Delta_{c-1,i}\Delta_{c,i}\prod_{j\neq i}\Delta_{c,j}^{a_{ij}}=1,\quad \text{if } c \text{ is hollow and } i:=i_c,\tag{2.18}$$

which holds for $i_c\in\pm I$. If c is solid, Corollary 2.14 yields $\Delta_{c-1,i_c}=t_c\Delta_{c,i_c}$. Thus,

(i) For each solid $c\in J_\beta$, $t_c=\Delta_{c-1,i_c}/\Delta_{c,i_c}$ is a ratio of two grid minors.
(ii) For each solid $c\in J_\beta$, the grid minors $(\Delta_{c-1,j})_{j\in\pm I}$ are Laurent monomials in the grid minors $(\Delta_{c,k})_{k\in\pm I}$ and the chamber minor $\Delta_c=\Delta_{c-1,i_c}$.
(iii) For each hollow $c\in[m]\setminus J_\beta$, the grid minors $(\Delta_{c-1,j})_{j\in\pm I}$ are Laurent monomials in the grid minors $(\Delta_{c,k})_{k\in\pm I}$.
(iv) Every grid minor $\Delta_{c,j}$ is a Laurent monomial in the solid chamber minors $(\Delta_e)_{e\in J_\beta}$.

We have already shown (i)–(iii), and (iv) follows from (ii)–(iii). This implies the result. ☐

2.7. **Almost positive sequences and Deodhar hypersurfaces.** Here we discuss the complement of the Deodhar torus and its irreducible components, the Deodhar hypersurfaces. We first introduce additional combinatorics, which we will use to describe an open subset of the Deodhar hypersurface.

Recall that we have $u_{c-1}=\min(u_c,s_{i_c}^-u_cs_{i_c}^+)$ and $w_{c-1}=\max(w_c,s_{i_c^*}^-w_cs_{i_c}^+)$ for all $c\in[m]$.

Definition 2.16. Let $e\in J_\beta$. Let $u_m^{\langle e\rangle}:=w_\circ$, and for $c=m,m-1,\dots,1$, define

$$u_{c-1}^{\langle e\rangle}:=\begin{cases}\max(u_e^{\langle e\rangle},s_{i_e}^-u_e^{\langle e\rangle}s_{i_e}^+), & \text{if } c=e,\\ \min(u_c^{\langle e\rangle},s_{i_c}^-u_c^{\langle e\rangle}s_{i_c}^+), & \text{otherwise.}\end{cases}$$

We call the sequence $\mathbf{u}^{\langle e\rangle}:=(u_0^{\langle e\rangle},\dots,u_m^{\langle e\rangle})$ the $\langle e\rangle$-*almost positive sequence.* We set $w_c^{\langle e\rangle}:=w_\circ u_c^{\langle e\rangle}$ for all $c\in[0,m]$, and write $\mathbf{w}^{\langle e\rangle}=w_\circ\mathbf{u}^{\langle e\rangle}:=(w_\circ u_0^{\langle e\rangle},\dots,w_\circ u_m^{\langle e\rangle})$.

Definition 2.17. We say that $e\in J_\beta$ is *mutable* if $u_0^{\langle e\rangle}=\mathrm{id}$. Otherwise, e is *frozen*. We let J_β^{mut} (resp., J_β^{fro}) denote the set of mutable (resp., frozen) indices.

Definition 2.18. Let $e\in J_\beta$. Define the *open Deodhar hypersurface* $\tilde{V}_e^\circ\subset\mathcal{Y}_\beta$ by

$$\tilde{V}_e^\circ:=\{(X_\bullet,Y_\bullet)\in\mathcal{Y}_\beta\mid X_c\xLeftarrow{w_c^{\langle e\rangle}}Y_c\quad\text{for all } c\in[0,m]\}.\tag{2.19}$$

Define the *(closed) Deodhar hypersurface* $\tilde{V}_e\subset\mathcal{Y}_\beta$ to be the closure of $\tilde{V}_e^\circ$.

It follows that an index $e\in J_\beta$ is mutable (resp., frozen) if and only if $\tilde V_e^\circ\subset\mathring{\mathcal{Y}}_\beta$ (resp., $\tilde V_e^\circ\cap\mathring{\mathcal{Y}}_\beta=\emptyset$). If e is mutable then the G-action on $\tilde V_e^\circ$ is free. In this case, we set $V_e^\circ:=\tilde V_e^\circ/G$ and let V_e be the closure of V_e° in $\mathring{R}_\beta$.

Proposition 2.19. *The closed subset $\mathcal{Y}_\beta\setminus\tilde T_\beta$ is a union of the Deodhar hypersurfaces $\tilde V_e$ for $e\in J_\beta$. Each $\tilde V_e$ is irreducible and has codimension one in $\mathcal{Y}_\beta$, and the hypersurfaces $\tilde V_e,\tilde V_{e'}$ are distinct for distinct $e,e'\in J_\beta$.*

Proof. We prove the second sentence. As explained in the proof of Lemma 2.7, $\mathcal{Y}_\beta$ is an iterated fiber bundle over G/U_+, where each fiber is either $\mathbb{C}$ (if c is hollow) or $\mathbb{C}^\times$ (if c is solid). Similarly, $\tilde V^\circ$ is an iterated fiber bundle over G/U_+, where each fiber is either $\mathbb{C}$, $\mathbb{C}^\times$, or (in the case of the crossing $c=e$) a point. It follows that $\tilde V_e$ is an irreducible subvariety of $\mathcal{Y}_\beta$ of codimension one: we have $\dim\mathcal{Y}_\beta=\dim(G/U_+)+m$ and $\dim\tilde V_e=\dim\tilde V_e^\circ=\dim(G/U_+)+m-1$. For distinct $e,e'\in J_\beta$, any point $(X_\bullet,Y_\bullet)$ in $\tilde V_{e'}^\circ$ satisfies $Z_{e-1}\in\mathring{\mathcal{X}}_{w_{e-1}}$. Meanwhile, any point $(X_\bullet,Y_\bullet)$ in $\tilde V_e$ satisfies $Z_{e-1}\in\mathcal{X}_{w_{e-1}^{\langle e\rangle}}$. Since $w_{e-1}^{\langle e\rangle}<w_{e-1}$ in the Bruhat order, we see that $\tilde V_{e'}^\circ\not\subset\tilde V_e$, and thus $\tilde V_{e'}\neq\tilde V_e$.

We now prove the first sentence. First, recall from Lemma 2.7 that $\tilde T_\beta$ is open in $\mathcal{Y}_\beta$, so $\mathcal{Y}_\beta\setminus\tilde T_\beta$ is closed. For each $e\in[m]$, we introduce an auxiliary subset

$$\tilde V_{\geq e}:=\{(X_\bullet,Y_\bullet)\in\mathcal{Y}_\beta\,|\,X_c\overset{w_c}{\Longleftarrow}Y_c\quad\text{for all } c\geq e,\quad\text{and}\quad X_{e-1}\overset{w'}{\Longleftarrow}Y_{e-1}\text{ for } w'\neq w_{e-1}.\}.\tag{2.20}$$

Recall that if e is hollow then $X_e\overset{w_e}{\Longleftarrow}Y_e$ implies $X_{e-1}\overset{w_{e-1}}{\Longleftarrow}Y_{e-1}$. Thus, $\tilde V_{\geq e}$ is empty unless $e\in J_\beta$. For $e\in J_\beta$, the element w' in (2.20) must be equal to $w_{e-1}^{\langle e\rangle}$ because there are only two possibilities for the relative position of (X_{e-1},Y_{e-1}) given that $X_e\overset{w_e}{\Longleftarrow}Y_e$.

Let $(X_\bullet,Y_\bullet)\in\mathcal{Y}_\beta\setminus\tilde T_\beta$. Then (2.6) must fail for $(X_\bullet,Y_\bullet)$ for some index $c\in[0,m]$. We always have $X_m\overset{w_m}{\Longleftarrow}Y_m$, so there exists a unique $e\in[m]$ such that $(X_\bullet,Y_\bullet)\in\tilde V_{\geq e}$. Applying an iterated fiber bundle argument as above, we see that $\tilde V_e^\circ$ is an open dense subset of $\tilde V_{\geq e}$, and therefore

$$\tilde V_e^\circ\subset\tilde V_{\geq e}\subset\tilde V_e.\tag{2.21}$$

Thus, $(X_\bullet,Y_\bullet)\in\tilde V_e$. We have shown that $\mathcal{Y}_\beta\setminus\tilde T_\beta=\bigcup_{e\in J_\beta}\tilde V_e$. □

2.8. **Cluster variables.** By Proposition 2.19, the irreducible components of $\mathcal{Y}_\beta\setminus\tilde T_\beta$ are the Deodhar hypersurfaces $\tilde V_e$, $e\in J_\beta$. For a grid minor $\Delta_{c,k}$ and $e\in J_\beta$, we denote by $\mathrm{ord}_{V_e}\Delta_{c,k}\in\mathbb{Z}$ the order of vanishing of $\Delta_{c,k}$ on the hypersurface $\tilde V_e$; cf. Corollary 2.13. Since $\Delta_{c,k}$ is regular on $\mathcal{Y}_\beta$, we have that $\mathrm{ord}_{V_e}\Delta_{c,k}\geq 0$. In this section, we utilize properties of $\mathrm{ord}_{V_e}\Delta_{c,k}$ to define a new basis of characters of T_β, the cluster variables.

We have the following basic unitriangularity property.

Proposition 2.20. *For $e\in J_\beta$ solid, $c\in[0,m]$, and $k\in\pm I$, we have*

$$\mathrm{ord}_{V_e}\Delta_{c,k}=\begin{cases}0, & \text{if } e\leq c;\\ 1, & \text{if } (c,k)=(e-1,i_e),\ \text{i.e., } \Delta_{c,k}=\Delta_e.\end{cases}\tag{2.22}$$

Proof. Suppose that $e\leq c$. Let $(X_\bullet,Y_\bullet)$ be a generic point in $\tilde V_e$. Then we have $X_c\overset{w_c}{\Longleftarrow}Y_c$, and thus $\Delta_{c,k}(X_\bullet,Y_\bullet)\neq 0$. It follows that $\mathrm{ord}_{V_e}\Delta_{c,k}=0$ when $e\leq c$.

Suppose now that $(c,k)=(e-1,i_e)$. Recall that we have introduced an open dense subset $\tilde V_{\geq e}\subset\tilde V_e$ in (2.20). The subset $\tilde T_\beta\cup\tilde V_{\geq e}\subset\tilde T_\beta\cup\tilde V_e$ is thus also open dense, and recall that for $(X_\bullet,Y_\bullet)\in\tilde T_\beta\cup\tilde V_{\geq e}$, we have $X_{e'}\overset{w_{e'}}{\Longleftarrow}Y_{e'}$ for all $e'\geq e$.

We may parameterize $\mathcal{Y}_\beta$ using parameters $(\mathbf{t}',X_m=Y_m)$ as in (2.7). We specialize all of these parameters except for $t:=t'_e$ to some fixed generic values, and view the resulting tuple $(X_\bullet,Y_\bullet)=(X_\bullet(t),Y_\bullet(t))$ as a function of t. Since the parameters $\mathbf{t}'_{>e}:=(t'_{e+1},\dots,t'_m,X_m=Y_m)$ are generic, we have $(X_\bullet,Y_\bullet)\in\tilde T_\beta\cup\tilde V_{\geq e}$ (and therefore $X_e\overset{w_e}{\Longleftarrow}Y_e$) for all $t\in\mathbb{C}$.

Assume that $i_e \in I$. Thus, we have $Z_e \in U_+ \dot{w}_e h U_+$ for $h = h_e^+ \in H$. The proof of Lemma 2.12 implies that $\dot{w}_e^{-1} Z_e \in U_- h U_+$. Let us write $\dot{w}_e^{-1} Z_e = y_- h y_+$ for $(y_-, y_+) \in U_- \times U_+$. Since $e \in J_\beta$ is solid, we have $w_{e-1} = w_e$. We find

$$\Delta_{e-1,k}(X_\bullet, Y_\bullet) = \Delta_{w_{e-1}\omega_k,\omega_k}(Z_e z_k(t)) = \Delta_{\omega_k,\omega_k}(\dot{w}_e^{-1} Z_e z_k(t)) = \Delta_{\omega_k,\omega_k}(y_- h y_+ z_k(t)).$$

Recall that $z_k(t) = x_k(t)\dot{s}_k$. Let $\Psi := \Phi^+ \setminus \{\alpha_k\}$ and let $U_+(\Psi) := (\dot{s}_k^{-1} U_+ \dot{s}_k) \cap U_+$ be the corresponding *root subgroup*; see [Hum75, Theorem 26.3]. We have $x_k(-t) U_+(\Psi) x_k(t) \subset U_+(\Psi)$ by [Hum75, Lemma 32.5]. Next, we have $\dot{s}_k^{-1} U_+(\Psi) \dot{s}_k \subset U_+(\Psi)$, since s_k permutes Ψ. Using (2.3), we can factorize $y_+ = x_k(p) y_+'$ for some $p \in \mathbb{C}$ and $y_+' \in U_+(\Psi)$. (Here p depends only on the parameters in $\mathbf{t}'_{>e}$ and not on t.) We therefore get $y_+' x_k(t) \dot{s}_k \in x_k(t) \dot{s}_k U_+$. Using (2.12), we get

$$\Delta_{e-1,k}(X_\bullet, Y_\bullet) = \Delta_{\omega_k,\omega_k}(y_- h x_k(p) y_+' x_k(t) \dot{s}_k) = \Delta_{\omega_k,\omega_k}(h x_k(p+t) \dot{s}_k).$$

It is clear that if $p+t=0$ then $\Delta_{e-1,k}(X_\bullet, Y_\bullet) = 0$. If $p+t \neq 0$, applying the first identity in (2.10) to $x_k(p+t)\dot{s}_k$ and using (2.14), we find

$$\Delta_{e-1,k}(X_\bullet, Y_\bullet) = (p+t)\Delta_{\omega_k,\omega_k}(h). \tag{2.23}$$

Thus, (2.23) holds regardless of whether $p+t=0$, and we have $p+t=0$ if and only if the condition $X_{e-1} \overset{w_{e-1}}{\Longleftarrow} Y_{e-1}$ fails, i.e., $(X_\bullet, Y_\bullet) \in \tilde{V}_{\geq e}$. By (2.23), since $\Delta_{\omega_k,\omega_k}(h) \neq 0$, we have $p+t=0$ if and only if $\Delta_{e-1,k}(X_\bullet, Y_\bullet) = 0$. Since $\Delta_{e-1,k}$ is of degree 1 in t, we find that $\mathrm{ord}_{V_e} \Delta_{e-1,k} \leq 1$. On the other hand, we have shown that $\Delta_{e-1,k}$ vanishes on $\tilde{V}_{\geq e}$, and thus on $\tilde{V}_e$ (cf. (2.21)), so $\mathrm{ord}_{V_e} \Delta_{e-1,k} \geq 1$. □

The integers $\mathrm{ord}_{V_e} \Delta_{c,k}$ are nonnegative. Our next result shows that whether $\mathrm{ord}_{V_e} \Delta_{c,k}$ is zero or positive is determined by the almost positive subexpression $\mathbf{u}^{\langle e \rangle}$. The stronger result that $\mathrm{ord}_{V_e} \Delta_{c,k} \in \{0,1\}$ holds when $G = \mathrm{SL}_n$ [GLSBS22, Proposition 7.10]. The precise value of $\mathrm{ord}_{V_e} \Delta_{c,k}$ for G of arbitrary type is given in Section 7.

Proposition 2.21. *For all $c \in [0,m]$, $e \in J_\beta$, and $k \in I$, we have*

$$\mathrm{ord}_{V_e} \Delta_{c,k} = 0 \iff u_c \omega_k = u_c^{\langle e \rangle} \omega_k \quad \textit{and} \quad \mathrm{ord}_{V_e} \Delta_{c,-k} = 0 \iff u_c^{-1} \omega_k = (u_c^{\langle e \rangle})^{-1} \omega_k.$$

Proof. Let $\leq$ denote the Bruhat order on W and the quotient order on the orbit $W\omega_k$ of the fundamental weight ω_k. Comparing Definitions 2.4 and 2.16, we see that $u_c \leq u_c^{\langle e \rangle}$ and $w_c \geq w_c^{\langle e \rangle}$ for all $c \in [0,m]$. Thus $w_c \omega_k \geq w_c^{\langle e \rangle} \omega_k$ for all $c \in [0,m]$ and $k \in I$.

For $(X_\bullet, Y_\bullet) \in \tilde{V}_e^\circ$, we have $Z_c \in \mathring{\mathcal{X}}_{w_c^{\langle e \rangle}} \subset \mathcal{X}_{w_c}$ for all $c \in [0,m]$, because $w_c^{\langle e \rangle} \leq w_c$. Recall that $\Delta_{c,k}(X_\bullet, Y_\bullet) = \Delta_{w_c\omega_k,\omega_k}(Z_c)$. It is well known that the function $\Delta_{w_c\omega_k,\omega_k}(Z_c)$, when restricted to $Z_c \in \mathcal{X}_{w_c}$, does not vanish at $Z_c \in \mathring{\mathcal{X}}_{w_c^{\langle e \rangle}}$ if and only if $w_c^{\langle e \rangle} \omega_k = w_c \omega_k$; this can be shown by e.g. adapting the proof of [FZ99, Proposition 2.4]. Similarly, we consider $\Delta_{c,-k}(X_\bullet, Y_\bullet) = \Delta_{w_\circ \omega_k, u_c^{-1}\omega_k}(Z_c)$ and observe that this function does not vanish at $Z_c \in \mathring{\mathcal{X}}_{w_\circ u_c^{\langle e \rangle}} \subset \mathcal{X}_{w_\circ u_c}$ if and only if $u_c^{-1}\omega_k = (u_c^{\langle e \rangle})^{-1}\omega_k$. □

Corollary 2.22. *The $J_\beta \times J_\beta$ matrix $M_\beta = (\mathrm{ord}_{V_e} \Delta_c)_{c,e \in J_\beta}$ is upper unitriangular.*

Inverting the matrix M_β, we arrive at the following definition, which is crucial for our analysis; cf. Proposition-Definition 1.3. Recall from Proposition 2.15 that a character on T_β is just a Laurent monomial in the solid chamber minors $\{\Delta_c\}_{c \in J_\beta}$.

Definition 2.23. For $c \in J_\beta$, the *cluster variable* x_c is the character of T_β satisfying

$$\mathrm{ord}_{V_e} x_c = \begin{cases} 1, & \text{if } c = e, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } e \in J_\beta. \tag{2.24}$$

We denote the *cluster* by $\mathbf{x}_\beta = \{x_c\}_{c \in J_\beta}$.

The conditions (2.24) are equivalent to

$$\Delta_{c,k} = \prod_{e \in J_\beta} x_e^{\mathrm{ord}_{V_e} \Delta_{c,k}} \quad \text{for all } c \in J_\beta \text{ and } k \in \pm I. \tag{2.25}$$

Corollary 2.24.

(1) *For each* $c\in J_\beta$*, there exists a unique character* $x_c\in X^*(T_\beta)$ *satisfying the conditions* (2.24).
(2) *For* $c\in J_\beta$*, the character* $x_c\in X^*(T_\beta)$ *extends to a regular function on* $\mathring{R}_\beta$, $\mathring{\mathcal{Y}}_\beta$, $\mathcal{Y}_\beta$.
(3) *For* $c\in J_\beta^{\mathrm{fro}}$*, the frozen cluster variable* x_c *is invertible in* $\mathbb{C}[\mathring{R}_\beta]$ *and* $\mathbb{C}[\mathring{\mathcal{Y}}_\beta]$.
(4) *The cluster variables in* $\mathbf{x}_\beta$ *are irreducible and algebraically independent.*

Proof. The existence and uniqueness of x_c follows from the invertibility of M_β: writing $M_\beta^{-1}=(m_{e,c})_{e,c\in J_\beta}$, we have $x_c=\prod_{e\in J_\beta}\Delta_c^{m_{e,c}}$. For $c\in J_\beta$, x_c is a rational function on $\mathcal{Y}_\beta$ which does not have a pole on $\tilde{T}_\beta$ or on $\tilde{V}_e$ for all $e\in J_\beta$. This implies that x_c is regular on $\mathring{R}_\beta$, $\mathring{\mathcal{Y}}_\beta$, $\mathcal{Y}_\beta$. By Proposition 2.19, x_c is irreducible. Since $\mathbf{x}_\beta=\{x_c\}_{c\in J_\beta}$ is a basis of the character lattice of T_β, we see that the cluster variables in $\mathbf{x}_\beta$ are algebraically independent. Finally, for $c\in J_\beta^{\mathrm{fro}}$, the function $1/x_c$ is regular on $\mathring{\mathcal{Y}}_\beta\setminus\tilde{V}_c^\circ$, and as we mentioned after Definition 2.18, we have $\tilde{V}_c^\circ\cap\mathring{\mathcal{Y}}_\beta=\emptyset$ for $c\in J_\beta^{\mathrm{fro}}$. ☐

Recall from Proposition 2.20 that $\mathrm{ord}_{V_e}\Delta_{c,k}$ can only be nonzero when $e>c$; see also (2.25). The next result follows from the parametrization (2.7) and will be used later in the proof. We denote $\mathrm{ord}_{V_e}\Delta_{c,k}$ by $\mathrm{ord}_{V_e}^{(\beta)}\Delta_{c,k}$ to emphasize dependence on β.

