BRAID VARIETY CLUSTER STRUCTURES, II: GENERAL TYPE

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Abstract. We show that braid varieties for any complex simple algebraic group G are cluster varieties. This includes open Richardson varieties inside the flag variety G/B.

1. Introduction

This is one of two papers concerned with the construction of cluster structures on braid varieties. In the present paper, we work in the setting of a general simple algebraic group G and construct cluster seeds using algebraic geometry. In the companion paper [GLSBS22], joint with David Speyer, we give an alternative proof in the special case $G=\mathrm{SL}_n$, using the combinatorics of plabic graphs and surfaces. The current work is logically independent of [GLSBS22], which, however, ultimately produces the same cluster structure in the case $G=\mathrm{SL}_n$.

Let G be a complex, simple, simply-connected algebraic group, $B_\pm$ opposing Borel subgroups, $U_\pm$ their unipotent radicals, $H:=B_+\cap B_-$ the torus, I the vertex set of the Dynkin diagram, W the Weyl group with simple generators $s_i, i\in I$, and denote by $\dot{w}$ the lift of $w\in W$ to G as in (2.1). Let $w_\circ \in W$ denote the longest element and $i\mapsto i^*$ the action of $w_\circ$ on I. Let $\alpha_i,\alpha_i^\vee,\omega_i$ for $i\in I$ denote the simple roots, simple coroots, fundamental weights, respectively, and let $A=(a_{ij})_{i,j\in I}$ be the Cartan matrix given by $a_{ij}:=\langle \alpha_i,\alpha_j^\vee\rangle$. Denote $d_i:=2/(\alpha_i,\alpha_i)$ so that $d_ia_{ij}=d_ja_{ji}$.

1.1. Double braid varieties.

A *double braid word* $\beta=i_1i_2\cdots i_m$ is a word in the alphabet $\pm I$. For $i\in I$, we set $(-i)^*:=-i^*$. For $i\in\pm I$, define

$$s_i^+:=\begin{cases} s_i, & \text{if } i>0,\\ \mathrm{id}, & \text{if } i<0,\end{cases}\qquad s_i^-:=\begin{cases} \mathrm{id}, & \text{if } i>0,\\ s_{-i}, & \text{if } i<0.\end{cases} \tag{1.1}$$

A *weighted flag* is an element $F=gU_+\in G/U_+$. Two weighted flags (F,F') are *weakly w-related* (resp., *strictly w-related*) if there exist $g\in G$ and $h\in H$ (resp., $g\in G$) such that $(gF,gF')=(U_+,h\dot{w}U_+)$ (resp., $(gF,gF')=(U_+,\dot{w}U_+)$). We write this as $F\xRightarrow{w}F'$ (resp., $F\xrightarrow{w}F'$).

Suppose that the Demazure product of β is $w_\circ$; see (2.4). We consider tuples $(X_\bullet,Y_\bullet)$ of weighted flags satisfying the relative position conditions

$$\begin{array}{ccccccc} X_0 & \xleftarrow{s^+_{i_1}} & X_1 & \xleftarrow{s^+_{i_2}} & \cdots & \xleftarrow{s^+_{i_m}} & X_m \\ \Uparrow{\scriptstyle w_\circ} & & & & & & \| \\ Y_0 & \xrightarrow[s^-_{i_1^*}]{} & Y_1 & \xrightarrow[s^-_{i_2^*}]{} & \cdots & \xrightarrow[s^-_{i_m^*}]{} & Y_m. \end{array} \tag{1.2}$$

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The group G acts on such tuples by acting on each individual weighted flag, and this action is free (Remark 2.15). The *double braid variety* $\mathring{R}_\beta$ is a complex, affine, irreducible variety defined as the quotient modulo the G-action of the configuration space of tuples $(X_\bullet, Y_\bullet)$ satisfying (1.2).

Double braid varieties include open Richardson varieties [Rie98, KLS14, Lec16], open positroid varieties [KLS13], double Bott-Samelson cells [SW21], the strata in [WY07], and the braid varieties of [Mel19, CGGS20]; see [GLSBS22] for further discussion. For each β, we construct a cluster seed Σ_β. Our main result settles conjectures of [Lec16, CGGS21] and generalizes work of [BFZ05, GL19, Ing19, SW21].

Theorem 1.1. *The coordinate ring of $\mathring{R}_\beta$ is isomorphic to the cluster algebra $\mathcal{A}_\beta = \mathcal{A}(\Sigma_\beta)$.*

It would be interesting to compare our construction and the cluster-categorical approach of [GLS06, BIRS09, Lec16, Mén22, CK22], as was done in type A in [SSB22].

Remark 1.2. At the final stages of completing our construction, we learned that a cluster structure for braid varieties was independently announced in a recent preprint [CGG$^+$22]. We thank the authors of [CGG$^+$22] for updating us on their progress. It would be interesting to understand the relation between our approach and their Legendrian-geometric viewpoint.