Lemma 2.25. *The integer* $\mathrm{ord}_{V_e}\Delta_{c,k}$ *only depends on* $i_{c+1},\dots,i_m$*. That is, suppose that* $\beta=i_1i_2\cdots i_m$ *and* $\beta'=i'_1i'_2\cdots i'_{m'}$ *are two double braid words in the alphabet* $\pm I$ *such that for some* $c\in[m]$ *and* $c'\in[m']$ *(with* $m-c=m'-c'$*), we have* $i_{c+1}\cdots i_m=i'_{c'+1}\cdots i'_{m'}$*. Then we have*

$$\mathrm{ord}_{V_e}^{(\beta)}\Delta_{c,k}=\mathrm{ord}_{V_{e'}}^{(\beta')}\Delta_{c',k}$$

for all $k\in\pm I$, $e>c$*, and* $e'>c'$ *such that* $m-e=m'-e'$.

2.9. **A two-form on the braid variety, and a seed.** We now introduce a two-form, which, together with the cluster variables, determines an exchange matrix via (1.3). At the end of the section, we put everything together to define a seed for the braid variety.

We start by introducing a family of 1-forms on T_β.

For $i,j\in\pm I$, recall that $a_{ij}=0$ if i,j have different signs, and $a_{ij}=a_{(-i)(-j)}$ otherwise, and that $d_i:=d_{|i|}$. For each $c\in[0,m]$ and $i\in\pm I$, we set

$$L_{c,i}:=\mathrm{dlog}\left(\prod_{k\in\pm I}\Delta_{c,k}^{a_{ik}}\right)=\frac{1}{2}\sum_{k\in\pm I}a_{ik}\mathrm{dlog}\Delta_{c,k}=\frac{1}{2}\sum_{k\in\pm I,\ e\in J_\beta}a_{ik}\left(\mathrm{ord}_{V_e}^{(\beta)}\Delta_{c,k}\right)\mathrm{dlog}x_e. \tag{2.26}$$

Consider the following 2-forms on T_β:

$$\omega_{\beta,c}:=\mathrm{sign}(i)\ d_i\ L_{c-1,i}\wedge L_{c,i}\quad\text{for } c\in[m] \text{ and } i:=i_c,\qquad\text{and}\qquad \omega_\beta:=\sum_{c\in[m]}\omega_{\beta,c}. \tag{2.27}$$

Note that $\mathrm{dlog}\Delta_{c-1,j}\wedge\mathrm{dlog}\Delta_{c,k}$ only contributes to $\omega_{\beta,c}$ if $j=k$ or $|j|,|k|$ are adjacent in the Dynkin diagram for G. Written in terms of $\mathrm{dlog}\,\Delta_{c-1,j}\wedge\mathrm{dlog}\,\Delta_{c,k}$, $\omega_{\beta,c}$ is essentially the same as [SW21, Definition of $\epsilon_{(n)}$], which generalizes [BFZ05, Definitions 2.2, 2.3].

Remark 2.26. By (2.26), the form ω_β has constant coefficients when expressed in terms of $(\mathrm{dlog}x_c\wedge\mathrm{dlog}_e)_{c,e\in J_\beta}$.

Since T_β is open dense in $\mathring{R}_\beta$, the forms ω_β and $\omega_{\beta,c}$ are rational 2-forms on $\mathring{R}_\beta$. Though it is not apparent from the above formula, it will follow from our main result (Theorem 1.1) combined with the results of [Mul12] that ω_β extends to a regular 2-form on the entire $\mathring{R}_\beta$.

Recall that (2.18) holds for $i_c\in\pm I$. Taking dlog of both sides of (2.18), we get

$$L_{c-1,i_c}+L_{c,i_c}=0\quad\text{if } c \text{ is hollow.} \tag{2.28}$$

Thus, $\omega_{\beta,c}=0$ for all $c\in[m]\setminus J_\beta$, which implies the following result.

Corollary 2.27. *We have* $\omega_\beta=\sum_{c\in J_\beta}\omega_{\beta,c}$.

The next proposition follows immediately from Corollaries 2.10 and 2.24.

Proposition 2.28. *Let $\beta \in (\pm I)^m$ be such that $\delta(\beta) = w_\circ$. The tuple*

$$\Sigma_\beta := (T_\beta, \mathbf{x}_\beta, \mathbf{d}_\beta, \omega_\beta) \tag{2.29}$$

is an abstract seed on $\mathring{R}_\beta$ in the sense of Definition 3.1 below.

3. Cluster algebras

Cluster algebras were discovered by Fomin and Zelevinsky [FZ02]. We consider skew-symmetrizable cluster algebras, relying on formalism similar to [FG09].

3.1. Background.

Definition 3.1. A rank n and dimension $n+m$ *(abstract) seed* is a quadruple $\Sigma = (T, \mathbf{x}, \mathbf{d}, \omega)$, where

1. T is a complex algebraic torus of dimension $n+m$,
2. $\mathbf{x} = (x_1, \dots, x_{n+m})$ is an ordered basis of $X^*(T)$, where $x_1, \dots, x_n$ (resp., $x_{n+1}, \dots, x_{n+m}$) are mutable (resp., frozen) variables,
3. $\mathbf{d} = (d_1, \dots, d_{n+m})$ is a collection of positive integers,
4. ω is a 2-form on T of the form
$$\omega = \sum_{i \le j} d_j \tilde{B}_{ij} \mathrm{dlog} x_i \wedge \mathrm{dlog} x_j = \sum_{i \le j} d_i \tilde{B}_{ji} \mathrm{dlog} x_j \wedge \mathrm{dlog} x_i, \tag{3.1}$$
where $\tilde{B}_{ii} = 0$ for $i \in [n+m]$ and $\tilde{B}_{ij} \in \mathbb{Q}$ for all $i, j \in [n+m]$.

The matrix $\tilde{B} = (\tilde{B}_{ij})_{(i,j) \in [n+m] \times [n]}$ is the usual $(n+m) \times n$ *extended exchange matrix* in the theory of cluster algebras. We have $d_j \tilde{B}_{ij} = -d_i \tilde{B}_{ji}$ for $i, j \in [n+m]$; in particular, the top $n \times n$ *principal part* B of $\tilde{B}$ is skew-symmetrizable.

Definition 3.2. Let $\Sigma = (T, \mathbf{x}, \mathbf{d}, \omega)$ be a seed and k a mutable index. We say that Σ is *integral at k* if $\tilde{B}_{jk} \in \mathbb{Z}$ for all $j \in [n+m]$. In this case, we define

$$x'_k := \frac{\prod_{\tilde{B}_{jk} > 0} x_j^{\tilde{B}_{jk}} + \prod_{\tilde{B}_{jk} < 0} x_j^{-\tilde{B}_{jk}}}{x_k}. \tag{3.2}$$

The *mutation* of Σ in the direction k is the seed $\mu_k(\Sigma) = (T', \mathbf{x}', \mathbf{d}, \omega')$ where T' is the algebraic torus with basis of characters $\mathbf{x}' = (x_1, \dots, x'_k, \dots, x_{n+m})$ and the 2-form ω' on T' is the pullback of ω via the natural rational map $T' \to T$.

We say that Σ is *integral* if it is integral at each $k \in [n]$, i.e., if $\tilde{B}_{ik} \in \mathbb{Z}$ for all $i \in [n+m]$ and $k \in [n]$.

Remark 3.3. As discussed in [FG09], ω' may be expressed in the form (3.1) using another matrix $\tilde{B}' = \mu_k(\tilde{B})$ obtained from $\tilde{B}$ via the usual cluster mutation of exchange matrices as defined in e.g. [FWZ16, Definition 2.7.6]. In particular, if Σ is integral (resp., integral at k) then so is $\mu_k(\Sigma)$.

For the rest of this subsection, we assume that all seeds are integral. Following [LS22, Section 5.1], a seed Σ is *really full rank* if the columns of $\tilde{B}$ span $\mathbb{Z}^n$ over $\mathbb{Z}$. We will prove this for the seeds Σ_β from (2.29) in Corollary 6.7.

Let X be an irreducible complex algebraic variety of dimension $n+m$. A *seed on X* is an abstract seed $\Sigma = (T, \mathbf{x}, \mathbf{d}, \omega)$ together with an identification $T \subset X$ of T with an open dense subset of X. The inclusion $T \hookrightarrow X$ induces an identification of the field $\mathbb{C}(X)$ of rational functions on X with the field of rational functions $\mathbb{C}(\mathbf{x}) := \mathbb{C}(x_1, \dots, x_{n+m})$ in the initial cluster variables $\mathbf{x}$. In practice, we abuse notation and write $\mathbf{x}$ for a tuple of elements in $\mathbb{C}(X)$.

The *cluster algebra* $\mathcal{A}(\Sigma)$ is the subring of $\mathbb{C}(\mathbf{x})$ generated by all cluster variables together with inverses of frozen variables. We let $\mathcal{V}(\Sigma) := \mathrm{Spec}(\mathcal{A}(\Sigma))$ denote the *cluster variety*. We say that (X, Σ) *is a cluster variety* if X is an affine variety and the coordinate ring $\mathbb{C}[X]$ is identified with $\mathcal{A}(\Sigma)$ under the identification $\mathbb{C}(X) \cong \mathbb{C}(\mathbf{x})$.

We will need the following property of cluster variables.

Proposition 3.4 ([GLS13, Theorem 3.1]). *Each cluster variable is an irreducible element of $\mathcal{A}(\Sigma)$.*

Our proofs will utilize some notions on cluster algebras that we now recall.

Definition 3.5. Let Σ be an abstract seed of rank n and dimension $n+m$, and let $F \subset [n]$. The *freezing of* Σ *at* F, denoted $\Sigma^{\setminus F}$, is the seed obtained from Σ by declaring the variables $\{x_c\}_{c\in F}$ to be frozen. It is a seed of rank $n-|F|$ and dimension $n+m$. For $k\in[n]$, we denote $\Sigma^{\setminus k} := \Sigma^{\setminus \{k\}}$.

To a seed Σ we associate a directed graph $\tilde{\Gamma}(\Sigma)$ with vertex set $[n+m]$ and an arrow $i\to j$ whenever $\tilde{B}_{ij}>0$. We let $\Gamma := \Gamma(\Sigma)$ be the *mutable part* of $\tilde{\Gamma}(\Sigma)$, i.e., the induced subgraph of $\tilde{\Gamma}(\Sigma)$ with vertex set $[n]$. We say that a mutable index $s\in[n]$ is a *sink* if it has no outgoing arrows in Γ. Let $N_s^{\mathrm{in}}(\Gamma)$ denote the set of vertices of Γ having an arrow to s, and denote $\widehat{N}_s^{\mathrm{in}}(\Gamma) := N_s^{\mathrm{in}}(\Gamma)\cup\{s\}$. The following definition is a variation of locally-acyclic seeds [Mul13] and Louise seeds [LS22]; see also [GL24, Section 5.4 and Remark 5.14].

Definition 3.6. The class of *sink-recurrent seeds* is defined recursively as follows.

- Any seed Σ such that $\Gamma(\Sigma)$ has no arrows is sink-recurrent.
- Any seed that is mutation equivalent to a sink-recurrent seed is sink-recurrent.
- Suppose that Σ is a seed with a sink $s\in[n]$ such that the seeds $\Sigma^{\setminus s}$ and $\Sigma^{\setminus \widehat{N}_s^{\mathrm{in}}(\Gamma(\Sigma))}$ are sink-recurrent. Then Σ is sink-recurrent.

The *upper cluster algebra* [BFZ05] $\mathcal{U}(\Sigma)\subset\mathbb{C}[\mathbf{x}^{\pm1}]$ is the intersection $\mathbb{C}[\mathbf{x}^{\pm1}]\cap\bigcap_{k\in[n]}\mathbb{C}[\mu_k(\mathbf{x})^{\pm1}]$.

Proposition 3.7. *Suppose that* Σ *is a sink-recurrent seed. Then* $\mathcal{A}(\Sigma)=\mathcal{U}(\Sigma)$.

Proof. It follows from induction and [Mul13, Lemma 5.3] that sink-recurrent seeds are locally acyclic in the sense of [Mul13, Mul14]. By [Mul14, Theorem 2], we have $\mathcal{A}(\Sigma)=\mathcal{U}(\Sigma)$. □

We will also need the notion of quasi-equivalence of seeds, which was first studied in [Fra16]. We adapt the definition of [Fra16] to our conventions.

Definition 3.8. Two seeds $\Sigma=(T,\mathbf{x},\mathbf{d},\omega)$ and $\tilde{\Sigma}=(\tilde{T},\tilde{\mathbf{x}},\tilde{\mathbf{d}},\tilde{\omega})$ of rank n and dimension $n+m$ are *quasi-equivalent*, denoted $\Sigma\sim\tilde{\Sigma}$, if the following conditions are satisfied:

(1) $T=\tilde{T}$, $\mathbf{d}=\tilde{\mathbf{d}}$, $\omega=\tilde{\omega}$;
(2) the sublattice of $X^*(T)$ spanned by the frozen variables $x_{n+1},\dots,x_{n+m}$ coincides with the sublattice spanned by $\tilde{x}_{n+1},\dots,\tilde{x}_{n+m}$;
(3) for each $k\in[n]$, we have $\tilde{x}_k=x_kM_k$, where M_k is a Laurent monomial in $x_{n+1},\dots,x_{n+m}$.

It is easy to see that if Σ is integral and $\Sigma\sim\tilde{\Sigma}$ then $\tilde{\Sigma}$ is integral. The following is also straightforward to check.

Lemma 3.9. *If* Σ *and* $\tilde{\Sigma}$ *are quasi-equivalent seeds then* $\mu_k(\Sigma)\sim\mu_k(\tilde{\Sigma})$ *for all mutable* k.

Corollary 3.10. *Suppose two seeds* $\Sigma,\tilde{\Sigma}$ *are quasi-equivalent. Then they define the same cluster algebra* $\mathcal{A}(\Sigma)=\mathcal{A}(\Sigma')\subset\mathbb{C}(T)$.

Proof. It follows from Lemma 3.9 that each cluster variable in $\mathcal{A}(\Sigma)$ differs from the corresponding cluster variable in $\mathcal{A}(\tilde{\Sigma})$ by a factor equal to a Laurent monomial in the frozen variables. □

3.2. **Deletion-contraction.** We give an inductive criterion for a pair (X,Σ) to be a sink-recurrent cluster variety, which is a key part of our proof of Theorem 1.1. In Section 4.8, we will apply this criterion to the seeds Σ_β from (2.29). See [GL24, Corollary 5.15] for a different application suggesting our nomenclature.

Assumption 3.11. Throughout this section, we let $\Sigma=(T,\mathbf{x},\mathbf{d},\omega)$ be an abstract seed of rank n and dimension $n+m$. Let $\Gamma:=\Gamma(\Sigma)$. We assume that Σ is sink-recurrent, with sink s in Γ such that Σ is integral at s. Further, we assume that there exists a frozen index f such that $\tilde{B}_{fs}=\pm 1$ and $\tilde{B}_{fj}=0$ for $j\in[n]\setminus\{s\}$. Suppose that the exchange relation for x_s in Σ is given by $x_s x'_s=M_1+x_f M_2$ for some monomials M_1,M_2 in $\{x_j\}_{j\in[n+m]\setminus\{s,f\}}$.

Definition 3.12. Suppose that s has $q:=|N_s^{\mathrm{in}}(\Gamma)|$ mutable neighbors. The *contraction* $\Sigma^{/s}=(T^{/s},\mathbf{x}^{/s},\mathbf{d}^{/s},\omega^{/s})$ is a seed of rank $n-q-1$ and dimension $n+m-2$ defined as follows.

(1) $\mathbf{x}^{/s}$ is obtained from $\mathbf{x}$ by omitting x_s and x_f and declaring the indices in $N_s^{\mathrm{in}}(\Gamma)$ to be frozen.
(2) $T^{/s}$ is an algebraic torus with character lattice generated by $\mathbf{x}^{/s}$.
(3) $\mathbf{d}^{/s}$ is obtained by restricting the sequence $\mathbf{d}$ to the set $[n+m]\setminus\{s,f\}$.
(4) $\omega^{/s}$ is obtained from ω by writing it in the form (3.1) and substituting $\operatorname{dlog} x_s:=0$ and $\operatorname{dlog} x_f:=\operatorname{dlog} M_1-\operatorname{dlog} M_2$.

The *deletion*[3] $\Sigma^{\setminus s}$ is the seed of rank $n-1$ and dimension $n+m$ obtained by declaring x_s to be frozen (cf. Definition 3.5).

In our next result, we use the following notation. If the same cluster variable x is viewed as a function on two different varieties U and U' (for example, $U=\mathcal{V}(\Sigma)$ and $U'=\mathcal{V}(\Sigma^{\setminus s})$), we denote the corresponding two functions by $x|_U$ and $x|_{U'}$.

Theorem 3.13 (Deletion-contraction recurrence)**.** *Let X be an affine, normal, irreducible, complex algebraic variety, and let $\Sigma=(T,\mathbf{x},\mathbf{d},\omega)$ be a seed on X with a sink s satisfying Assumption 3.11. Assume that all cluster variables in $\mathbf{x}\sqcup\{x'_s\}$ are regular on X. Define subvarieties $W:=\{x_s\neq 0\}$ and $V:=\{x_s=0\}$ of X. Suppose that we have isomorphisms $W\cong\mathcal{V}(\Sigma^{\setminus s})$ and $V\cong V_1\times V_2$ with $V_1:=\operatorname{Spec}(\mathbb{C}[x'_s])\cong\mathbb{C}$ and $V_2:=\mathcal{V}(\Sigma^{/s})$. Let $p_1:V\to V_1$, $p_2:V\to V_2$, $\iota_W:W\hookrightarrow X$, and $\iota_V:V\hookrightarrow X$ denote the natural projections and inclusions. Suppose that:*

- *for each cluster variable x of $\Sigma^{\setminus s}$, we have $\iota_W^*(x|_X)=x|_W$;*
- *for each cluster variable x of $\Sigma^{/s}$, we have $\iota_V^*(x|_X)=p_2^*(x|_{V_2})$;*
- *for the cluster variable x'_s, we have $\iota_V^*(x'_s|_X)=p_1^*(x'_s|_{V_1})$.*

Then (X,Σ) is a cluster variety.

Proof. First, the condition that $\Sigma^{\setminus s}$ is integral is included in the assumption that $(W,\Sigma^{\setminus s})$ is a cluster variety. Since Σ is integral at s, this implies Σ is also integral.

Let $j\in[n]\setminus\{s\}$ be a mutable index. Clearly, the (pullback under ι_W of the) exchange relation for x_j in Σ coincides with the exchange relation for x_j in $\Sigma^{\setminus s}$. Thus, the mutated variable x'_j is regular on W. Next, assume that $j\notin N_s^{\mathrm{in}}(\Gamma)$. By assumption, j is not connected to s,f in Γ, and thus the terms involving $\operatorname{dlog} x_j$ are unchanged when passing from ω to $\omega^{/s}$. Thus, the pullback of the exchange relation for x_j under ι_V is still the exchange relation for x_j in $\Sigma^{/s}$, and therefore the mutated variable x'_j is regular on V. For $j\in N_s^{\mathrm{in}}(\Gamma)$, we claim that x'_j must also be regular on V. Indeed, by (3.2), x'_j is regular on V if x_j^{-1} is regular on V. But $\iota_V^*(x_j|_X)=p_2^*(x_j|_{V_2})$, and $x_j|_{V_2}$ is a frozen variable in $\Sigma^{/s}$, so indeed x_j is invertible on V. It follows that for all $j\in[n]\setminus\{s\}$, the mutated variable x'_j is a regular function on X since it is regular on both V and W. For $j=s$, x'_s is regular on X by assumption.

Next, we show that $\mathbb{C}[X]\subset\mathcal{U}(\Sigma)$. This is equivalent to constructing inclusions $T\hookrightarrow X$ and $\mu_j(T)\hookrightarrow X$ for all $j\in[n]$. Since Σ is a seed on X, we have $T\subset X$. For the tori $\mu_j(T)$, we show that the subset $X_j\subset X$ where the regular functions in $\mu_j(\mathbf{x})$ are all non-vanishing is isomorphic to an algebraic torus $\mu_j(T)\cong(\mathbb{C}^\times)^{n+m}$ via the map $\varphi_j:X_j\to\mu_j(T)$ sending $y\in X_j$ to $z:=(x_1(y),\dots,x'_j(y),\dots,x_{n+m}(y))$.

[3]The terminology "deletion-contraction" comes from an analogous construction for matroids and hyperplane arrangements. Note that, in the case of cluster seeds, "deletion" corresponds to *freezing* x_s (thus deleting s from the mutable part Γ) and not to deleting x_s from the seed.

If $j\in[n]\setminus s$ then we have $X_j\subset W$, and thus the statement follows since W is a cluster variety. So let $j=s$. Consider the torus $\mu_s(T)\cong(\mathbb{C}^\times)^{n+m}$. Let $p:=M_1+x_fM_2\in\mathbb{C}[X]$ be the exchange binomial for x_s (cf. Assumption 3.11). Since p does not involve x_s and x'_s, we can also view p as a regular function on $\mu_s(T)$ compatible with pullback under φ_s. Let $z=(z_1,\dots,z_{n+m})\in\mu_s(T)$. Our goal is to show that z has a unique preimage under φ_s. Suppose first that $p(z)\neq 0$. Then $\varphi_s^{-1}(z)\subset T$, and the result follows. Suppose now that $p(z)=0$. Then $\varphi_s^{-1}(z)\subset V$. Recall that $V\cong V_1\times V_2$. Since $x'_s=z_s$, the first coordinate $p_1\circ\varphi_s^{-1}(z)$ of the preimage is uniquely determined by z. The second coordinate $p_2\circ\varphi_s^{-1}(z)$ of the preimage is uniquely determined by $(z_i)_{i\in[n+m]\setminus\{s,f\}}$. We have shown that z has a unique preimage under φ_s, which completes the proof of the inclusion $\mathbb{C}[X]\subset\mathcal{U}(\Sigma)$. The statement of the theorem now follows from Proposition 3.14 below. □

Proposition 3.14. *Let X be an affine, normal, irreducible, complex algebraic variety, and let $\Sigma=(T,\mathbf{x},\mathbf{d},\omega)$ be an integral sink-recurrent seed on X. Suppose that $\mathbb{C}[X]\subset\mathcal{U}(\Sigma)$. Then (X,Σ) is a cluster variety.*

Proof. The inclusions $\mathbb{C}[X]\subset\mathbb{C}[x_1^{\pm1},\dots,(x'_j)^{\pm1},\dots,x_{n+m}^{\pm1}]$ give tori $\mu_j(T)\cong X_j\subset X$ as in the proof of Theorem 3.13. By a standard argument, this implies that the complement of $T\cup\bigcup_{j\in[n]}X_j$ has codimension greater than or equal to two in X; see [Zel00, Section 3], [BFZ05, Proof of Theorem 2.10], or [GLSBS22, Lemmas 9.5–9.8]. Since X is normal, we have $\mathbb{C}[X]=\mathbb{C}[T\cup\bigcup_{j\in[n]}X_j]=\mathcal{U}(\Sigma)$. By assumption, Σ is sink-recurrent, so we are done by Proposition 3.7. □

Remark 3.15. If the seed $\Sigma^{\setminus s}$ is really full rank then it follows from Assumption 3.11 that Σ is really full rank. Indeed, row f of the exchange matrix of Σ contains a single nonzero entry equal to ±1 in column s. The exchange matrix of $\Sigma^{\setminus s}$ is obtained from that of Σ by removing column s. This implies that if $\Sigma^{\setminus s}$ is really full rank then so is Σ.

4. Double braid moves

Recall from Section 2 that we have constructed a single seed Σ_β for each double braid variety $\mathring{R}_\beta$. In this section, we first study natural isomorphisms between braid varieties corresponding to *double braid moves*, and show that pullbacks along them are well-behaved on seeds. In Section 4.8, we show how to apply Theorem 3.13 on deletion-contraction to $\mathring{R}_\beta$ for β of a particular form. Finally, in Section 4.9, we use double braid moves and deletion-contraction to prove Theorem 1.1 in simply-laced types.

Double braid moves are defined as follows:

(B1) $ij\leftrightarrow ji$ if $i,j\in\pm I$ have different signs;
(B2) $ij\leftrightarrow ji$ if $i,j\in\pm I$ have the same sign and $(s_{|i|}s_{|j|})^2=1$;
(B3) $\underbrace{iji\dots}_{m_{ij}\text{ letters}}\leftrightarrow\underbrace{jij\dots}_{m_{ij}\text{ letters}}$ if $i,j\in\pm I$ have the same sign and $(s_{|i|}s_{|j|})^{m_{ij}}=1$ with $m_{ij}\geq 3$;
(B4) $\beta_0 i\leftrightarrow\beta_0(-i^*)$ for $i\in\pm I$ and $\beta_0\in(\pm I)^{m-1}$;
(B5) $i\beta_0\leftrightarrow(-i)\beta_0$ for $i\in\pm I$ and $\beta_0\in(\pm I)^{m-1}$.

If double braid words β and β' are related by one of the moves (B1)–(B5), there is a natural isomorphism $\phi:\mathring{R}_\beta\xrightarrow{\sim}\mathring{R}_{\beta'}$, discussed below.

Definition 4.1. Suppose that β and β' are related by one of the moves (B1)–(B3). If this move involves indices $l,l+1,\dots,r$, the isomorphism ϕ sends $(X_\bullet,Y_\bullet)\in\mathring{R}_\beta$ to the unique tuple $(X'_\bullet,Y'_\bullet)\in\mathring{R}_{\beta'}$ such that $X'_c=X_c$ and $Y'_c=Y_c$ for $0\leq c<l$ or $r\leq c\leq m$. The remaining weighted flags $X'_l,\dots,X'_{r-1},Y'_l,\dots,Y'_{r-1}$ are uniquely determined by Lemma 2.1.

For the moves (B4) and (B5), the isomorphism ϕ is described in Sections 4.6 and 4.7, respectively. The main result of this section is the following.

Theorem 4.2. *Suppose that β and β' are related by one of the moves* (B1)–(B5)*. If $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety then so is $(\mathring{R}_{\beta'},\Sigma_{\beta'})$.*

We then use Theorem 4.2 and Theorem 3.13 to prove Theorem 1.1; see Theorem 4.10 and Sections 4.9 and 6.3.

The proof of Theorem 4.2 will occupy Sections 4–6. Along the way, we will construct a seed $\Sigma' = (T', \mathbf{x}', \mathbf{d}', \omega')$ obtained from $\Sigma_\beta = (T, \mathbf{x}, \mathbf{d}, \omega)$ by one or several mutations, followed by a relabeling. We will show the following for moves (B1)–(B5):

- (F) The 2-form is invariant: $\phi^* \omega_{\beta'} = \omega_\beta$.
- (Q) Suppose that $(\mathring{R}_\beta, \Sigma_\beta)$ is a cluster variety. Then the seeds Σ' and $\phi^*\Sigma_{\beta'}$ are quasi-equivalent.

Here, for a seed $\Sigma_{\beta'} = (T_{\beta'}, \mathbf{x}_{\beta'}, \mathbf{d}_{\beta'}, \omega_{\beta'})$, $\phi^*\Sigma_{\beta'} = (T^*, \mathbf{x}^*, \mathbf{d}^*, \omega^*)$ is an abstract seed on $\mathring{R}_\beta$ defined by

$$T^* := \phi^{-1}(T_{\beta'}), \quad \mathbf{x}^* := \phi^* \mathbf{x}_{\beta'}, \quad \mathbf{d}^* := \mathbf{d}_{\beta'}, \quad \text{and} \quad \omega^* := \phi^* \omega_{\beta'}. \tag{4.1}$$

Note that (Q) immediately implies Theorem 4.2.

Definition 4.3. A (B1)–(B3) move is *solid* if all indices involved are solid. For $i, j \in I$, the (B1) move $(-i)j \leftrightarrow j(-i)$ on indices $c, c+1$ is *special* if $u_c s_i = s_j u_c$ and *solid-special* if it is both solid and special. A (B3) move with $m_{ij} > 3$ is *long*; all other moves are *short*. Finally, a (B1)–(B5) move is a *mutation move* if it involves at least one cluster mutation; otherwise it is a *non-mutation move*.

Remark 4.4. As we will show in Section 4.1, a solid-special (B1) move corresponds to a single mutation, at the rightmost index involved in the move. The move (B3) involving q solid indices corresponds to a sequence of $\binom{q-1}{2}$ mutations on the rightmost $m_{ij} - 2$ indices involved in the move (see Sections 4.4 and 6). We will show that all other moves are non-mutation moves.

We will show (F), (Q) for short moves directly. This will complete the proof of Theorem 4.2 in simply-laced types. We then use this and folding to show (F), (Q) for long moves in Sections 5 and 6.

Throughout the rest of this section, we fix β, β' related by a short move and thus an isomorphism $\phi : \mathring{R}_\beta \xrightarrow{\sim} \mathring{R}_{\beta'}$. For a rational function or a form f on $\mathring{R}_{\beta'}$, we use the shorthand $f^* := \phi^* f$.

Remark 4.5. If all indices involved in a move (B1)–(B3) are hollow, then the statements (F), (Q) follow trivially; cf. Corollary 2.27.

4.1. **Mutation move: (B1), solid-special.** Consider the case of a solid-special move (B1) on indices $c, c+1$. Since both indices are solid, we denote $u := u_{c-1} = u_c = u_{c+1}$ and $w := w_{c-1} = w_c = w_{c+1}$. The indices $i, j \in \pm I$ are of opposite signs; we assume that $i \in -I$ and $j \in I$ as the other case is similar. The solid-special condition yields

$$u < s_{|i|} u = u s_j \quad \text{and} \quad s_{|i|^*} w = w s_j < w. \tag{4.2}$$

To show (F), we will utilize the following relation among grid minors.

Proposition 4.6 ([FZ99, Theorem 1.17]). *We have*[4]

$$\Delta_{c,j} \Delta^*_{c,j} = \Delta_{c+1,j} \Delta_{c-1,j} + \prod_{k \neq j} \Delta_{c,k}^{-a_{jk}}. \tag{4.3}$$

Proof. We may choose $t, t' \in \mathbb{C}$ such that $Z_c = Z_{c+1} z_j(t)$, $Z^*_c = \bar{z}_{|i|^*}(t')^{-1} Z_{c+1}$, and $Z_{c-1} = Z^*_{c-1} = \bar{z}_{|i|^*}(t')^{-1} Z_{c+1} z_j(t) = \dot{s}_{|i|^*} x_{|i|^*}(t') Z_{c+1} x_j(t) \dot{s}_j$. Let $Z := x_{|i|^*}(t') Z_{c+1} x_j(t)$. By [FZ99, Theorem 1.17], we have

$$\Delta_{w\omega_j, s_j\omega_j}(Z) \Delta_{ws_j\omega_j, \omega_j}(Z) = \Delta_{ws_j\omega_j, s_j\omega_j}(Z) \Delta_{w\omega_j, \omega_j}(Z) + \prod_{k \neq j} \Delta_{ws_j\omega_k, \omega_k}(Z)^{-a_{jk}}. \tag{4.4}$$

Using properties of generalized minors from Section 2.6, one can check that each term of (4.3) equals the corresponding term of (4.4). For example, we have

$$\Delta_{c,j} = \Delta_{w\omega_j, \omega_j}(Z_{c+1} x_j(t) \dot{s}_j) = \Delta_{\omega_j, \omega_j}(\dot{w}^{-1} Z_{c+1} x_j(t) \dot{s}_j) = \Delta_{\omega_j, \omega_j}(\dot{w}^{-1} Z \dot{s}_j) = \Delta_{w\omega_j, s_j\omega_j}(Z),$$

where we have used $\dot{w}^{-1} x_{|i|^*}(t') \in U_- \dot{w}^{-1}$; cf. (2.12) and (4.2). For $\Delta_{c,k}^{-a_{jk}}$, $k \neq j$, we additionally used that $s_j \omega_k = \omega_k$. $\square$

[4] Our Cartan matrix $a_{ij} := \langle \alpha_i, \alpha_j^\vee \rangle$ is the transpose of that of [FZ99]; see [FZ99, Equation (2.27)].

We shall use the following analog of [GLSBS22, Lemma 8.10].

Lemma 4.7. *For $e\in[0,m]$ and $-i,j\in I$ such that $u_e s_j = s_{|i|}u_e$, we have*

$$\prod_{k\in\pm I}\Delta_{e,k}^{a_{ik}}=\prod_{k\in\pm I}\Delta_{e,k}^{\epsilon a_{jk}} \quad \text{and} \quad L_{e,i}=\epsilon L_{e,j}, \quad \text{where} \quad \epsilon:=\begin{cases}1, & \text{if } u_e<u_e s_j,\\ -1, & \text{if } u_e>u_e s_j.\end{cases} \tag{4.5}$$

Proof. We have $\alpha_j=\sum_{k\in I}a_{jk}\omega_k$ and similarly for $\alpha_{|i|}$. By (2.16), the first identity in (4.5) therefore becomes $(h_e^-)^{\alpha_{|i|}}=(h_e^+)^{\epsilon\alpha_j}$, which follows from the assumption $u_e\alpha_j=\epsilon\alpha_{|i|}$ together with $h_e^-=u_e\cdot h_e^+$; cf. (2.8). Taking dlog of both sides, we obtain the second identity. □

Remark 4.8. Equations (4.3) and (4.5) are true as stated in the case $i,-j\in I$ as well.

Proof of (F) *for* (B1), *solid-special.* Only the terms $\omega_{\beta,c}$ and $\omega_{\beta,c+1}$ change when applying the move (B1). Note that we must have $d_i=d_j$ because the simple roots $\alpha_{|i|},\alpha_j$ are related by the action of $u\in W$ (which preserves the lengths of roots). Applying (4.5), we get

$$\begin{aligned}\frac{1}{d_j}(\omega_\beta-\omega^*_{\beta'})&=\frac{1}{d_j}\left(\omega_{\beta,c}+\omega_{\beta,c+1}-\omega^*_{\beta',c}-\omega^*_{\beta',c+1}\right)\\ &=-L_{c-1,i}\wedge L_{c,i}+L_{c,j}\wedge L_{c+1,j}-L^*_{c-1,j}\wedge L^*_{c,j}+L^*_{c,i}\wedge L^*_{c+1,i}\\ &=-L_{c-1,j}\wedge L_{c,j}+L_{c,j}\wedge L_{c+1,j}-L_{c-1,j}\wedge L^*_{c,j}+L^*_{c,j}\wedge L_{c+1,j}\\ &=(L_{c,j}+L^*_{c,j})\wedge(L_{c-1,j}+L_{c+1,j}).\end{aligned}$$

For $e\in\{c-1,c,c+1\}$, let $M_e:=\prod_{k\neq j}\Delta_{e,k}^{-a_{jk}}$. Thus, M_c is the third term in (4.3). By (2.9), we have $h_c^+=\alpha_j^\vee(t_{c+1})h_{c+1}^+$ and $h_{c-1}^-=\alpha_{|i|}^\vee(t_c)h_c^-$. This implies that $h_{c-1}^+=\alpha_j^\vee(t_ct_{c+1})h_{c+1}^+$ since $h_c^-=u\cdot h_c^+$. Thus, we have $M:=M_{c-1}=M_c=M_{c+1}$. Since $M^*_{c+1}=M_{c+1}$, we get that $M=M^*_{c-1}=M^*_c=M^*_{c+1}$. Set $A:=\frac{\Delta_{c,j}\Delta^*_{c,j}}{M}$ and $B:=\frac{\Delta_{c-1,j}\Delta_{c+1,j}}{M}$. Then (4.3) gives $A=B+1$. Thus, $dA=dB$, and so $\operatorname{dlog}A\wedge\operatorname{dlog}B=0$. It remains to prove that $\operatorname{dlog}A=L_{c,j}+L^*_{c,j}$ and $\operatorname{dlog}B=L_{c-1,j}+L_{c+1,j}$. Indeed, by (2.26), for $e\in\{c-1,c,c+1\}$, we have

$$L_{e,j}=\frac{1}{2}\operatorname{dlog}\left(\frac{\Delta_{e,j}^2}{M_e}\right)=\operatorname{dlog}\left(\frac{\Delta_{e,j}}{M^{1/2}}\right) \quad \text{and} \quad L^*_{e,j}=\frac{1}{2}\operatorname{dlog}\left(\frac{(\Delta^*_{e,j})^2}{M^*_e}\right)=\operatorname{dlog}\left(\frac{\Delta^*_{e,j}}{M^{1/2}}\right), \tag{4.6}$$

since $M_e=M^*_e=M$. Using the additivity of dlog, we get $\operatorname{dlog}A=L_{c,j}+L^*_{c,j}$ and $\operatorname{dlog}B=L_{c-1,j}+L_{c+1,j}$. □

Proof of (Q) *for* (B1), *solid-special.* We do not use the assumption that $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety until the last paragraph of this proof. Let $x:=x_{c+1}$ and $V:=V_{c+1}$. Applying Propositions 2.20 and 2.21, we see that

$$\operatorname{ord}_V\Delta_{c,j}=\operatorname{ord}_V\Delta_{c,i}=1 \quad \text{and} \quad \operatorname{ord}_V\Delta_{e,k}=0 \quad \text{for } (e,k)\in[0,m]\times(\pm I)\setminus\{(c,j),(c,i)\}. \tag{4.7}$$

In particular, $\operatorname{dlog}x$ appears in ω_β only in the terms $L_{c,j}$ and $L_{c,i}$ in $\omega_{\beta,c+1}=d_jL_{c,j}\wedge L_{c+1,j}$ and $\omega_{\beta,c}=-d_iL_{c-1,i}\wedge L_{c,i}$, respectively. Recall from (4.5) that we actually have $L_{c,j}=L_{c,i}$. Using this and $d_i=d_j$, we find

$$\omega_\beta-\omega_{\text{rest}}=\omega_{\beta,c}+\omega_{\beta,c+1}=d_jL_{c,j}\wedge(L_{c+1,j}+L_{c-1,i}),$$

where $\omega_{\text{rest}}:=\sum_{e\in J_\beta\setminus\{c,c+1\}}\omega_{\beta,e}$ by Corollary 2.27. Expanding the forms $L_{e,k}$ in terms of $(\operatorname{dlog}x_{e'})_{e'\in J_\beta}$ via (2.26), we see from (4.7) that $\operatorname{dlog}x$ appears in $L_{c,j}$ with coefficient 1 and that $\operatorname{dlog}x$ does not appear in ω_{rest}. Using (4.6), we get

$$\omega_\beta-\omega'_{\text{rest}}=d_j\operatorname{dlog}x\wedge(L_{c+1,j}+L_{c-1,i})=d_j\operatorname{dlog}x\wedge\left(\operatorname{dlog}(\Delta_{c+1,j}\Delta_{c-1,j})-\operatorname{dlog}\prod_{k\neq j}\Delta_{c,k}^{-a_{jk}}\right), \tag{4.8}$$

where ω'_{rest} is a linear combination of terms $\operatorname{dlog}x_e\wedge\operatorname{dlog}x_{e'}$ for $x_e,x_{e'}\neq x$.

By Proposition 2.20, a cluster variable x_e for $e\in J_\beta$ may appear on the right-hand side of (4.8) only for $e\geq c$. Moreover, we have already observed that $d_c=d_i=d_j=d_{c+1}$. Let us denote $p_e:=\operatorname{ord}_{V_e}(\Delta_{c+1,j}\Delta_{c-1,j})$ and $q_e:=\operatorname{ord}_{V_e}\prod_{k\neq j}\Delta_{c,k}^{-a_{jk}}$. Clearly, $p_e,q_e\geq 0$.

Expanding ω_β as a linear combination of terms $\mathrm{dlog} x_e \wedge \mathrm{dlog} x_{e'}$ for $e<e'$ (resp., $e>e'$) via (2.26), we see from (3.1) that for each $e \in J_\beta \setminus \{c+1\}$, $d_j \tilde{B}_{e,c+1}$ equals the coefficient of $\mathrm{dlog} x_e \wedge \mathrm{dlog} x_{c+1}$, regardless of whether $e<c+1$ or $e>c+1$. Since $x=x_{c+1}$ does not appear in ω'_{rest}, we see from (4.8) that $d_j \tilde{B}_{e,c+1}=q_e-p_e$ for all $e\in J_\beta\setminus\{c+1\}$. In fact, this identity also holds for $e=c+1$ since in this case $\tilde{B}_{e,c+1}=q_e=p_e=0$.