One application of Theorem 1.1 is that a *curious Lefschetz theorem* (see [HRV08, LS16], [GLSBS22, Theorem 10.1], and [GL20, Theorem 1.5]) holds for double braid varieties; see Theorem 6.8. In the case of open Richardson varieties, this implies that the doubly-graded extension group $\mathrm{Ext}_{\mathcal{O}}(M_w, M_v)$ of two Verma modules in Category $\mathcal{O}$ satisfies curious Lefschetz; cf. [GL20, Section 1.11].

1.2. **The seed.** We give an informal summary of our construction; see Section 2 for details and [GLSBS22, Gal23] for examples. We introduce an open dense algebraic torus $T_\beta \subset \mathring{R}_\beta$ called the *Deodhar torus*, so named for its relation to the Deodhar decomposition of Richardson varieties [Deo85, MR04]. It is defined by requiring the weighted flags X_c, Y_c to be weakly w_c-related, where $w_c \in W$ is maximal possible subject to (1.2) (for each $c = 0, 1, \ldots, m$). The complement $\mathring{R}_\beta \setminus T_\beta$ is a union of irreducible *mutable Deodhar hypersurfaces* $\{V_c \mid c \in J_\beta^{\mathrm{mut}}\}$. We define a partial compactification of $\mathring{R}_\beta$ so that the complement of T_β in it also includes *frozen Deodhar hypersurfaces* $\{V_c \mid c \in J_\beta^{\mathrm{fro}}\}$. We let $J_\beta := J_\beta^{\mathrm{fro}} \sqcup J_\beta^{\mathrm{mut}}$. The following definition, suggested by David Speyer, is key to our approach.

Proposition-Definition 1.3. For $c \in J_\beta$, define the *cluster variable* x_c to be the unique character of T_β that vanishes to order one on V_c and has neither a pole nor a zero on V_e for $e \in J_\beta \setminus \{c\}$. We denote the cluster by $\mathbf{x}_\beta = \{x_c\}_{c \in J_\beta}$.

We show that the cluster variables form a basis of the character lattice of T_β, and that they extend to regular functions on $\mathring{R}_\beta$. We consider a 2-form ω_β on $\mathring{R}_\beta$, defined in terms of certain generalized minors as a sum of local contributions for each letter of β. This is similar in spirit to [BFZ05, FG06, Ing19, SW21]. We introduce integers $\mathbf{d} = (d_i)_{i \in J_\beta}$ and expand ω_β in the basis of cluster variables:

$$\omega_\beta = \sum_{c,e \in J_\beta:\ c \le e} d_e \tilde{B}_{ce} \mathrm{dlog} x_c \wedge \mathrm{dlog} x_e = \sum_{c,e \in J_\beta:\ c \le e} d_c \tilde{B}_{ec} \mathrm{dlog} x_e \wedge \mathrm{dlog} x_c. \tag{1.3}$$

The coefficients $\tilde{B}_{ce}$ define a $J_\beta \times J_\beta^{\mathrm{mut}}$ integer matrix $\tilde{B} := (\tilde{B}_{ce})$. The principal $J_\beta^{\mathrm{mut}} \times J_\beta^{\mathrm{mut}}$ part of the matrix $\tilde{B}$ is skew-symmetrizable, with symmetrizer $\mathrm{diag}(d_c \mid c \in J_\beta^{\mathrm{mut}})$. Therefore $\Sigma_\beta := (\mathbf{x}_\beta, \tilde{B})$ is a seed of a cluster algebra $\mathcal{A}(\Sigma_\beta)$ of geometric type. The content of Theorem 1.1 is that $\mathcal{A}(\Sigma_\beta) = \mathbb{C}[\mathring{R}_\beta]$.

1.3. **Overview of the proof.** The proof that Σ_β provides a cluster structure for $\mathring{R}_\beta$ is obtained in two steps. First, we develop a notion of deletion-contraction for cluster seeds in Section 3.3. We apply this in the case when $\beta = ii\beta'$ starts with a repeated letter; the braids $i\beta'$ and β' correspond to deletion and contraction, respectively. Using deletion-contraction, we deduce that if Theorem 1.1 holds for $i\beta'$ and β' then it holds for β.

Second, in Section 4, we describe moves $\beta \sim \beta'$ on double braid words that induce natural isomorphisms $\mathring{R}_\beta \cong \mathring{R}_{\beta'}$. In Theorem 4.2, we show that the isomorphisms $\mathring{R}_\beta \cong \mathring{R}_{\beta'}$ give rise to a sequence of mutations connecting the corresponding seeds Σ_β and $\Sigma_{\beta'}$. Theorem 1.1 then follows by induction. In Section 4, we prove Theorem 4.2 in the simply-laced case (i.e., for G of type A, D, E); the seeds Σ_β and $\Sigma_{\beta'}$ either coincide or are related by a single mutation. The proof of Theorem 4.2 in the multiply-laced case is achieved via folding in Sections 5 and 6; the seeds Σ_β and $\Sigma_{\beta'}$ are related by a sequence of mutations. This generalizes a result of Fock and Goncharov [FG06, Theorem 3.5]. Finally, in Section 7, we give an algorithm, implemented in [Gal23], for computing our seeds using only root-system combinatorics.