Combining (4.2) and Definition 2.16, we see that $u_e^{\langle c+1\rangle}=u_e$ for all $e\neq c$. In particular, by Definition 2.17, the cluster variable x is mutable. Thus, the mutated variable $x':=x'_{c+1}$ satisfies

$$xx' = \prod_{e\in J_\beta : p_e>q_e} x_e^{p_e-q_e} + \prod_{e\in J_\beta : q_e>p_e} x_e^{q_e-p_e}. \tag{4.9}$$

We have $V_e=V_e^*$ and $x_e=x_e^*$ for all $e\in J_\beta\setminus\{c+1\}$. Let $V^*:=V^*_{c+1}$ and $x^*:=x^*_{c+1}$. A generic point $(X_\bullet,Y_\bullet)\in V$ satisfies $X_{c-1}\overset{ws_j}{\Longleftarrow}Y_{c+1}$ and $X_{c+1}\overset{w}{\Longleftarrow}Y_{c-1}$, while a generic point $(X_\bullet,Y_\bullet)\in V^*$ satisfies $X_{c+1}\overset{ws_j}{\Longleftarrow}Y_{c-1}$ and $X_{c-1}\overset{w}{\Longleftarrow}Y_{c+1}$. Thus, $V\neq V^*$.

For $e\in J_\beta$, applying ord_{V_e} to both sides of (4.3), we get

$$\mathrm{ord}_{V_e}\Delta_{c,j}+\mathrm{ord}_{V_e}\Delta^*_{c,j}\geq \min(p_e,q_e). \tag{4.10}$$

For $e=c+1$, we have $\mathrm{ord}_V\Delta_{c,j}=1$, $\mathrm{ord}_V\Delta^*_{c,j}=0$ (since $V\neq V^*$), and $p_{c+1}=q_{c+1}=0$ by (4.7). Similarly, $\mathrm{ord}_{V^*}\Delta^*_{c,j}=1$, $\mathrm{ord}_{V^*}\Delta_{c,j}=0$, and the order of vanishing of $\Delta^*_{c+1,j}\Delta^*_{c-1,j}$ and $\prod_{k\neq j}(\Delta^*_{c,k})^{-a_{jk}}$ at V^* is zero.

Dividing both sides of (4.3) by $\prod_{e\in J_\beta\setminus\{c+1\}} x_e^{\min(p_e,q_e)}$, we get

$$xx^*\prod_{e\in J_\beta\setminus\{c+1\}} x_e^{r_e} = \prod_{e\in J_\beta : p_e>q_e} x_e^{p_e-q_e} + \prod_{e\in J_\beta : q_e>p_e} x_e^{q_e-p_e},$$

where $r_e:=\mathrm{ord}_{V_e}\Delta_{c,j}+\mathrm{ord}_{V_e}\Delta^*_{c,j}-\min(p_e,q_e)\geq 0$. By (4.9), we get

$$x' = x^*\prod_{e\in J_\beta\setminus\{c+1\}} x_e^{r_e}. \tag{4.11}$$

Now, assume that $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety. We get from Proposition 3.4 that the mutated cluster variable x' is irreducible in $\mathbb{C}[\mathring{R}_\beta]$. The function x^* vanishes on $V^*\subset\mathring{R}_\beta$ and therefore is not a unit in $\mathbb{C}[\mathring{R}_\beta]$. It follows that $r_e=0$ for all mutable e, i.e.,

$$\mathrm{ord}_{V_e}\Delta_{c,j}+\mathrm{ord}_{V_e}\Delta^*_{c,j}=\min(p_e,q_e)\quad \text{for } e\in J_\beta^{\mathrm{mut}}. \tag{4.12}$$

Thus, x' and x^* differ by a monomial in the frozen variables: we have

$$x' = x^*\prod_{e\in J_\beta^{\mathrm{fro}}} x_e^{r_e}. \tag{4.13}$$

We claim that the mutated seed $\Sigma':=\mu_{c+1}\Sigma_\beta$ is quasi-equivalent to the pulled back seed $\Sigma^*_{\beta'}$. Recall that $\Sigma^*_{\beta'}=(T^*,\mathbf{x}^*,\mathbf{d}^*,\omega^*)$ was defined in (4.1) while $\Sigma'=(T',\mathbf{x}',\mathbf{d}',\omega')$ was defined in Definition 3.2. To show that these seeds are quasi-equivalent, we check each condition in Definition 3.8. We have $\mathbf{d}^*=\mathbf{d}'$ since $d_i=d_j$. We have $\omega'=\omega_\beta$ by Definition 3.2 and $\omega^*=\omega_\beta$ by (F). The tori $T^*=T'$ are both obtained as the subset of $\mathring{R}_\beta$ where the cluster variables in $\{x_e\}_{e\in J_\beta\setminus\{c+1\}}\cup\{x'\}$ are nonzero in view of (4.13). Thus, condition (1) in Definition 3.8 is satisfied. The set of frozen variables has not changed, so condition (2) is satisfied trivially. Condition (3) is satisfied by (4.13). □

4.2. **Non-mutation move: (B1), not solid-special.** We continue to assume that the move involves indices $c,c+1$, and that $i\in -I$, $j\in I$.

4.2.1. (B1), *special, non-solid.* Suppose that at least one of the indices is hollow, and that the move is special. Then it follows that $c+1$ is hollow and c is solid in both β and β'. By (2.28), $L_{c,j}=-L_{c+1,j}$ and $L^*_{c,i}=-L^*_{c+1,i}$. Applying (4.5) with $\epsilon=1$ for $e=c-1,c$ and $\epsilon=-1$ for $e=c+1$ and using $d_i=d_j$, we obtain

$$\frac{\omega_{\beta,c}}{d_j} = -L_{c-1,i}\wedge L_{c,i} = L_{c-1,i}\wedge L_{c+1,i} = L^*_{c-1,i}\wedge L^*_{c+1,i} = -L^*_{c-1,j}\wedge L^*_{c+1,j} = L^*_{c-1,j}\wedge L^*_{c,j} = \frac{\omega^*_{\beta',c}}{d_i},$$

which proves (F). The clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ are identical, which proves (Q).

4.2.2. (B1), *non-special.* We start by introducing a formalism for working with the forms $L_{e,k}$. Let $\lambda := \sum_{k\in I} b_k\omega_k$ with $b_k \in \mathbb{Q}$, and let h be an H-valued rational function on $\mathring{R}_\beta$. We introduce a rational 1-form

$$\mathrm{dlog} h^\lambda := \sum_{k\in I} b_k \mathrm{dlog}(h^{\omega_k}).$$

It is clear that

$$\mathrm{dlog} h^{\lambda_1+\lambda_2} = \mathrm{dlog} h^{\lambda_1} + \mathrm{dlog} h^{\lambda_2} \quad \text{and} \quad \mathrm{dlog}(h_1h_2)^\lambda = \mathrm{dlog} h_1^\lambda + \mathrm{dlog} h_2^\lambda. \tag{4.14}$$

For $e \in [0,m]$ and $k \in I$, Lemma 2.12 gives

$$L_{e,k} = \mathrm{dlog}(h_e^+)^{\alpha_k/2} = \mathrm{dlog}(h_e^-)^{u_e\alpha_k/2}, \quad L_{e,-k} = \mathrm{dlog}(h_e^-)^{\alpha_k/2} = \mathrm{dlog}(h_e^+)^{u_e^{-1}\alpha_k/2}. \tag{4.15}$$

Finally, suppose that $h_1 = h_2\alpha_k^\vee(t)$. Then we have

$$\mathrm{dlog} h_1^\lambda = \mathrm{dlog} h_2^\lambda + \langle \lambda, \alpha_k^\vee\rangle \mathrm{dlog} t. \tag{4.16}$$

Proof of (F) *and* (Q) *for* (B1)*, non-special.* Suppose as before that the move involves indices $c, c+1$, and that $i \in -I$, $j \in I$. Assume first that both $c, c+1$ are solid, and let $u := u_{c-1} = u_c = u_{c+1}$. Let $a := \langle u^{-1}\alpha_{|i|}/2, \alpha_j^\vee\rangle$ and $a' := \langle u\alpha_j/2, \alpha_{|i|}^\vee\rangle$. Using (4.15)–(4.16) and (2.9), we get

$$L_{c,i} = L_{c+1,i} + a\mathrm{dlog} t_{c+1}, \qquad L_{c-1,i} = L_{c,i} + \mathrm{dlog} t_c, \qquad L_{c,j} = L_{c+1,j} + \mathrm{dlog} t_{c+1}; \tag{4.17}$$

$$L^*_{c,j} = L^*_{c+1,j} + a'\mathrm{dlog} t^*_{c+1}, \qquad L^*_{c-1,j} = L^*_{c,j} + \mathrm{dlog} t^*_c, \qquad L^*_{c,i} = L^*_{c+1,i} + \mathrm{dlog} t^*_{c+1}. \tag{4.18}$$

Since the move is non-special, the coroots $\alpha_j^\vee$ and $u^{-1}\alpha_{|i|}^\vee$ are linearly independent, which implies $t^*_c = t_{c+1}$ and $t^*_{c+1} = t_c$. Note also that we have $L^*_{c+1,i} = L_{c+1,i}$ and $L^*_{c+1,j} = L_{c+1,j}$. Using (4.17)–(4.18) to express each 1-form $L_{e,k}$ in terms of $L_{c+1,i}$, $L_{c+1,j}$, $\mathrm{dlog} t_c$, and $\mathrm{dlog} t_{c+1}$, we find

$$\omega_{\beta,c} + \omega_{\beta,c+1} - \omega^*_{\beta',c} - \omega^*_{\beta',c+1} = (d_ja' - d_ia)\mathrm{dlog} t_c \wedge \mathrm{dlog} t_{c+1}.$$

Since $d_ja' = d_ia$, we get that $\omega_\beta = \omega^*_{\beta'}$. The clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ differ by a relabeling $c \leftrightarrow c+1$.

Suppose now that one of $c, c+1$ is hollow. For instance, let $c \notin J_\beta$ and $c+1 \in J_\beta$. By Corollary 2.9, we have $h_c^+ = h_{c-1}^+$, and thus $L_{c,j} = L_{c-1,j}$. Similarly, $L^*_{c,j} = L^*_{c+1,j}$. Recall that $L^*_{c\pm1,j} = L_{c\pm1,j}$. Thus, $\omega_{\beta,c+1} = \omega^*_{\beta',c}$, and so $\omega_\beta = \omega^*_{\beta'}$. The case where $c \in J_\beta$ and $c+1 \notin J_\beta$ is similar. The clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ differ by a relabeling $c \leftrightarrow c+1$. For the case $c, c+1 \notin J_\beta$, see Remark 4.5. □

4.3. **Non-mutation move: (B2).** Suppose that the move involves indices $c, c+1$. We have $\omega_{\beta,c} = \omega^*_{\beta',c+1}$ and $\omega_{\beta,c+1} = \omega^*_{\beta',c}$, so $\omega_\beta = \omega^*_{\beta'}$. The chamber minors satisfy $\Delta_c = \Delta^*_{c+1}$ and $\Delta_{c+1} = \Delta^*_c$. Thus, the clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ differ by a relabeling $c \leftrightarrow c+1$. This shows (F) and (Q).

4.4. **Mutation move: (B3), solid, short.** We proceed analogously to the case of solid-special (B1) in Section 4.1. Suppose that the move $\beta \to \beta'$, $iji \to jij$, involves indices $c-1, c, c+1$, and that all three indices are solid. Suppose in addition that $i, j \in I$; the case $i, j \in -I$ is similar.

For (F), we will use the following relation among grid minors.

Proposition 4.9 ([FZ99, Theorem 1.16(1)])**.** *We have*

$$\Delta_{c,i}\Delta^*_{c,j} = \Delta_{c+1,i}\Delta_{c-2,j} + \Delta_{c-2,i}\Delta_{c+1,j}. \tag{4.19}$$

Proof. We have $Z_{c-2} = Z_{c+1}z_i(t_1)z_j(t_2)z_i(t_3)$ for some $t_1, t_2, t_3 \in \mathbb{C}$. We have $z_i(t_1)z_j(t_2)z_i(t_3) = z_j(t_3)z_i(t'_2)z_j(t_1)$ for $t'_2 := t_1t_3 - t_2$, which can be checked inside SL_3. Thus, $Z_{c-1} = Z_{c+1}z_i(t_1)z_j(t_2)$, $Z_c = Z_{c+1}z_i(t_1)$, and $Z^*_c = Z_{c+1}z_j(t_3)$. Let $Z := Z_{c-2}(\dot{s}_i\dot{s}_j\dot{s}_i)^{-1}$. Let $w := w_{c-1} = w_c = w_{c+1}$. By [FZ99, Theorem 1.16(1)],

$$\Delta_{w\omega_i,s_i\omega_i}(Z)\Delta_{w\omega_j,s_j\omega_j}(Z) = \Delta_{w\omega_i,\omega_i}(Z)\Delta_{w\omega_j,s_is_j\omega_j}(Z) + \Delta_{w\omega_i,s_js_i\omega_i}(Z)\Delta_{w\omega_j,\omega_j}(Z). \tag{4.20}$$

Similarly to the proof of Proposition 4.6, we observe that each term in (4.19) equals the corresponding term in (4.20). □

Proof of (F) *for* (B3)*, solid, short.* Let $\tilde{\beta} = \beta w_\circ$ and $\tilde{\beta}' = \beta' w_\circ$. By definition, $\sum_{j\in J_{\tilde{\beta}}} \omega_j(\tilde{\beta})$ and $\sum_{j\in[m]}\omega_j(\beta)$ are identical when expressed in terms of the symbols $\Delta_{c,i}$. It is known ([SW21, Proposition 3.25] or [BFZ05]) that Proposition 4.9 implies $\omega_{\tilde{\beta}} = \omega_{\tilde{\beta}'}$. Since the same identity for grid minors in Proposition 4.9 holds on $\mathring{R}_\beta$, we deduce that $\omega_\beta = \omega_{\beta'}$. □

Proof of (Q) *for* (B3)*, solid, short.* Let $x := x_{c+1}$ and $V := V_{c+1}$. By Propositions 2.20 and 2.21, x is mutable, $\operatorname{ord}_V \Delta_{c-1,i} = \operatorname{ord}_V \Delta_{c,i} = 1$, and $\operatorname{ord}_V \Delta_{e,k} = 0$ when $e \notin \{c-1, c\}$ or $k \in I\setminus\{i\}$.

Note that $d_i = d_j$. Collecting the terms of $\omega_{\beta,c-1} + \omega_{\beta,c} + \omega_{\beta,c+1}$ involving $\operatorname{dlog} x$, we get

$$d_i \operatorname{dlog} x \wedge \left(L_{c+1,i} - L_{c-2,i} + \frac{1}{2}(L_{c-1,j} - L_{c,j}) \right). \tag{4.21}$$

Applying (2.26) and using Corollary 2.14, we get

$$L_{c+1,i} - L_{c-2,i} = \operatorname{dlog}(\Delta_{c+1,i}) - \operatorname{dlog}(\Delta_{c-2,i}) + \frac{1}{2}\operatorname{dlog}(\Delta_{c-2,j}) - \frac{1}{2}\operatorname{dlog}(\Delta_{c+1,j});$$
$$L_{c-1,j} - L_{c,j} = \operatorname{dlog}(\Delta_{c-1,j}) - \operatorname{dlog}\Delta_{c,j} = \operatorname{dlog}(\Delta_{c-2,j}) - \operatorname{dlog}\Delta_{c+1,j}.$$

Thus, (4.21) becomes $d_i \operatorname{dlog} x \wedge (\operatorname{dlog}(\Delta_{c+1,i}\Delta_{c-2,j}) - \operatorname{dlog}(\Delta_{c-2,i}\Delta_{c+1,j}))$. The rest of the proof is entirely analogous to the argument for solid-special (B1) given at the end of Section 4.1, using (4.19) in place of (4.3). □

4.5. **Non-mutation move: (B3), non-solid, short.** Suppose that at least one of the indices $c-1, c, c+1$ is hollow. By Remark 4.5, we may assume that there are either one or two hollow indices in $\{c-1, c, c+1\}$. Explicitly, underlining the hollow crossings, the possible moves are $i\underline{j}i \leftrightarrow ji\underline{j}$ and $\underline{i}\underline{j}\underline{i} \leftrightarrow \underline{j}\underline{i}j$ (or the moves obtained from these by swapping the roles of i and j).

For $l \in \{i,j\}$ and $e \in J_\beta$, let us denote

$$A_l := \operatorname{dlog} \prod_{k\neq i,j} \Delta_{c+1,k}^{a_{lk}}, \quad B_l := \operatorname{dlog}\Delta_{c+1,l}, \quad \text{and} \quad T_e := \operatorname{dlog} t_e. \tag{4.22}$$

Using (2.17)–(2.18), we can express the dlogs of grid minors $\Delta_{e,l}$ for $l \in \{i,j\}$ and $e \in \{c-1, c, c+1\}$ in the symbols (4.22). Using $\operatorname{dlog}\Delta^*_{c-2,l} = \operatorname{dlog}\Delta_{c-2,l}$ for $l \in \{i,j\}$, we express T^*_e in terms of $T_{e'}$ for all indices $e \in \{c-1, c, c+1\}$ which are solid in β'. Thus, we can express the forms $\omega_{\beta,e}$, $\omega^*_{\beta',e}$, $e \in \{c-1, c, c+1\}$ in terms of the symbols (4.22). Using a straightforward computation, we check $\omega_\beta = \omega^*_{\beta'}$.

We observe using Corollary 2.14 that the clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ differ by a relabeling, which shows (Q).

4.6. **Non-mutation move: (B4).** Suppose that $i \in I$. The isomorphism $\phi : \mathring{R}_\beta \xrightarrow{\sim} \mathring{R}_{\beta'}$ sending $(X_\bullet, Y_\bullet) \mapsto (X'_\bullet, Y'_\bullet)$ is given by $X'_{m-1} = X'_m = Y'_m := X_{m-1}$, $Y'_{m-1} := Y_{m-1}$, and $(X'_c, Y'_c) := (X_c, Y_c)$ for all $0 \le c < m-1$. The last crossing in $\beta_0 i$ and $\beta_0(-i^*)$ is always hollow, and thus the statements (F) and (Q) follow trivially.

4.7. **Non-mutation move: (B5).** Suppose that $i \in I$. The isomorphism $\phi : \mathring{R}_\beta \xrightarrow{\sim} \mathring{R}_{\beta'}$ sending $(X_\bullet, Y_\bullet) \mapsto (X'_\bullet, Y'_\bullet)$ is defined as follows. For $c \in [m]$, we set $(X'_c, Y'_c) := (X_c, Y_c)$ and $X'_0 := X'_1$. Note that $Y_0 = Y_1 = Y'_1$ and recall $X_0 \overset{w_\circ}{\Longrightarrow} Y_0$. We let Y'_0 be the unique weighted flag satisfying $X_0 \overset{w_\circ s_{i^*}}{\Longrightarrow} Y'_0 \xrightarrow{s_{i^*}} Y_0$. It follows that $X'_0 \overset{w_\circ}{\Longleftarrow} Y'_0$ and $Y'_0 \xrightarrow{s_{i^*}} Y'_1$, so $(X'_\bullet, Y'_\bullet) \in \mathring{R}_{\beta'}$. The inverse map is defined similarly: X_0 is the unique weighted flag satisfying $X'_0 \xrightarrow{s_i} X_0 \overset{s_i w_\circ}{\Longrightarrow} Y'_0$.

The statement (F) is trivial if the first crossing of β is hollow. If the first crossing of β is solid, we have

$$X'_0 = X'_1 = X_1 \xrightarrow{s_i} X_0 \overset{s_i w_\circ}{\Longrightarrow} Y'_0 \xrightarrow{s_{i^*}} Y_0 = Y_1 = Y'_1.$$

It follows that after acting on all these flags by some $g \in G$, we can find $t, t' \in \mathbb{C}$ and $h \in H$ such that

$$X'_0 = X'_1 = X_1 = \dot{w}_\circ \dot{s}_i h \bar{z}_i(t) U_+, \quad X_0 = \dot{w}_\circ \dot{s}_i h U_+, \quad Y'_0 = U_+, \quad Y_0 = Y_1 = Y'_1 = z_{i^*}(t') U_+.$$

Here, we have $X_0 \overset{s_i w_\circ}{\Longrightarrow} Y'_0$ and thus $X_0 \overset{w_\circ s_i}{\Longleftarrow} Y'_0$, and we have used a representative $\dot{w}_\circ \dot{s}_i$ of $w_\circ s_i$ in $N_G(H)$. Let us denote $h_0 := h_0^+ = h_0^-$ and $h_0^* := (h_0^+)^* = (h_0^-)^*$. We have $Z_0 = Y_0^{-1} X_0 = z_{i^*}(t')^{-1} \dot{w}_\circ \dot{s}_i h$, and thus, proceeding as in the proof of Lemma 2.8, we get $h_0=h$. Similarly, $Z'_0=(Y'_0)^{-1}X'_0=\dot{w}_\circ\dot{s}_ih\bar{z}_i(t)$, so $h_0^*=s_i\cdot h$. Applying (4.15), we find

$$L^*_{0,-i}=\mathrm{dlog}(s_i\cdot h_0)^{\alpha_i/2}=\mathrm{dlog}(h_0)^{-\alpha_i/2}=-L_{0,i}.$$

Applying (4.5) for $e=0,1$, we obtain $L_{0,i}=L_{0,-i}$ and $L_{1,i}=L_{1,-i}$. Recall that $L_{1,i}=L^*_{1,i}$. Thus, we get $\omega_{\beta,1}=\omega^*_{\beta',1}$, and therefore $\omega_\beta=\omega^*_{\beta'}$, finishing the proof of (F).

We now prove (Q). Let $\beta=i\beta_0$ and $\beta'=(-i)\beta_0$. If the first crossing is hollow, the claim is trivial. Suppose that the first crossing is solid. We have $\Delta_{c,k}=\Delta^*_{c,k}$ for all $c\geq 1$ and $k\in\pm I$. Thus, $x_c=x_c^*$ for all $c\in J_\beta$ such that $c>1$. Since $h_0^*=s_i\cdot h_0$, Lemma 2.12 implies that $x_1^*=x_1^{-1}M$, where M is a Laurent monomial in the grid minors $\Delta_{0,k}$ for $k\neq i$ of the same sign as i. It follows from Propositions 2.20 and 2.21 that M is a Laurent monomial in the frozen variables other than x_1. This shows (Q).

4.8. **Deletion-contraction for double braid varieties.** Theorem 4.2 tells us that it is enough to show $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety for a single β in a double-braid-move equivalence class. We now explain how the cluster algebraic results from Section 3.2 apply to $(\mathring{R}_\beta,\Sigma_\beta)$ for a special shoice of β.

Theorem 4.10. *Let $i\in I$ and consider a double braid word $\beta=ii\beta'$ on positive letters. If $(\mathring{R}_{i\beta'},\Sigma_{i\beta'})$ and $(\mathring{R}_{\beta'},\Sigma_{\beta'})$ are sink-recurrent cluster varieties, then $(\mathring{R}_\beta,\Sigma_\beta)$ is a sink-recurrent cluster variety.*

Proof. Suppose first that at least one of the first two crossings in β is hollow, in which case 1 must be solid and 2 must be hollow. Consider an arbitrary point $(X_\bullet,Y_\bullet)\in\mathring{R}_\beta$. Since the letters in β are positive, we have $Y_0=Y_1=\cdots=Y_m=X_m$. Since $w_2\leq w_\circ s_i$ and $w_0=w_\circ$, we must have $X_1\overset{w_\circ}{\Longleftarrow}X_m$ and $X_2\overset{w_\circ s_i}{\Longleftarrow}X_m$. It follows that $h_1^\pm$ and $h_2^\pm$ are regular functions on $\mathring{R}_\beta$. Choose a representative $Z_2=\dot{w}_\circ h_2^-\dot{s}_i^{-1}$ as in (2.8), and let $t,t'\in\mathbb{C}$ be such that $Z_1=Z_2z_i(t)$ and $Z_0=Z_2z_i(t)z_i(t')$. Thus, t,t' are regular functions on $\mathring{R}_\beta$. Proceeding as in the proof of Lemma 2.8, we find $h_1^+=h_2^-$ and $h_0^+=h_2^-\alpha_i^\vee(t')$, where h_0^+,h_1^+ are regular on $\mathring{R}_\beta$. It follows that $\Delta_{0,i}=t'\Delta_{1,i}$. For any $e\in J_\beta$ such that $e>1$, the function x_e depends on $Z_2,Z_3,...,Z_m$ but does not depend on t,t'. By Proposition 2.20, we have $\Delta_{0,i}=x_1M$ for some monomial M in $\{x_e\}_{e>1}$. The Deodhar hypersurface V_1 is clearly given by the equation $t'=0$. We conclude that $x_1=t'$. We thus have an isomorphism

$$r:\mathring{R}_\beta\xrightarrow{\sim}\mathring{R}_{i\beta'}\times\mathbb{C}^\times,\quad (X_\bullet,Y_\bullet)\mapsto((X_1,...,X_m,Y_1,...,Y_m),x_1). \tag{4.23}$$

Moreover, since $\Delta_{1,i}=M$ involves only frozen variables, we see that 1 is connected to only frozen indices in $\tilde{\Gamma}(\Sigma_\beta)$. It follows that the principal parts of Σ_β and $\Sigma_{i\beta'}$ agree, and therefore $(\mathring{R}_\beta,\Sigma_\beta)$ is a sink-recurrent cluster variety. Moreover, if $\Sigma_{i\beta'}$ is really full rank then so is Σ_β.

Suppose now that the first two crossings are both solid. Our goal is to apply Theorem 3.13. Recall from Proposition 2.5 that $\mathring{R}_\beta$ is affine, smooth (and thus normal), and irreducible. We now show that Σ_β is sink-recurrent. Let $\Gamma:=\Gamma(\Sigma_\beta)$. The seed $\Sigma_\beta^{\setminus 2}$ is obtained from $\Sigma_{i\beta'}$ by adding an isolated frozen variable x_1, and $\Sigma_\beta^{\setminus\widehat{N}_s^{\mathrm{in}}(\Gamma)}$ is obtained from $\Sigma_{\beta'}$ by adding isolated frozen variables x_1 and x_2, so both of these seeds are sink-recurrent.

Next, the variable x_1 is frozen in Σ_β. We claim that $\tilde{B}_{12}=1$, and $\tilde{B}_{1c}=0$ for mutable $c>2$. Indeed, since the first two crossings are solid, we have $u_0=u_1=u_2=\mathrm{id}$. Moreover, since $i_1=i_2=i$, by Definition 2.16, we get that $u_0^{\langle c\rangle}=u_1^{\langle c\rangle}$ for all $c\in J_\beta$ such that $c>2$. (For $c=2$, we have $u_0^{\langle 2\rangle}=\mathrm{id}$ and $u_1^{\langle 2\rangle}=s_i$.) Let $c>2$ be mutable so that $u_0^{\langle c\rangle}=\mathrm{id}$. Then $u_1^{\langle c\rangle}=\mathrm{id}$, and thus by Proposition 2.21, we have $\mathrm{ord}_{V_c}\Delta_{0,k}=\mathrm{ord}_{V_c}\Delta_{1,k}=0$ for all $k\in\pm I$. Thus, $\mathrm{dlog}x_c$ does not contribute to $L_{1,i}$, and so $\tilde{B}_{1c}=0$. By Propositions 2.20 and 2.21, we have $\mathrm{ord}_{V_1}\Delta_{0,i}=\mathrm{ord}_{V_2}\Delta_{1,i}=1$, $\mathrm{ord}_{V_1}\Delta_{0,j}=\mathrm{ord}_{V_2}\Delta_{1,j}=0$ for all $j\in I\setminus\{i\}$, and $\mathrm{ord}_{V_1}\Delta_{1,j}=\mathrm{ord}_{V_2}\Delta_{0,j}=0$ for all $j\in I$. Thus, $\tilde{B}_{12}=1$.

Next, by Corollary 2.14, the sum of terms of ω_β involving x_2 is clearly of the form $d_i\mathrm{dlog}x_2\wedge\mathrm{dlog}M$ for a Laurent monomial M in $\mathbf{x}$, and thus Σ_β is integral at 2. We have shown that Σ_β satisfies Assumption 3.11.

Next, we show that the mutated cluster variable x_2' is regular on $\mathring{R}_\beta$. We apply the moves $ii\beta' \xrightarrow{\text{(B5)}} (-i)i\beta' \xrightarrow{\text{(B1)}} i(-i)\beta'$. Denote $\dot{\mathbf{x}} := \mathbf{x}_{(-i)i\beta'}$ and $\ddot{\mathbf{x}} := \mathbf{x}_{i(-i)\beta'}$. It follows from the argument in Section 4.1 that $\dot{x}_2$ is mutable in $\Sigma_{(-i)i\beta'}$, and by (4.11), its mutation $\dot{x}_2'$ is regular on $\mathring{R}_{(-i)i\beta'}$, as it equals the pullback $\ddot{x}_2^*$ times a monomial in the other cluster variables in $\dot{\mathbf{x}}$ with nonnegative exponents. As explained in Section 4.7, the seeds Σ_β and $\Sigma_{(-i)i\beta'}$ are quasi-equivalent. By Lemma 3.9, we find that the mutation x_2' differs from $\dot{x}_2'$ by a unit (cf. part (3) of Corollary 2.24), and thus x_2' is regular on $\mathring{R}_\beta$.

Let $W := \{x_2 \neq 0\}$ $V := V_2 = \{x_2 = 0\}$ be the open-closed covering of $\mathring{R}_\beta$ coming from x_2. Our final goal is to construct isomorphisms

$$W \cong \mathring{R}_{i\beta'} \times \mathbb{C}^\times \cong \mathcal{V}(\Sigma_\beta^{\setminus 2}) \quad \text{and} \quad V \cong V_1 \times V_2 = \operatorname{Spec}(\mathbb{C}[x_2']) \times \mathring{R}_{\beta'} \cong \mathbb{C} \times \mathcal{V}(\Sigma_\beta^{/2})$$

satisfying the conditions of Theorem 3.13. Recall that by Proposition 2.21, for $e \in J_\beta$, $e > 2$, we have $\operatorname{ord}_{V_e} \Delta_{1,i} = 0$ if and only if $\operatorname{ord}_{V_e} \Delta_{0,i} = 0$. Moreover, the same proposition implies $\operatorname{ord}_{V_2} \Delta_{0,i} = 0$. It follows by Proposition 2.20 that $\Delta_{1,i}$ is equal to x_2 times a monomial in the frozen variables, and that $\Delta_{0,i}$ is equal to x_1 times a monomial in the same set of frozen variables. Since W is the complement of V_2, we see that $(X_\bullet, Y_\bullet) \in W$ if and only if $X_1 \overset{w_\circ}{\Longleftarrow} Y_1 = X_m$. Thus, h_1^+ is a regular function on W. We choose a representative $Z_1 = \dot{w}_\circ h_1^+$ and let $t \in \mathbb{C}$ be such that $Z_0 = Z_1 z_i(t')$. Then we get $t' = \Delta_{0,i}/\Delta_{1,i} = Mx_1/x_2$, where M is a Laurent monomial in the frozen variables other than x_1. Similarly to (4.23), we let $r : W \to \mathring{R}_{i\beta'} \times \mathbb{C}^\times$ be the map sending $(X_\bullet, Y_\bullet)$ to $((X_1, \dots, X_m, Y_1, \dots, Y_m), Mx_1/x_2)$. By assumption, we have $\mathring{R}_{i\beta'} \cong \mathcal{V}(\Sigma_{i\beta'})$. The frozen index 1 is only connected to other frozen indices in $\tilde{\Gamma}(\Sigma_\beta)$. Thus, the seed $\Sigma_\beta^{\setminus 2}$ is obtained from $\Sigma_{i\beta'}$ by adding an isolated frozen vertex, and therefore $\mathcal{V}(\Sigma_\beta^{\setminus 2}) \cong \mathcal{V}(\Sigma_{i\beta'}) \times \mathbb{C}^\times$. Adjusting the isolated frozen variable by a Laurent monomial in the other frozen variables, we see that the pullbacks of $x_1, \dots, x_{n+m}$ under the inclusion $\mathcal{V}(\Sigma_\beta^{\setminus 2}) \cong W \hookrightarrow X$ are indeed the same-named cluster variables in $\Sigma_\beta^{\setminus 2}$. This verifies the assumptions on ι_W in Theorem 3.13.

Now suppose that $(X_\bullet, Y_\bullet) \in V$. We have $X_0 \overset{w_\circ}{\Longleftarrow} X_m$ but not $X_1 \overset{w_\circ}{\Longleftarrow} X_m$, so we must have $X_1 \overset{w_\circ s_i}{\Longleftarrow} X_m$, and therefore $X_2 \overset{w_\circ}{\Longleftarrow} X_m$. Consider the map $r : V \to \mathring{R}_{\beta'} \times \mathbb{C}$ sending $(X_\bullet, Y_\bullet)$ to $((X_2, \dots, X_m, Y_2, \dots, Y_m), x_2')$. We claim that this map is an isomorphism. To construct an inverse, we need to show how to recover X_0, X_1, Y_0, Y_1 from the image of r. We have $Y_0 = Y_1 = X_m$. Also, X_1 is uniquely determined by $Y_1 = Y_2$ and X_2, since $Y_1 \overset{s_i w_\circ}{\Longleftarrow} X_1 \xleftarrow{s_i} X_2$. It remains to recover X_0. Note also that we can recover the cluster variables x_e, $e > 2$, as well as the mutated cluster variable x_2', from the image of r. Since $(X_\bullet, Y_\bullet) \in V$, the frozen variable x_1 is also recovered from the exchange relation $0 = x_2 x_2' = M_1 + x_1 M_2$ for x_2.

In order to recover X_1, we apply moves $ii\beta' \xrightarrow{\text{(B5)}} (-i)i\beta' \xrightarrow{\text{(B1)}} i(-i)\beta'$ as we did above. Let $(\ddot{X}_\bullet, \ddot{Y}_\bullet)$ denote the image of $(X_\bullet, Y_\bullet)$ in $\mathring{R}_{i(-i)\beta'}$ under this isomorphism ϕ, and let $\ddot{\mathbf{x}} := \mathbf{x}_{i(-i)\beta'}$. As in Section 4.7, let Y_0' be the unique weighted flag satisfying $X_0 \overset{w_\circ s_{i^*}}{\Longrightarrow} Y_0' \xrightarrow{s_{i^*}} Y_0$. Then $\ddot{X}_2 = X_2$, and $\ddot{Y}_2 = Y_2$, and $(\ddot{X}_1, \ddot{X}_0, \ddot{Y}_0, \ddot{Y}_1) = (X_2, X_1, Y_0', Y_0')$. We have that Y_0' is uniquely determined by X_2, Y_2, and $\ddot{\mathbf{x}}$: if $\ddot{x}_2 = 0$ then Y_0' is uniquely determined by $Y_2 \xleftarrow{s_{i^*}} Y_0' \overset{w_\circ s_{i^*}}{\Longleftarrow} X_2$; otherwise, we have $X_2 \overset{w_\circ}{\Longleftarrow} Y_0'$, and the values of $\ddot{\mathbf{x}}$ uniquely fixes the $U_+ \times U_+$-double coset $\ddot{Z}_1 := (Y_0')^{-1} X_2$ which determines Y_0'. The weighted flag X_0 is then uniquely determined by $X_1 \xrightarrow{s_i} X_0 \overset{s_i w_\circ}{\Longrightarrow} Y_0'$. It thus suffices to show that $\ddot{\mathbf{x}}$ is uniquely determined by the image of r. For $e \in J_\beta$, $e > 2$, we have $\ddot{x}_e = x_e$. Moreover, $\ddot{x}_1 = M/x_1$ for some monomial M in the frozen variables x_e other than x_1 (all of which must satisfy $e > 2$ since x_2 is mutable). Finally, by (4.11), $\ddot{x}_2$ differs from x_2' by a monomial in the cluster variables other than x_2. We are done with verifying the assumptions on ι_V in Theorem 3.13.

We have verified all conditions in Assumption 3.11 and Theorem 3.13. Thus, $(\mathring{R}_\beta, \Sigma_\beta)$ is a cluster variety. We have already shown that it is sink-recurrent. $\square$

4.9. **Proof of Theorem 1.1 for G simply-laced.** We proceed by induction on the number m of indices in β. Recall that we always assume $\delta(\beta)=w_\circ$. The base case is $m=\ell(w_\circ)$, where all indices are hollow. The cluster algebra is $\mathcal{A}_\beta=\mathbb{C}$ and the braid variety $\mathring{R}_\beta$ is a point.

Suppose now that $m>\ell(w_\circ)$. Applying (B1) and (B4), we can assume that all letters of β belong to I. Since G is simply-laced, all braid moves are automatically short. Applying (B2)–(B3), we may therefore transform β into a braid word of the form $\beta_1 ii\beta_2$ for some braid words β_1,β_2 and $i\in I$. We can also apply *conjugation moves* to β: if $\beta=j\beta_0$, the conjugation move consists of the moves

$$\beta=j\beta_0 \xrightarrow{\text{(B5)}} (-j)\beta_0 \xrightarrow{\text{(B1)}} \cdots \xrightarrow{\text{(B1)}} \beta_0(-j) \xrightarrow{\text{(B4)}} \beta_0 j^*. \tag{4.24}$$

Applying conjugation moves, we may further transform β into the form $\beta' := ii\beta_2\beta_1^*$, where β_1^* is obtained from β_1 by applying the map $j\mapsto j^*$ to each letter. Utilizing the inductive hypothesis and applying Theorem 4.10 to β', we find that $(\mathring{R}_{\beta'},\Sigma_{\beta'})$ is a cluster variety. It follows from Theorem 4.2 (for short moves) and Corollary 3.10 that $(\mathring{R}_\beta,\Sigma_\beta)$ is therefore also a cluster variety. □

Remark 4.11. It follows from our proof that the seed Σ_β is really full rank when G is simply-laced. Indeed, this property is preserved under moves (B1)–(B5), and by Remark 3.15 is compatible with deletion-contraction.

Finally, we show that for G simply-laced, double braid moves correspond to mutation equivalence.

Proposition 4.12. *Suppose that G is simply-laced and β,β' are related by a braid move* (B1)–(B4)*. The seeds $\Sigma_\beta,\Sigma^*_{\beta'}$ are mutation equivalent (up to relabeling cluster variables).*

Proof. By Theorem 1.1 for simply-laced G, $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety. By (Q), there is a seed Σ', which differs from Σ_β by mutation and possibly relabeling, such that $\Sigma'\sim\Sigma^*_{\beta'}$. We claim that these seeds are actually identical. Indeed, choose a double braid word β_0 such that all cluster variables of β,β' become mutable in $\tilde{\beta}:=\beta_0\beta$, $\tilde{\beta}':=\beta_0\beta'$; cf. Lemma 2.25. Let $\tilde{\Sigma}'$ be obtained from $\Sigma_{\tilde{\beta}}$ using the same mutations and relabeling by which Σ' was produced from Σ_β. Now, $(\mathring{R}_{\tilde{\beta}},\Sigma_{\tilde{\beta}})$ is also a cluster variety, so by (Q), $\tilde{\Sigma}'\sim\Sigma^*_{\tilde{\beta}'}$. Since all frozen variables of Σ' are mutable in $\tilde{\Sigma}'$, it follows that the seeds Σ' and $\Sigma^*_{\beta'}$ are identical. □

5. Folding

Before completing the proof in Section 6, we review some background on folding. We first compare Deodhar geometry (Section 2) in the case of a multiply-laced group G to the case of the "unfolded" simply-laced group $\dot{G}$. We then review folding on seeds.

5.1. **Pinnings.** Let G be a complex, simple, simply-connected algebraic group. Choose a pinning $(H,B_+,B_-,x_i,y_i;i\in I)$. Then there exists an algebraic group $\dot{G}$ of simply-laced type with pinning $(\dot{H},\dot{B}_+,\dot{B}_-,\dot{x}_{i'},\dot{y}_{i'};i'\in\dot{I})$; see [Lus94, §1.6]. We have a bijection $\sigma:\dot{I}\to\dot{I}$ which extends to an automorphism $\sigma:\dot{G}\to\dot{G}$, and a map $\iota:G\to\dot{G}$ which yields algebraic group isomorphisms

$$\iota:G\xrightarrow{\sim}\dot{G}^\sigma,\quad H\xrightarrow{\sim}\dot{H}^\sigma,\quad B_\pm\xrightarrow{\sim}(\dot{B}_\pm)^\sigma,\quad U_\pm\xrightarrow{\sim}(\dot{U}_\pm)^\sigma.$$

The maps $gB_+\mapsto\iota(g)\dot{B}_+$ and $gU_+\mapsto\iota(g)\dot{U}_+$ induce isomorphisms of varieties:

$$\iota:G/B_+\xrightarrow{\sim}(\dot{G}/\dot{B}_+)^\sigma\quad\text{and}\quad G/U_+\xrightarrow{\sim}(\dot{G}/\dot{U}_+)^\sigma. \tag{5.1}$$

For the first isomorphism, see [Lus94, §8.8]. The surjectivity and injectivity of the second map follow from that of the first by a straightforward computation.

For an element $i\in I$, we denote by $\boldsymbol{i}\subset\dot{I}$ the associated σ-orbit, i.e., the orbit under the cyclic group generated by σ. We also let $-\boldsymbol{i}:=\{-i'\mid i'\in\boldsymbol{i}\}\subset-\dot{I}$. We let $\{\dot{\alpha}_{i'}\mid i'\in\dot{I}\}$, $\{\dot{\alpha}^\vee_{i'}\mid i'\in\dot{I}\}$, and $\{\dot{\omega}_{i'}\mid i'\in\dot{I}\}$ be the simple roots, simple coroots, and fundamental weights of the root system of $\dot{G}$.

Letting $\dot a_{i'j'} := \langle \dot\alpha_{i'}, \dot\alpha^\vee_{j'}\rangle$ be the entries of the associated Cartan matrix (and setting $\dot a_{(-i')(-j')} := \dot a_{i'j'}$ and $\dot a_{(\pm i')(\mp j')} := 0$ as before), we have

$$d_i = |\boldsymbol{i}| \quad \text{and} \quad a_{ij} = \sum_{j'\in \boldsymbol{j}} \dot a_{i'j'} \quad \text{for all } i,j\in \pm I \text{ and } i'\in \boldsymbol{i}. \tag{5.2}$$

The Coxeter generators of the Weyl group $\dot W$ of $\dot G$ are denoted by $\{\tilde s_{i'} \mid i'\in \dot I\}$. Restricting ι to the normalizer of H, we get a group isomorphism

$$\iota : W \xrightarrow{\sim} \dot W^\sigma, \quad s_i \mapsto \prod_{i'\in\boldsymbol{i}} \tilde s_{i'}. \tag{5.3}$$

Here the order inside $\boldsymbol{i}$ is immaterial since the corresponding elements $\tilde s_{i'}$ commute. It follows that the longest element $w_\circ \in W$ gets mapped under (5.3) to the longest element $\tilde w_\circ$ of $\dot W$, because $\sigma : \dot W \to \dot W$ preserves Coxeter length and therefore $\tilde w_\circ \in \dot W^\sigma$. The following result is immediate.

Lemma 5.1. *Let $B_1, B_2 \in G/U_+$. If $B_1 \xrightarrow{w} B_2$ then $\iota(B_1) \xrightarrow{\iota(w)} \iota(B_2)$. If $B_1 \xRightarrow{w} B_2$ then $\iota(B_1) \xRightarrow{\iota(w)} \iota(B_2)$. In particular, if $B_1 \xRightarrow{w_\circ} B_2$ then $\iota(B_1) \xRightarrow{\tilde w_\circ} \iota(B_2)$.*

5.2. **Braid varieties.** Let $\beta = i_1 i_2 ... i_m \in (\pm I)^m$ be a double braid word. Let $\tilde\beta = i'_1 i'_2 ... i'_{\tilde m} \in (\pm \dot I)^{\tilde m}$ be obtained by concatenating the letters in $\boldsymbol{i_1}, \boldsymbol{i_2}, ..., \boldsymbol{i_m}$ (choosing the order inside each $\boldsymbol{i_c}$ arbitrarily), where $\tilde m := |\boldsymbol{i_1}| + |\boldsymbol{i_2}| + \cdots + |\boldsymbol{i_m}|$. We let $\lambda_\beta : [\tilde m] \to [m]$ denote the unique order-preserving map satisfying $|\lambda_\beta^{-1}(c)| = |\boldsymbol{i_c}|$ for all $c \in [m]$. It is clear that an index $c\in[m]$ is solid (resp., hollow) if and only if all indices in $\lambda_\beta^{-1}(c)$ are solid (resp., hollow). In other words, the set $\dot J_{\tilde\beta}$ of solid crossings for $\tilde\beta$ is given by

$$\dot J_{\tilde\beta} = \lambda_\beta^{-1}(J_\beta). \tag{5.4}$$

Let $\mathring{\mathcal{Y}}'_\beta$ be the variety of tuples $(\dot X_\bullet, \dot Y_\bullet)$ of weighted flags in $\dot G/\dot U_+$ satisfying

$$\begin{tikzcd}
\dot X_0 \arrow[Leftarrow, d, "\tilde w_\circ"] & \dot X_1 \arrow[l, "\iota(s^+_{i_1})"'] & \cdots \arrow[l, "\iota(s^+_{i_2})"'] & \dot X_m \arrow[l, "\iota(s^+_{i_m})"'] \arrow[d, equal] \\
\dot Y_0 \arrow[r, "\iota(s^-_{i_1^*})"'] & \dot Y_1 \arrow[r, "\iota(s^-_{i_2^*})"'] & \cdots \arrow[r, "\iota(s^-_{i_m^*})"'] & \dot Y_m.
\end{tikzcd}$$

Let $\mathcal{Y}'_\beta$ be obtained by omitting the condition $\dot X_0 \xLeftarrow{\tilde w_\circ} \dot Y_0$. Lemma 2.1 yields isomorphisms $\mathring{\mathcal{Y}}'_\beta \cong \mathring{\mathcal{Y}}_{\tilde\beta}$ and $\mathcal{Y}'_\beta \cong \mathcal{Y}_{\tilde\beta}$. Let $\mathring{R}'_\beta$ be the quotient of $\mathring{\mathcal{Y}}'_\beta$ by the free $\dot G$-action. Then $\mathring{R}'_\beta \cong \mathring{R}_{\tilde\beta}$.

The map σ acts on the varieties $\mathring{\mathcal{Y}}'_\beta$, $\mathcal{Y}'_\beta$, and $\mathring{R}'_\beta$ termwise by acting on each $\dot X_c$ and $\dot Y_c$. Let $T'_\beta \subset \mathring{R}'_\beta$ be the image of the Deodhar torus $T_{\tilde\beta} \subset \mathring{R}_{\tilde\beta}$ under the isomorphism $\mathring{R}_{\tilde\beta} \cong \mathring{R}'_\beta$. We have the following straightforward result.

Proposition 5.2. *Applying ι termwise yields isomorphisms*

$$\mathring{\mathcal{Y}}_\beta \xrightarrow{\sim} (\mathring{\mathcal{Y}}'_\beta)^\sigma, \quad \mathcal{Y}_\beta \xrightarrow{\sim} (\mathcal{Y}'_\beta)^\sigma, \quad \mathring{R}_\beta \xrightarrow{\sim} (\mathring{R}'_\beta)^\sigma, \quad \text{and} \quad T_\beta \xrightarrow{\sim} (T'_\beta)^\sigma. \tag{5.5}$$

5.3. **Grid minors.** Recall that we have the character and cocharacter lattices $X^*(H) := \operatorname{Hom}(H, \mathbb{C}^\times)$, $X_*(H) := \operatorname{Hom}(\mathbb{C}^\times, H)$. The map $\iota : H \to \dot H$ induces a map $\iota_* : X_*(H) \to X_*(\dot H)$ sending $\alpha_i^\vee \mapsto \sum_{i'\in\boldsymbol{i}} \dot\alpha^\vee_{i'}$ for $i\in I$, so that $\iota(\alpha_i^\vee(t)) = \prod_{i'\in\boldsymbol{i}} \dot\alpha^\vee_{i'}(t)$ for $t\in\mathbb{C}^\times$. It also induces a map $\iota^* : X^*(\dot H) \to X^*(H)$ sending $\dot\omega_{i'} \mapsto \omega_i$ for all $i\in I$ and $i'\in\boldsymbol{i}$, so that $\iota(h)^{\dot\omega_{i'}} = h^{\omega_i}$ for $h\in H$. It follows that for all $g\in G$, $v,w\in W$, $i\in I$, and $i'\in\boldsymbol{i}$, we have

$$\Delta_{v\omega_i, w\omega_i}(g) = \Delta_{\iota(v)\dot\omega_{i'}, \iota(w)\dot\omega_{i'}}(\iota(g)). \tag{5.6}$$

Let $(X_\bullet, Y_\bullet) \in \mathring{R}'_\beta$. As usual, for $c = 0,1,\dots,m$, we denote $Z_c := Y_c^{-1} X_c$. Let $\tilde u_c := \iota(u_c)$ and $\tilde w_c := \iota(w_c)$. For $i'\in\dot I$, consider analogs of grid minors for $\mathring{R}'_\beta$:

$$\Delta'_{c,i'}(X_\bullet, Y_\bullet) = \Delta_{\tilde w_c \dot\omega_{i'}, \dot\omega_{i'}}(Z_c), \qquad \Delta'_{c,-i'}(X_\bullet, Y_\bullet) = \Delta_{\tilde w_\circ \dot\omega_{i'}, \tilde u_c^{-1}\dot\omega_{i'}}(Z_c). \tag{5.7}$$

Comparing (5.6)–(5.7) to Definition 2.11, we find that the grid minors on $\mathring{R}_\beta$ are pullbacks of the minors defined in (5.7): for $c=0,1,\dots,m$, $i\in\pm I$, and $i'\in\boldsymbol{i}$, we have

$$\iota^*\Delta'_{c,i'}=\Delta_{c,i}. \tag{5.8}$$

Using Corollary 2.14, we obtain the following description of chamber minors on $\mathring{R}_{\tilde\beta}$, which we denote by $\tilde\Delta_{c'}$, $c'\in\dot J_{\tilde\beta}$; cf. (5.4).

Lemma 5.3. *Let $c'\in\dot J_{\tilde\beta}$ be a solid crossing for $\tilde\beta$. Set $i':=i'_{c'}$ and $c:=\lambda_\beta(c')\in J$. Then the isomorphism $\mathring{R}_{\tilde\beta}\cong\mathring{R}'_\beta$ sends the chamber minor $\tilde\Delta_{c'}$ to $\Delta'_{c-1,i'}$, and we have $\iota^*\Delta'_{c-1,i'}=\Delta_c$.*

5.4. 2**-form.** Our next goal is to show that the two-form also folds.

Lemma 5.4. *Let ω'_β be the pullback of the 2-form $\omega_{\tilde\beta}$ on $\mathring{R}_{\tilde\beta}$ under the isomorphism $\mathring{R}'_\beta\cong\mathring{R}_{\tilde\beta}$. We have $\iota^*\omega'_\beta=\omega_\beta$.*

Proof. Recall from (2.26)–(2.27) that we have 1-forms $L_{c,i}=\frac{1}{2}\sum_{k\in\pm I}a_{ik}\,\mathrm{dlog}\Delta_{c,k}$ on $\mathring{R}_\beta$ for $(c,i)\in[m]\times(\pm I)$, and that for $c\in[m]$ and $i:=i_c$, we set $\omega_c(\beta):=\mathrm{sign}(i)d_iL_{c-1,i}\wedge L_{c,i}$. For $i'\in\pm\dot I$, introduce a 1-form $L'_{c,i'}:=\frac{1}{2}\sum_{j'\in\pm\dot I}\dot a_{i'j'}\mathrm{dlog}\Delta'_{c,j'}$. By Corollary 2.14, we have

$$\omega'_\beta=\sum_{c\in J_\beta}\mathrm{sign}(i_c)\sum_{i'\in\boldsymbol{i_c}}L'_{c-1,i'}\wedge L'_{c,i'}. \tag{5.9}$$

Next, applying (5.8) and (5.2), we see that for all $c\in[m]$, $i\in\pm I$, and $i'\in\boldsymbol{i}$, we have

$$\iota^*L'_{c,i'}=\iota^*\left(\frac{1}{2}\sum_{j'\in\pm\dot I}\dot a_{i'j'}\mathrm{dlog}\Delta'_{c,j'}\right)=\frac{1}{2}\sum_{j\in\pm I}\left(\sum_{j'\in\boldsymbol{j}}\dot a_{i'j'}\right)\mathrm{dlog}\Delta_{c,j}=L_{c,i}. \tag{5.10}$$

The result follows by combining (5.9)–(5.10) with (5.2). □

5.5. **Folding seeds.** We briefly review the notion of folding seeds, following [FWZ16, Section 4.4], though translating into our conventions.

Definition 5.5. Let $\dot\Sigma=(\dot T,\dot x,\dot{\mathbf{d}},\dot\omega)$ be a seed with $\dot{\mathbf{d}}=(1,\dots,1)$, with mutable indices $\dot J^{\mathrm{mut}}$ and frozen indices $\dot J^{\mathrm{fro}}$. Let σ be a bijection acting on $\dot J:=\dot J^{\mathrm{mut}}\sqcup\dot J^{\mathrm{fro}}$. Let J be the set of σ-orbits, and for $j\in J$, we denote the corresponding orbit by $\boldsymbol{j}$. An orbit is *mutable* (resp., *frozen*) if it consists entirely of mutable (resp., frozen) indices. The bijection σ also acts on the set of cluster variables by $\sigma(\dot x_{j'})=\dot x_{\sigma(j')}$. We call $\dot\Sigma$ *weakly σ-admissible*[5] if:

1. Every orbit is either mutable or frozen.
2. The 2-form ω is invariant under the σ-action.
3. For all $a',a''\in\dot J^{\mathrm{mut}}$ in the same σ-orbit, $\dot{\tilde B}_{a'a''}=0$, where $\dot{\tilde B}$ is the exchange matrix of $\dot\Sigma$.

Part (1) implies a natural decomposition $J=J^{\mathrm{mut}}\sqcup J^{\mathrm{fro}}$. The map σ also acts on the torus $\dot T$ by permuting coordinates. Notice that $\dot T^\sigma$ is isomorphic to $(\mathbb{C}^\times)^{|J|}$. We denote by $\iota:\dot T^\sigma\hookrightarrow\dot T$ the inclusion map.

Definition 5.6. Suppose $\dot\Sigma$ is weakly σ-admissible, with notation as in Definition 5.5. The *folded seed* is a seed with index set $J=J^{\mathrm{mut}}\sqcup J^{\mathrm{fro}}$, defined as $\iota^*\dot\Sigma=\Sigma:=(T,\mathbf{x},\mathbf{d},\omega)$ where

- $T=\dot T^\sigma$;
- $\mathbf{x}=(x_j)_{j\in J}$, where for $j\in J$, $x_j:=\iota^*\dot x_{j'}$ for any $j'\in\boldsymbol{j}$;
- $d_j=|\boldsymbol{j}|$ for $j\in J$;
- $\omega=\iota^*\dot\omega$.

[5] Our notion of weak σ-admissibility differs from the notion of admissibility in [FWZ16, Definition 4.4.1] in that we do not require that for any a',a'' in the same orbit and any k' mutable, $\dot{\tilde B}_{a'k'}\tilde B_{a''k'}\geq 0$.

Note that x_j is well-defined, since $\iota^* \dot{x}_{j'} = \iota^* \dot{x}_{\sigma(j')}$ for all $j' \in \boldsymbol{j}$. The exchange matrix $\tilde{B}$ of Σ is therefore written in terms of the exchange matrix $\dot{\tilde{B}}$ of $\dot{\Sigma}$ as

$$\tilde{B}_{ab} = \sum_{a' \in \boldsymbol{a}} \dot{\tilde{B}}_{a'b'}, \quad \text{where } b' \in \boldsymbol{b} \text{ is arbitrary.} \tag{5.11}$$

In particular, if $\dot{\Sigma}$ is integral then so is Σ. For the rest of this subsection, we assume that $\dot{\Sigma}$ and Σ are integral.

For weakly σ-admissible $\dot{\Sigma}$ and $j \in J^{\mathrm{mut}}$, we denote by $\mu_{\boldsymbol{j}} \dot{\Sigma} = \mu_{\boldsymbol{j}}(\dot{\Sigma})$ the result of mutating $\dot{\Sigma}$ once at each index in $\boldsymbol{j}$, and call $\mu_{\boldsymbol{j}}$ an *orbit-mutation*. Note that $\mu_{\boldsymbol{j}} \dot{\Sigma}$ does not depend on the order of mutation. The seed $\mu_{\boldsymbol{j}} \dot{\Sigma}$ may not be weakly σ-admissible; we introduce the following notion to avoid such mutations.

Definition 5.7. Let $\dot{\Sigma}$ be a weakly σ-admissible seed and $j \in J^{\mathrm{mut}}$. We call $\mu_{\boldsymbol{j}}$ *quasi-admissible* if for all $k \in J^{\mathrm{mut}}$, we have $\tilde{B}_{k'j'} \tilde{B}_{k''j'} \geq 0$ for all $k', k'' \in \boldsymbol{k}$ and $j' \in \boldsymbol{j}$.

The name "quasi-admissible" is justified by the following proposition.

Proposition 5.8. *Let $\dot{\Sigma}$ be a weakly σ-admissible seed and $j \in J^{\mathrm{mut}}$. If $\mu_{\boldsymbol{j}}$ is quasi-admissible, then $\mu_{\boldsymbol{j}}(\dot{\Sigma})$ is weakly σ-admissible and*

$$\mu_j(\iota^* \dot{\Sigma}) \sim \iota_j^*(\mu_{\boldsymbol{j}} \dot{\Sigma}),$$

where ι_j is an inclusion of the associated tori.

Proof. We use the notation of Definitions 5.5 and 5.6. In particular, let $\Sigma := \iota^* \dot{\Sigma}$ be the folded seed on index set J. Since the exchange matrix $\dot{\tilde{B}}$ of $\dot{\Sigma}$ is skew-symmetric, it is equivalent to a quiver $\dot{Q}$; we will use the two interchangeably.

It is clear that $\mu_{\boldsymbol{j}} \dot{\Sigma}$ satisfies condition (1) of Definition 5.5. Because there are no arrows between vertices in $\boldsymbol{j}$, mutating at all vertices of $\boldsymbol{j}$ shows that the exchange matrix of $\mu_{\boldsymbol{j}} \dot{\Sigma}$ satisfies $(\mu_{\boldsymbol{j}} \dot{\tilde{B}})_{a'b'} = (\mu_{\boldsymbol{j}} \dot{\tilde{B}})_{\sigma(a')\sigma(b')}$ for $a', b' \in \dot{J}$. The assumption that $\mu_{\boldsymbol{j}}$ is quasi-admissible implies that for $k \in J^{\mathrm{mut}}$, $(\mu_{\boldsymbol{j}} \dot{\tilde{B}})_{k'k''} = 0$ for all $k', k'' \in \boldsymbol{k}$. Thus, $\mu_{\boldsymbol{j}} \dot{\Sigma}$ is weakly σ-admissible.

Let $\Sigma_1 := \mu_j(\Sigma)$ and $\Sigma_2 := \iota_j^*(\mu_{\boldsymbol{j}} \dot{\Sigma})$. Let $\mathbf{y}$ and $\mathbf{z}$ be the clusters in Σ_1 and Σ_2, respectively. For $k \neq j$, $y_k = z_k$ because both are equal to the cluster variable x_k of Σ.

To analyze the relationship between y_j and z_j, we need the following notions. Let $a, k \in J$ and choose $a' \in \boldsymbol{a}$, $k' \in \boldsymbol{k}$. We call a path $a' \to k' \to a''$ in $\dot{Q}$ a *bad path* if a', a'' are in the same orbit; condition (2) of Definition 5.7 implies that no bad path in $\dot{Q}$ begins or ends in a mutable orbit. Let $P_{k'}$ be a maximal (by inclusion) collection of arrow-disjoint bad paths with middle vertex k'.

In $\dot{\Sigma}$, for $j' \in \boldsymbol{j}$, the mutation $\dot{x}'_{j'}$ of $\dot{x}_{j'}$ is defined by the exchange relation

$$\dot{x}_{j'} \dot{x}'_{j'} = M'N' + M''N'', \quad \text{where}$$

$$N' := \prod_{(a' \to j' \to a'') \in P_{j'}} \dot{x}_{a'} \qquad \text{and} \qquad N'' := \prod_{(a' \to j' \to a'') \in P_{j'}} \dot{x}_{a''},$$

and M', M'' are the appropriate monomials in the cluster variables of $\dot{\Sigma}$. Notice that if $\dot{x}_{a'}$ appears in M' and $\dot{x}_{b'}$ appears in M'' for $a' \in \boldsymbol{a}$, $b' \in \boldsymbol{b}$, then $\boldsymbol{a} \neq \boldsymbol{b}$, by the maximality of $P_{j'}$. Notice also that by assumption, N' and N'' are monomials in the frozen variables. We set $N := \iota^*(N') = \iota^*(N'')$. Using (5.11), we have

$$\iota^*(\dot{x}'_{j'}) = N \frac{\iota^* M' + \iota^* M''}{x_j}, \qquad \text{and} \qquad \mu_j(x_j) = \frac{\iota^* M' + \iota^* M''}{x_j}. \tag{5.12}$$

This shows that the tori and the lattices spanned by the frozens of Σ_1, Σ_2 agree, and that cluster variables differ by Laurent monomials in frozens. The multipliers $\mathbf{d}$ of both seeds are the same by definition. The 2-forms of the two seeds agree by the functoriality of pullbacks. $\square$

6. Proof of Theorem 1.1 for G multiply-laced

Fix multiply-laced braid words β,β' related by a long braid move (B3), so that
$$\beta=\beta_1\underbrace{iji...}_{m_{ij}\text{ letters}}\beta_2=\beta_1\delta\beta_2 \quad\text{and}\quad \beta'=\beta_1\underbrace{jij...}_{m_{ij}\text{ letters}}\beta_2=\beta_1\delta'\beta_2.$$
By Definition 4.1, we have an isomorphism $\phi:\mathring{R}_\beta\xrightarrow{\sim}\mathring{R}_{\beta'}$. The goal of this section is to show (F), (Q), and thus Theorem 4.2, for the geometrically defined seeds Σ_β and $\phi^*\Sigma_{\beta'}$, and a particular mutation Σ' of Σ_β (defined in (6.1)). We will then show Theorem 1.1 for G multiply-laced in Section 6.3, and discuss consequences of Theorem 1.1 in Section 6.4

Note that we already have shown Theorems 1.1 and 4.2 for simply-laced braid varieties.

6.1. **Proof of (F) for long braid moves.** Let $\tilde\beta_1,\tilde\beta_2,\tilde\delta,\tilde\delta'$ be lifts of $\beta_1,\beta_2,\delta,\delta'$, respectively, following the conventions of Section 5.2. Define $\tilde\beta:=\tilde\beta_1\tilde\delta\tilde\beta_2$, $\tilde\beta':=\tilde\beta_1\tilde\delta'\tilde\beta_2$ which are lifts of β and β', respectively. There is at least one sequence of short braid moves (B3) relating $\tilde\beta$ and $\tilde\beta'$; fix such a sequence of braid moves and denote the corresponding isomorphism of braid varieties by $\tilde\phi:\mathring{R}_{\tilde\beta}\xrightarrow{\sim}\mathring{R}_{\tilde\beta'}$.

Throughout this section, we abuse notation and use ι to denote the compositions
$$\iota:\mathring{R}_\beta\xrightarrow{\sim}(\mathring{R}'_\beta)^\sigma\hookrightarrow\mathring{R}'_\beta\xrightarrow{\sim}\mathring{R}_{\tilde\beta}\qquad\text{and}\qquad\iota:\mathring{R}_{\beta'}\xrightarrow{\sim}(\mathring{R}'_{\beta'})^\sigma\hookrightarrow\mathring{R}'_{\beta'}\xrightarrow{\sim}\mathring{R}_{\tilde\beta'}.$$

Lemma 6.1. *We have the equality $\iota\circ\phi=\tilde\phi\circ\iota$.*

Proof. Observe that the words $\tilde\delta,\tilde\delta'$ are reduced. Thus, the sequence of moves (B3) from $\tilde\delta$ to $\tilde\delta'$ fixes the weighted flags to the left and right of the indices involved in $\tilde\delta,\tilde\delta'$, and all weighted flags in between are uniquely determined; cf. Definition 4.1. □

Lemma 6.2. *We have $\omega_\beta=\phi^*\omega_{\beta'}$; that is,* (F) *holds for β,β'.*

Proof. We have $\omega_\beta=\iota^*\omega_{\tilde\beta}=\iota^*\tilde\phi^*\omega_{\tilde\beta'}=\phi^*\iota^*\omega_{\tilde\beta'}=\phi^*\omega_{\beta'}$, where we have used Lemma 5.4, (F) for $\tilde\beta,\tilde\beta'$, Lemma 6.1, and Lemma 5.4 again (in that order). □

6.2. **Proof of (Q) for long braid moves.** We continue to use the notation established earlier in this section. Without loss of generality, we assume that either $\delta=ijij$ (in the case when α_i,α_j form a root subsystem of type B_2 or C_2, where $|\boldsymbol{i}|=2$ and $|\boldsymbol{j}|=1$), or $\delta=121212$ (in the case of $G=G_2$, where $|\mathbf{1}|=3$ and $|\mathbf{2}|=1$).

The words δ,δ' involve indices $r+1,...,r+p$, and for convenience, we decrease all indices by r so that δ,δ' are supported on $1,...,p$. Similarly, we assume that $\tilde\delta,\tilde\delta'$ involve indices $1,...,\tilde p$. We define the seed
$$\Sigma':=\tau_{\text{fold}}\circ\mu_{\text{fold}}(\Sigma), \tag{6.1}$$
where τ_{fold} is a permutation and μ_{fold} is a sequence of mutations involving $1,...,p$. We list $\tau_{\text{fold}},\mu_{\text{fold}}$ in Table 1(a–b). In our tables, we only list the restriction of τ_{fold} to the solid crossings in $1,...,p$. We would like to show that Σ' and $\phi^*\Sigma_{\beta'}$ are quasi-equivalent. To do so, we will eventually fold the seeds $\Sigma_{\tilde\beta}$ and $\Sigma_{\tilde\beta'}$, and then establish a chain of quasi-equivalences involving the folded seeds, Σ' and $\phi^*\Sigma_{\beta'}$.

As a first step, we fix a particular sequence S of braid moves (B3) between the lifts $\tilde\beta,\tilde\beta'$, and thus also fix $\tilde\phi$. By Theorems 1.1 and 4.2, there is a corresponding mutation sequence μ_{braid} and relabeling τ_{braid} such that
$$\tau_{\text{braid}}\circ\mu_{\text{braid}}(\dot\Sigma_{\tilde\beta})=\tilde\phi^*\dot\Sigma_{\tilde\beta'}, \tag{6.2}$$
where the seeds $\dot\Sigma_{\tilde\beta},\dot\Sigma_{\tilde\beta'}$ are the seeds denoted $\Sigma_{\tilde\beta},\Sigma_{\tilde\beta'}$ in Section 2. The sequence S of braid moves is chosen so that μ_{braid} and τ_{braid} are as in Table 1(c–d).

We have the relabeling maps $\lambda_\beta:[\tilde m]\to[m]$ and $\lambda'_\beta:[\tilde m]\to[m]$ as in Section 5.2, where β,β' (resp., $\tilde\beta,\tilde\beta'$) are on m (resp., on $\tilde m$) letters. By construction, we can extend the action of σ from I to $\dot J_{\tilde\beta}$ and $\dot J_{\tilde\beta'}$. Specifically, for each letter i_c in β, σ permutes the letters $\boldsymbol{i_c}$ in the corresponding consecutive subword $\lambda_\beta^{-1}(c)$ of $\tilde\beta$, and similarly for $\tilde\beta'$.

Proposition 6.3. *Let $C\subset J_\beta$ be the set of indices c such that none of $c'\in\boldsymbol{c}$ is used in μ_{braid}, and let $\boldsymbol{C}:=\lambda_\beta^{-1}(C)$. Let $C'\subset J_{\beta'}$ and $\boldsymbol{C'}\subset \dot J_{\tilde\beta'}$ be defined similarly. Then*

$$\iota^*(\dot\Sigma_{\tilde\beta}^{\setminus\boldsymbol{C}})\sim\Sigma_\beta^{\setminus C}\qquad \text{and}\qquad \iota^*(\dot\Sigma_{\tilde\beta'}^{\setminus\boldsymbol{C'}})\sim\Sigma_{\beta'}^{\setminus C'}.$$

Proof. We focus on the first quasi-equivalence. By Lemma 6.2, it suffices to show that for each $e\in J_\beta\setminus C$ and $e'\in\lambda_\beta^{-1}(e)$, we have $\iota^*(\dot x_{e'})=x_e$. Let us fix such e,e'. Choose also $c\in[0,p]$, $k\in I$, and $k'\in\boldsymbol{k}$. It is enough to show the statement

$$\operatorname{ord}_{V_e}\Delta_{c,k}=\operatorname{ord}_{V'_{e'}}\Delta'_{c,k'},\tag{6.3}$$

where $V'_{e'}\subset\mathcal{Y}'_\beta$ is the Deodhar hypersurface corresponding to $\dot x_{e'}\in\mathbb{C}[\mathring{R}_{\tilde\beta}]\cong\mathbb{C}[\mathring{R}'_\beta]$ and $\Delta'_{c,k'}$ was defined in Section 5.3.

We observe that the hollow crossings in δ,δ' (and thus in $\tilde\delta,\tilde\delta'$) have a very special form: one of δ,δ' has hollow crossings in positions $[r+1,p]$, while the other one has hollow crossings in positions $[r,p-1]$, for some r; cf. Table 1(a–b). In this case, computing $\operatorname{ord}_{V_e}\Delta_{c,k}$ is straightforward. First, suppose that $e\le r$. Then all crossings in $[p]$ to the left of e are solid. It follows from Propositions 2.20 and 2.21 and Corollary 2.14 that for $c\in[0,p]$ and $k\in I$, we have $\operatorname{ord}_{V_e}\Delta_{c,k}=1$ if $(c,k)\in\{(e-1,i_e),(e-2,i_e)\}$ and $\operatorname{ord}_{V_e}\Delta_{c,k}=0$ otherwise. Applying the same argument to compute $\operatorname{ord}_{V'_{e'}}\Delta'_{c,k'}$, we obtain (6.3). It remains to consider the case $e=p$ when the hollow crossings are in positions $[r,p-1]$. The crossings $r-1$ and $r-2$ are solid, so Proposition 2.21 implies that $\operatorname{ord}_{V_e}\Delta_{c,k}=0$ for $k=i_{r-1}$, $c<r-1$ or $k=i_{r-2}$, $c<r-2$. Here $\{i_{r-1},i_{r-2}\}=\{i,j\}$. For $c=p-1$, we have $\operatorname{ord}_{V_e}\Delta_{c,k}=\langle\omega_k,\alpha_{i_p}^\vee\rangle$ by Propositions 2.20 and 2.21. Thus, for $c\in[r,p-1]$, Lemma 2.8 implies that $\operatorname{ord}_{V_e}\Delta_{c,k}=\langle\omega_k,s_{i_{c+1}}\cdots s_{i_{p-1}}\alpha_{i_p}^\vee\rangle$. We have thus determined the values $\operatorname{ord}_{V_e}\Delta_{c,k}$ for all $(c,k)\in[0,p]\times\{i,j\}$ except for $(c,k)=(r-1,i_{r-2})$. By Corollary 2.14, we have $\operatorname{ord}_{V_e}\Delta_{r-1,i_{r-2}}=\operatorname{ord}_{V_e}\Delta_{r,i_{r-2}}$. It is clear that we have $\operatorname{ord}_{V_e}\Delta_{c,k}=0$ for $k\in I\setminus\{i,j\}$. Computing $\operatorname{ord}_{V'_{e'}}\Delta'_{c,k'}$ via a similar argument, we obtain (6.3). □

For the remainder of the section, let $C,\boldsymbol{C},C',\boldsymbol{C'}$ be as in Proposition 6.3.

The sequence μ_{braid} is ill-adapted to folding, so we find another mutation sequence μ_{lift} relating $\dot\Sigma_{\tilde\beta}$ and a relabeling τ_{lift} of $\tilde\phi^*\dot\Sigma_{\tilde\beta'}$. Explicitly, μ_{lift} is a sequence of orbit-mutations lifting the sequence μ_{fold} from Table 1(a–b), and τ_{lift} is given in Table 1(e–f). Part (1) of the next result generalizes [FG06, Theorem 3.5], which concerns the "all solid" case.

Proposition 6.4. *Let μ_{lift} and τ_{lift} be as listed in Table 1(e–f). Then*

1. *$\tau_{\mathrm{braid}}\circ\mu_{\mathrm{braid}}(\dot\Sigma_{\tilde\beta})=\tau_{\mathrm{lift}}\circ\mu_{\mathrm{lift}}(\dot\Sigma_{\tilde\beta})$.*
2. *μ_{lift} is a sequence of quasi-admissible mutations of $\dot\Sigma_{\tilde\beta}^{\setminus\boldsymbol{C}}$.*

We delay the proof of Proposition 6.4 to the end of the section. Proposition 5.8 and part (2) of Proposition 6.4 together imply the following result.

Corollary 6.5. *We have $\iota^*(\mu_{\mathrm{lift}}\dot\Sigma_{\tilde\beta}^{\setminus\boldsymbol{C}})\sim\mu_{\mathrm{fold}}(\iota^*\dot\Sigma_{\tilde\beta}^{\setminus\boldsymbol{C}})$.*

Proof of (Q) *for long braid moves.* We have a string of quasi-equivalences:

$$\iota^*(\mu_{\mathrm{lift}}\dot\Sigma_{\tilde\beta}^{\setminus\boldsymbol{C}})\sim\mu_{\mathrm{fold}}(\iota^*\dot\Sigma_{\tilde\beta}^{\setminus\boldsymbol{C}})\sim\mu_{\mathrm{fold}}\Sigma_\beta^{\setminus C},$$

where the first quasi-equivalence is Corollary 6.5 and the second follows from Proposition 6.3 and Lemma 3.9. On the other hand,

$$\iota^*(\mu_{\mathrm{lift}}\dot\Sigma_{\tilde\beta}^{\setminus\boldsymbol{C}})=\iota^*(\tau_{\mathrm{lift}}^{-1}\circ\tilde\phi^*\dot\Sigma_{\tilde\beta'}^{\setminus\boldsymbol{C'}})=\pi_{\mathrm{fold}}^{-1}\circ\iota^*(\tilde\phi^*\dot\Sigma_{\tilde\beta'}^{\setminus\boldsymbol{C'}})=\tau_{\mathrm{fold}}^{-1}\circ\phi^*(\iota^*\dot\Sigma_{\tilde\beta'}^{\setminus\boldsymbol{C'}})\sim\tau_{\mathrm{fold}}^{-1}\circ\phi^*\Sigma_{\beta'}^{\setminus C'},$$

where the first equality holds by Proposition 6.4 and (6.2), the second holds by direct computation (cf. Table 1(c–f)), the third holds by Lemma 6.1, and the final quasi-equivalence follows from Proposition 6.3 and the fact that ϕ^* preserves quasi-equivalence.

Summarizing, we have that $\mu_{\mathrm{fold}}\Sigma_\beta^{\setminus C}$ is quasi-equivalent to (a relabeling of) $\phi^*\Sigma_{\beta'}^{\setminus C'}$. Notice that the cluster variables of $\mu_{\mathrm{fold}}\Sigma_\beta^{\setminus C}$, resp., $\phi^*\Sigma_{\beta'}^{\setminus C'}$ are equal to the cluster variables of $\mu_{\mathrm{fold}}\Sigma_\beta$, resp.,

$\phi^*\Sigma_{\beta'}$. By assumption $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety, so Proposition 3.4 implies the cluster variables of $\mu_{\mathrm{fold}}\Sigma_\beta$ are irreducible elements of $\mathbb{C}[\mathring{R}_\beta]$. On the other hand, by Corollary 2.24, the cluster variables in $\phi^*\Sigma_{\beta'}$ are irreducible elements of $\mathbb{C}[\mathring{R}_\beta]$. Thus, the cluster variables in $\mu_{\mathrm{fold}}\Sigma_\beta$ and $\phi^*\Sigma_{\beta'}$ can differ only by units and $\mu_{\mathrm{fold}}\Sigma_\beta$ is quasi-equivalent to (a relabeling of) $\phi^*\Sigma_{\beta'}$. □

Proof of Proposition 6.4. Recall that $\tilde\beta=\tilde\beta_1\tilde\delta\tilde\beta_2$ and $\tilde\beta'=\tilde\beta_1\tilde\delta'\tilde\beta_2$, and that we index the crossings of $\tilde\delta$ by $1,\ldots,\tilde p$. Let $J:=J_\beta\setminus C$ be the set of indices which are mutated in μ_{fold}.

We show part (1). By [GSV08, Theorem 4], a seed in $\mathcal{A}(\dot\Sigma_{\tilde\beta})$ is uniquely determined by its cluster, so we need only check (1) at the level of cluster variables. This is easy to check for the cluster variables $\{x_c : c\in C\}$ which are not touched by either mutation sequence.

Let $\dot\Sigma_{\tilde\beta}=(\dot{\mathbf{x}},\dot Q)$, and let $\dot Q_{\mathrm{res}}$ be the induced subquiver of $\dot Q$ on $\dot J:=\lambda_{\tilde\beta}^{-1}(J)$. Let $\dot Q^{\mathrm{fr}}_{\mathrm{res}}$ be the *framing* of $\dot Q_{\mathrm{res}}$; the extended exchange matrix of $\dot Q^{\mathrm{fr}}_{\mathrm{res}}$ is thus of size $2|\dot J|\times|\dot J|$ and the bottom $|\dot J|\times|\dot J|$ submatrix is the identity. We denote by $\dot\Sigma_{\mathrm{res}}$ the seed $(\dot{\mathbf{y}},\dot Q^{\mathrm{fr}}_{\mathrm{res}})$ for some cluster $\dot{\mathbf{y}}$. By [FZ07, Theorem 3.7], to show (1), it suffices to check that

$$\tau_{\mathrm{braid}}\circ\mu_{\mathrm{braid}}(\dot\Sigma_{\mathrm{res}})=\tau_{\mathrm{lift}}\circ\mu_{\mathrm{lift}}(\dot\Sigma_{\mathrm{res}}). \tag{6.4}$$

The relevant cluster variables in $\mu_{\mathrm{braid}}(\dot\Sigma_{\tilde\beta})$ and $\mu_{\mathrm{fold}}(\dot\Sigma)$ can then be obtained from those in (6.4) by specialization determined by $\dot Q$.

To check (6.4), recall the description of the orders of vanishing of the cluster variables in $\dot J$ from the proof of Proposition 6.3. This description only depends on which crossings in $\tilde\delta,\tilde\delta'$ are hollow, which in turn is determined by which crossings in $\tilde\beta_2$ are hollow. This implies that to compute $\dot Q_{\mathrm{res}}$, we may assume $\tilde\beta_2$ is a type A_3 braid word (in the B_2/C_2 case) or a type D_4 braid word (in the G_2 case) consisting entirely of hollow crossings. Applying the algorithm from Section 7 (to the simply-laced braids $\tilde\beta,\tilde\beta'$; cf. Remark 7.3), we get that $\dot Q_{\mathrm{res}}$ is as displayed in Table 2. Equation (6.4) may then be verified by computer.

Part (2) is also established by direct computation in $\dot Q_{\mathrm{res}}$. □

This completes the proof of Theorem 4.2 for long braid moves.

6.3. **Finishing the proof.** We now have shown Theorem 4.2 for all braid moves. Theorem 1.1 for multiply-laced G follows by the argument in Section 4.9. Repeating the proof of Proposition 4.12, we have the following.

Proposition 6.6. *Suppose β,β' are related by a braid move* (B1)–(B4)*. The seeds $\Sigma_\beta,\Sigma^*_{\beta'}$ are mutation equivalent (up to relabeling cluster variables).*

Continuing Remark 4.11, we obtain the following.

Corollary 6.7. *The seeds Σ_β are really full rank for all β.*

6.4. **Curious Lefschetz property.** We now briefly discuss the consequences of our results for the cohomology $H^*(\mathring{R}_\beta,\mathbb{C})$; we refer to [LS22] and [GLSBS22, Section 10.1] for further details. Recall from Proposition 2.5 that $\mathring{R}_\beta$ is a smooth, affine, complex algebraic variety of dimension $d:=d(\beta)$. The mixed Hodge structure [Del71] of $H^k(\mathring{R}_\beta,\mathbb{C})$ endows the the cohomology group with a *Deligne splitting* $H^k(\mathring{R}_\beta,\mathbb{C})=\bigoplus_{p,q}H^{k,(p,q)}(\mathring{R}_\beta,\mathbb{C})$. Since $\mathring{R}_\beta$ is a sink-recurrent cluster variety of really full rank, it follows from [LS22, Theorem 8.3] that $H^*(\mathring{R}_\beta,\mathbb{C})$ is of *mixed Tate type*, i.e., $H^{k,(p,q)}(\mathring{R}_\beta,\mathbb{C})=0$ for $p\neq q$. The form ω_β, which coincides with the *GSV form* [GSV10] in view of (3.1), defines an element $[\omega_\beta]\in H^{2,(2,2)}(\mathring{R}_\beta,\mathbb{C})$. When the dimension d of $\mathring{R}_\beta$ is even, we say that $\mathring{R}_\beta$ satisfies the *curious Lefschetz property* with respect to $[\omega_\beta]$ if the cup product induces isomorphisms

$$[\omega_\beta]^{d-p}: H^{p+s,(p,p)}(\mathring{R}_\beta,\mathbb{C})\xrightarrow{\sim}H^{2d-p+s,(2d-p,2d-p)}(\mathring{R}_\beta,\mathbb{C})$$

B_2/C_2:

$\delta \to \delta'$	μ_{fold}	τ_{fold}
$ijij \to jiji$	$\mu_{(4,3,4)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
$ij\underline{i}j \to jij\underline{i}$	μ_4	$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \end{pmatrix}$
$iji\underline{j} \to ji\underline{j}i$	μ_3	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$
$ij\underline{ij} \to j\underline{ij}i$	id	$\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$
$i\underline{ji}j \to ji\underline{ji}$	id	$\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$
$\underline{iji}j \to j\underline{iji}$	id	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$
$i\underline{jij} \to \underline{jij}i$	id	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(a)

G_2:

$\delta \to \delta'$	μ_{fold}	τ_{fold}
$121212 \to 212121$	$\mu_{(6,3,4,6,5,6,3,4,5,6)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$
$12121\underline{2} \to 2121\underline{2}1$	$\mu_{(3,4,5,3,4,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 6 \end{pmatrix}$
$1212\underline{1}2 \to 21212\underline{1}$	$\mu_{(6,3,4,6,3,4)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$
$1212\underline{12} \to 212\underline{12}1$	$\mu_{(4,3,4)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 6 & 3 \end{pmatrix}$
$121\underline{21}2 \to 2121\underline{21}$	$\mu_{(6,3,6)}$	$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
$121\underline{212} \to 21\underline{212}1$	μ_3	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 6 \end{pmatrix}$
$12\underline{121}2 \to 212\underline{121}$	μ_6	$\begin{pmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \end{pmatrix}$
$12\underline{1212} \to 2\underline{1212}1$	id	$\begin{pmatrix} 1 & 2 \\ 6 & 1 \end{pmatrix}$
$1\underline{2121}2 \to 21\underline{2121}$	id	$\begin{pmatrix} 1 & 6 \\ 2 & 1 \end{pmatrix}$
$\underline{12121}2 \to 2\underline{12121}$	id	$\begin{pmatrix} 6 \\ 1 \end{pmatrix}$
$1\underline{21212} \to \underline{21212}1$	id	$\begin{pmatrix} 1 \\ 6 \end{pmatrix}$

(b)

$[p] \setminus \dot{J}_{\tilde{\beta}}$	μ_{braid}	τ_{braid}
$\emptyset$	$\mu_{(4,5,6,4)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$
$\{4,5\}$	μ_6	$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 3 & 1 & 2 & 4 \end{pmatrix}$
$\{6\}$	$\mu_{(4,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 6 & 5 \end{pmatrix}$
$\{4,5,6\}$	id	$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 1 \end{pmatrix}$
$\{3,4,5\}$	id	$\begin{pmatrix} 1 & 2 & 6 \\ 2 & 3 & 1 \end{pmatrix}$
$\{1,2,3,4,5\}$	id	$\begin{pmatrix} 6 \\ 1 \end{pmatrix}$
$\{3,4,5,6\}$	id	$\begin{pmatrix} 1 & 2 \\ 6 & 5 \end{pmatrix}$

(c)

$[p] \setminus \dot{J}_{\tilde{\beta}}$	μ_{braid}	τ_{braid}
$\emptyset$	$\mu_{(8,9,5,6,7,8,11,10,9,5,12,6,10,8,5,11)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 9 & 6 & 5 & 7 & 11 & 12 & 8 & 10 \end{pmatrix}$
$\{12\}$	$\mu_{(8,5,6,7,8,11,9,5,6,10,8,5,11)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 1 & 12 & 6 & 5 & 7 & 11 & 10 & 8 \end{pmatrix}$
$[9,11]$	$\mu_{(8,5,6,7,8,12,5,6,8,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 12 \\ 2 & 3 & 4 & 1 & 9 & 6 & 5 & 7 & 8 \end{pmatrix}$
$[9,12]$	$\mu_{(6,7,5,6,8,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 11 & 12 & 5 & 10 \end{pmatrix}$
$[8,11]$	$\mu_{(6,5,7,12,6,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 5 & 8 \end{pmatrix}$
$[8,12]$	$\mu_{(5,6,7)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 10 & 11 & 12 \end{pmatrix}$
$[5,11]$	μ_{12}	$\begin{pmatrix} 1 & 2 & 3 & 4 & 12 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}$
$[5,12]$	id	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 11 & 12 & 1 \end{pmatrix}$
$[4,11]$	id	$\begin{pmatrix} 1 & 2 & 3 & 12 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
$[2,12]$	id	$\begin{pmatrix} 1 \\ 12 \end{pmatrix}$
$[1,11]$	id	$\begin{pmatrix} 12 \\ 1 \end{pmatrix}$

(d)

$[p] \setminus \dot{J}_{\tilde{\beta}}$	μ_{lift}	τ_{lift}
$\emptyset$	$\mu_{(6,4,5,6)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 5 & 4 \end{pmatrix}$

(e)

$[p] \setminus \dot{J}_{\tilde{\beta}}$	μ_{lift}	τ_{lift}
$\emptyset$	$\mu_{(12,5,6,7,8,12,9,10,11,12,5,6,7,8,9,10,11,12)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 & 10 & 11 & 12 & 9 \end{pmatrix}$
$\{12\}$	$\mu_{(5,6,7,8,9,10,11,5,6,7,8,9,10,11)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 & 10 & 11 & 12 \end{pmatrix}$
$[9,11]$	$\mu_{(12,5,6,7,8,12,5,6,7,8)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 & 9 \end{pmatrix}$
$[9,12]$	$\mu_{(8,5,6,7,8)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 10 & 11 & 12 & 5 \end{pmatrix}$
$[8,11]$	$\mu_{(5,6,7,12,5,6,7)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \end{pmatrix}$

(f)

TABLE 1. The mutation sequences μ_{fold}, μ_{braid}, μ_{lift}, and the relabelings τ_{fold}, τ_{braid}, τ_{lift} used in Section 6. Hollow crossings are underlined. We denote $\mu_{(a_1,\dots,a_r)} := \mu_{a_1} \circ \cdots \circ \mu_{a_r}$. For the case B_2/C_2, we denote $\boldsymbol{i} = \{i', i''\}$, $\boldsymbol{j} = \{j'\}$, $\tilde{\delta} = i'i''j'i'i''j'$, $\tilde{\delta}' = j'i'i''j'i'i''$; for G_2, we denote $\mathbf{1} = \{\dot{1}, \dot{3}, \dot{4}\}$, $\mathbf{2} = \{\dot{2}\}$, $\tilde{\delta} = \dot{1}\dot{3}\dot{4}\dot{2}\dot{1}\dot{3}\dot{4}\dot{2}\dot{1}\dot{3}\dot{4}\dot{2}$, $\tilde{\delta}' = \dot{2}\dot{1}\dot{3}\dot{4}\dot{2}\dot{1}\dot{3}\dot{4}\dot{2}\dot{1}\dot{3}\dot{4}$. In (e) and (f), the cases where μ_{lift} and μ_{braid} coincide are omitted; we define $\tau_{\text{lift}} := \tau_{\text{braid}}$ in those cases.

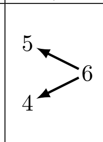
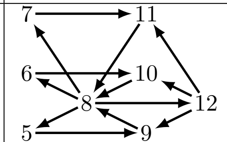
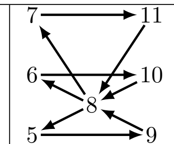
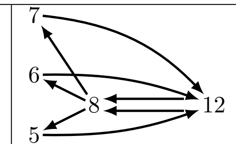
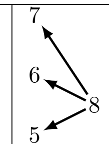
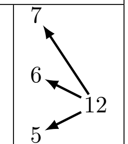
Type	B_2/C_2	G_2				
$\dot{Q}_{\mathrm{res}}$						
$[p]\setminus \dot{J}_{\tilde\beta}$	$\emptyset$	$\emptyset$	$\{12\}$	$[9,11]$	$[9,12]$	$[8,11]$

TABLE 2. The quivers $\dot{Q}_{\mathrm{res}}$ from Proposition 6.4 listed in the same order as in Table 1(e–f). In the cases where $\mu_{\mathrm{braid}} = \mu_{\mathrm{lift}}$, $\dot{Q}_{\mathrm{res}}$ has no arrows.

for all p and s. We also say that $\mathring{R}_\beta$ satisfies (the weaker) *curious Poincaré symmetry* if

$$\dim H^{p+s,(p,p)}(\mathring{R}_\beta,\mathbb{C}) = \dim H^{2d-p+s,(2d-p,2d-p)}(\mathring{R}_\beta,\mathbb{C}).$$

The following result is a consequence of the results of [LS22]; see [GLSBS22, Theorem 10.1].[6]

Theorem 6.8. *Even-dimensional double braid varieties $\mathring{R}_\beta$ satisfy the curious Lefschetz property with respect to $[\omega_\beta]$ and thus also curious Poincaré symmetry. Odd-dimensional $\mathring{R}_\beta$ satisfy curious Poincaré symmetry.*

Note that if $d = d(\beta)$ is odd, one can always add an isolated vertex to the seed Σ_β, and the corresponding variety $\mathring{R}_\beta \times \mathbb{C}^\times$ will satisfy the curious Lefschetz property.

7. Combinatorial algorithm

The exchange matrices for our seeds Σ_β are defined via Deodhar geometry. In particular, writing ω_β in terms of $\mathrm{dlog} x_e \wedge \mathrm{dlog} x_c$ requires knowing orders of vanishing along Deodhar hypersurfaces. In this section, we give an algorithm to compute the order of vanishing of Δ_c on V_e, which determines our cluster algebras.

Let $c \in [0,m]$. The function $h_c^\pm$ of Section 2.5 is an H-valued character on T_β. We may thus write $h_c^\pm = \prod_{e\in J_\beta} \gamma^\pm_{\beta,c,e}(x_e)$, where $\gamma^\pm_{\beta,c,e}$ are cocharacters of H satisfying $\gamma^-_{\beta,c,e} = u_c \cdot \gamma^+_{\beta,c,e}$. By (2.16) and (2.25), for all $c \in [0,m]$ and $k \in I$, we have

(7.1)

$$\Delta_{c,\pm k} = (h_c^\pm)^{\omega_k} = \prod_{e\in J_\beta} x_e^{\langle \omega_k, \gamma^\pm_{\beta,c,e}\rangle}, \quad \text{and so} \quad \mathrm{ord}^{(\beta)}_{V_e} \Delta_{c,\pm k} = \langle \omega_k, \gamma^\pm_{\beta,c,e}\rangle \quad \text{for all } e \in J_\beta,\ c \in [0,m],\ k \in I.$$

Lemma 7.1. *Let $c \in [0,m]$ and let $e \in J_\beta$ be such that $e \geq c$.*

1. *Suppose β' is obtained from β by removing the first $c-1$ letters. Then $\gamma^\pm_{\beta,c,e} = \gamma^\pm_{\beta',1,e-c+1}$.*
2. *Suppose β' is obtained from β by doing non-mutation moves* (B1) *and* (B4)*, only involving indices greater than c, and let e' be the image of e under the resulting identification of cluster seeds. Then we have $\gamma^\pm_{\beta,c,e} = \gamma^\pm_{\beta',c,e'}$.*
3. *Suppose β' is obtained from β by removing solid crossings greater than e. Then $\gamma^\pm_{\beta,c,e} = \gamma^\pm_{\beta',c,e}$.*

Proof. Part (1) follows from Lemma 2.25.

For part (2), as explained in Sections 4.2 and 4.6, applying non-mutation moves (B1) and (B4) results in a relabeling of the cluster variables. In particular, the pullbacks $V^*_{e'}$ and $\Delta^*_{c,\pm k}$ coincide with V_e and $\Delta_{c,\pm k}$, respectively. Thus, by (7.1), we get $\gamma^\pm_{\beta,c,e} = \gamma^\pm_{\beta',c,e'}$

We prove (3). Suppose β has a solid crossing $e' > e$ and assume that e' is the largest solid crossing in β. Similarly to Proposition 4.12, we may append a double braid word β_0 to the left of β and β' so

[6] While [LS22] work in the skew-symmetric setting, the curious Lefschetz theorem therein generalizes to the skew-symmetrizable case.

that $e\in J^{\rm mut}_\beta$; by part (1), this does not affect $\gamma^\pm_{\beta,c,e}$ or $\gamma^\pm_{\beta',c,e'}$. Applying non-mutation moves (B1) and (B4) if necessary, we may assume that $i_{e'}\in I$; by part (2), this preserves $\gamma^\pm_{\beta,c,e}$ and $\gamma^\pm_{\beta',c,e'}$. Since e' is the largest solid crossing, we may apply moves (B1)–(B3) involving hollow indices $e'+1,\dots,m$ until we have $i_{e'}=i_{e'+1}$; these moves do not affect $\gamma^\pm_{\beta,c,e}$ or $\gamma^\pm_{\beta',c,e'}$ by Remark 4.5. From now on, we assume $i_{e'}=i_{e'+1}\in I$.

Let β'' be obtained from β by removing the letter $i_{e'}$ from β. Let $W\subset \mathring{R}_\beta$ be the open subset obtained by removing the Deodhar hypersurface $V_{e'}$, if e' is mutable; otherwise, let $W:=\mathring{R}_\beta$. The projection $\pi:W\to \mathring{R}_{\beta''}$ given by forgetting the flags $(X_{e'},Y_{e'})$ is a fiber bundle with fiber $\mathbb{C}^\times$. We have $\pi^*(\Delta_c^{\beta''})=\Delta_c^\beta$ and π maps $V_e^\beta\cap W$ surjectively onto $V_e^{\beta''}$. (Here we use the superscript to refer to the braid variety on which Δ_c and V_e are defined.) Since both $V_e^\beta\subset\mathring{R}_\beta$ and $V_e^{\beta''}\subset\mathring{R}_{\beta''}$ are hypersurfaces, it follows that the order of vanishing of Δ_c^β on V_e^β is equal to that of $\Delta_c^{\beta''}$ on $V_e^{\beta''}$. Repeating this argument, we obtain (3). □

Using Lemma 7.1, we may assume that $c=1$, $e=m-1$ and $\beta=(-\mathbf{b}^{*\rm rev})\mathbf{a}kk$, where $\mathbf{a},\mathbf{b}$ are words in I, and $(-\mathbf{b}^{*\rm rev})$ is obtained by reversing $\mathbf{b}$ and applying the map $i\mapsto -i^*$ to each letter, and $k=i_e=i_m\in I$. Define

$$\gamma(\mathbf{a},k,\mathbf{b}):=\gamma^+_{(-\mathbf{b}^{*\rm rev})\mathbf{a}kk,1,m-1},$$

and let a and b denote the Demazure product of $\mathbf{a}$ and $\mathbf{b}$, respectively. Recall also from Section 2.3 that $*$ denotes Demazure product.

Proposition 7.2. *Suppose that Theorems 1.1 and 4.2 hold for G. Then the cocharacter $\gamma(\mathbf{a},k,\mathbf{b})$ satisfies, and is recursively defined by the following properties.*

I) *We have $\gamma(\mathbf{a},k,\mathbf{b})=0$ if $a=w_\circ$ or $b=w_\circ$.*
II) *The cocharacter $\gamma(\mathbf{a},k,\mathbf{b})=\gamma(a,k,b)$ only depends on the Demazure products a,b.*
III) *We have $\gamma(\mathbf{a},k,\emptyset)=a\alpha_k^\vee$ if $as_k>a$ and $\gamma(\mathbf{a},k,\emptyset)=0$ if $as_k<a$.*
IV) *Suppose that $\mathbf{a},\mathbf{b}$ are reduced. We let $\mathbf{a}'=i\mathbf{a}$ and $\mathbf{b}=\mathbf{b}'j$, where the Demazure products satisfy $a'=s_ia>a$ and $b=b's_j>b'$.*
   (1) *If $a'*s_k*b>a*s_k*b$, then $\gamma(\mathbf{a},k,\mathbf{b})=s_i\cdot\gamma(\mathbf{a}',k,\mathbf{b})$.*
   (2) *If $a*s_k*b>a*s_k*b'$, then $\gamma(\mathbf{a},k,\mathbf{b})=\gamma(\mathbf{a},k,\mathbf{b}')$.*
   (3) *If $w:=a'*s_k*b=a*s_k*b=a*s_k*b'$, write $\alpha^\vee=\alpha_i^\vee$ and $\beta^\vee=-w^{-1}\cdot\alpha_j^\vee$.*
      (3a) *Suppose that $\alpha^\vee\neq\beta^\vee$. Then $\gamma(\mathbf{a}',k,\mathbf{b})=\gamma(\mathbf{a},k,\mathbf{b}')+x\alpha^\vee+y\beta^\vee$ for $x,y\in\mathbb{Z}$, and we have $\gamma(\mathbf{a},k,\mathbf{b})=\gamma(\mathbf{a},k,\mathbf{b}')+y\beta^\vee$.*
      (3b) *Suppose that $\alpha^\vee=\alpha_i^\vee=\beta^\vee$. Then $\gamma(\mathbf{a},k,\mathbf{b})-\gamma(\mathbf{a}',k,\mathbf{b}')\in\mathbb{Z}\alpha_i^\vee$, and*

$$\langle\omega_i,\gamma(\mathbf{a},k,\mathbf{b})\rangle=-\langle\omega_i,\gamma(\mathbf{a}',k,\mathbf{b}')\rangle+\min\left(\langle\omega_i,\gamma(\mathbf{a}',k,\mathbf{b})\rangle+\langle\omega_i,\gamma(\mathbf{a},k,\mathbf{b}')\rangle,-\sum_{l\neq i}a_{il}\langle\omega_l,\gamma(\mathbf{a}',k,\mathbf{b}')\rangle\right).$$

Proof. We first argue that the stated properties determine $\gamma(\mathbf{a},k,\mathbf{b})$. By I) and III), we know $\gamma(\mathbf{a},k,\mathbf{b})$ when $a=w_\circ$ or $b=\mathrm{id}$. If $a\neq w_\circ$ and $b\neq\mathrm{id}$, property IV) allows us to express $\gamma(\mathbf{a},k,\mathbf{b})$ in terms of $\gamma(\mathbf{a}',k,\mathbf{b}),\gamma(\mathbf{a},k,\mathbf{b}'),\gamma(\mathbf{a}',k,\mathbf{b}')$ where $a'>a$ and $b'<b$. Thus, all values of $\gamma(\mathbf{a},k,\mathbf{b})$ are determined. We now prove I)–IV).

Suppose that $a=w_\circ$ or $b=w_\circ$. Then a generic point $(X_\bullet,Y_\bullet)$ in V_e satisfies $Y_0\xRightarrow{w_\circ}X_0$. It follows that Δ_1 does not vanish on V_e, establishing I).

We show II). It is clear that if $as_k<a$ then $\gamma(\mathbf{a},k,\mathbf{b})=0$. Suppose that $as_k>a$. We apply the moves $\beta\xrightarrow{\rm (B4)}(-\mathbf{b}^{*\rm rev})\mathbf{a}k(-k^*)\xrightarrow{\rm (B1)}(-\mathbf{b}^{*\rm rev})\mathbf{a}(-k^*)k\xrightarrow{\rm (B1)}\cdots\xrightarrow{\rm (B1)}(-\mathbf{b}^{*\rm rev})(-k^*)\mathbf{a}k$. Since $as_k>a$, these are non-mutation moves, and thus part (2) of Lemma 7.1 applies. We may now remove the solid crossings from $\mathbf{a}$ using part (3) of Lemma 7.1, and then reverse the procedure to put β back into its original form with solid crossings removed from $\mathbf{a}$.

We prove III). If $as_k < a$, we have already shown that $\gamma(\mathbf{a},k,\emptyset)=0$. Assume $as_k > a$. When $a=\mathrm{id}$, the result follows from Proposition 2.20. For $a \neq \mathrm{id}$, we apply induction, II), and the hollow case of Lemma 2.8.

We prove IV). For (1), adding the letter i in front of $(-\mathbf{b}^{*\mathrm{rev}})\mathbf{a}kk$ produces a new hollow crossing, and the claim follows from Lemmas 2.8 and 7.1. Similarly, for (2), the letter $-j^*$ is a hollow crossing in $(-\mathbf{b}^{*\mathrm{rev}})\mathbf{a}kk$. Case (3) holds if both i and $-j^*$ are solid crossings in the word $i(-j^*)(-(\mathbf{b}')^{*\mathrm{rev}})\mathbf{a}kk$. If swapping the order of i and $-j^*$ is a non-mutation move then we are in Case (3a), and the claim follows from Lemma 2.8 and the linear independence of $\alpha^\vee$ and $\beta^\vee$. If swapping the order of i and $-j^*$ is a mutation, then we are in Case (3b), and the claim follows from (4.12) and the assumption that Theorems 1.1 and 4.2 have been shown for G. $\square$

The algorithm has been implemented at [Gal23], where some examples can be found.

Remark 7.3. The logical dependencies in our proof are summarized as follows. In Section 4, we give a complete proof of Theorems 1.1 and 4.2 for the case when G is simply-laced. Thus, Proposition 7.2 applies in this case. The proof for the case when G is multiply-laced is given in Section 6; it depends on Proposition 7.2, but only invokes it for the simply-laced group $\dot{G}$.

The following result follows from our algorithm, but we have been unable to show it directly from Deodhar geometry.

Corollary 7.4. *Let $\iota : \mathring{R}_\beta \hookrightarrow \mathring{R}_{\tilde{\beta}}$ and $\lambda_\beta : [\tilde{m}] \to [m]$ be as in Sections 5.2 and 6. Then for each $e \in J_\beta$ and $e' \in \lambda_\beta^{-1}(e)$, we have $\iota^*(\dot{x}_{e'}) = x_e$.*

Proof. With notation as in the proof of Proposition 6.3, it suffices to show that $\mathrm{ord}_{V_e} \Delta_{c,k} = \mathrm{ord}_{V'_{e'}} \Delta'_{c,k'}$ for $c \le e$, $k \in I$, and $k' \in \boldsymbol{k}$. This follows from applying Proposition 7.2 to $\mathring{R}_\beta$ and $\mathring{R}_{\tilde{\beta}}$ separately. $\square$

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