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2. Deodhar geometry

We discuss the geometry of the double braid variety $\mathring{R}_\beta$ with the goal of defining a cluster seed on it. The ingredients of a cluster seed were outlined in Section 1.2. In Section 2.3, we construct a *Deodhar torus* $T_\beta \subset \mathring{R}_\beta$. In Section 2.7, we introduce a family $\mathbf{x}_\beta = \{x_c\}_{c \in J_\beta}$ of *cluster variables* and show that they are regular functions on $\mathring{R}_\beta$. Finally, in Section 2.8, we introduce a 2-form ω_β on T_β from which the $\tilde{B}$-matrix can be extracted via (1.3).

2.1. **Background.** For each $i \in I$, we fix a group homomorphism

$$\phi_i : \mathrm{SL}_2 \to G, \quad \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mapsto x_i(t), \quad \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mapsto y_i(t),$$

where $x_i(t), y_i(t)$ are the exponentiated Chevalley generators. The data $(H, B_+, B_-, x_i, y_i; i \in I)$ is a *pinning* of G; see [Lus94, Section 1.1].

Let Φ be the root system of G, with positive roots Φ^+ corresponding to B_+. Let $X^*(H) := \mathrm{Hom}(H, \mathbb{C}^\times)$ be the *character lattice* of H and $X_*(H) := \mathrm{Hom}(\mathbb{C}^\times, H)$ be the *cocharacter lattice* of H. Let $\{\alpha_i\}_{i \in I} \subset X^*(H)$ (resp., $\{\alpha_i^\vee\}_{i\in I} \subset X_*(H)$, $\{\omega_i\}_{i \in I} \subset X^*(H)$) be the simple roots (resp., simple coroots, fundamental weights) of Φ^+. We have a natural pairing $\langle \cdot, \cdot \rangle : X^*(H) \times X_*(H) \to \mathbb{Z}$ satisfying $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ and $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$, where $A = (a_{ij})_{i,j \in I}$ is the Cartan matrix of G.

Let the Weyl group W have simple generators $\{s_i\}_{i \in I}$, length function $\ell(\cdot)$, and identity $\mathrm{id} \in W$. For $i \in I$, we set

$$\dot{s}_i = \overline{s_i} := \phi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \dot{s}_i^{-1} = \overline{\overline{s_i}} := \phi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_l}$, where $l = \ell(w)$, we set

$$\dot{w} = \overline{w} := \overline{s_{i_1}} \cdot \overline{s_{i_2}} \cdots \overline{s_{i_l}}, \quad \overline{\overline{w}} := \overline{\overline{s_{i_1}}} \cdot \overline{\overline{s_{i_2}}} \cdots \overline{\overline{s_{i_l}}}. \tag{2.1}$$

The resulting product does not depend on the choice of the reduced expression. For $u \in W$ and $h \in H$, we set $u \cdot h := \dot{u} h \dot{u}^{-1} = \overline{u} h \overline{u}^{-1} = \overline{\overline{u}} h \overline{\overline{u}}^{-1}$. We also consider elements

$$z_i(t) := \phi_i \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} = x_i(t) \dot{s}_i, \quad \bar{z}_i(t) := \phi_i \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix} = x_i(-t) \dot{s}_i^{-1}. \tag{2.2}$$

For each $w \in W$, it is well known that the multiplication map gives rise to an isomorphism

$$(\dot{w}^{-1} U_+ \dot{w} \cap U_-) \times (\dot{w}^{-1} U_+ \dot{w} \cap U_+) \xrightarrow{\sim} \dot{w}^{-1} U_+ \dot{w}. \tag{2.3}$$

2.2. **Weighted flags.** Recall from Section 1.1 that a *weighted flag* is an element $F = gU_+ \in G/U_+$. Associated to a weighted flag F is the flag $\pi(F) = gB_+$, the image of F in G/B_+.

The following elementary facts can be found in e.g. [SW21, Appendix]; see also [GLSBS22, Section 6.2].

Lemma 2.1. *Let F, F', F'' be weighted flags. Suppose $v, w \in W$ with $\ell(vw) = \ell(v) + \ell(w)$.*

(1) $F \xrightarrow{\mathrm{id}} F'$ if and only if $F = F'$.

(2) If $F \xrightarrow{v} F' \xrightarrow{w} F''$, then $F \xrightarrow{vw} F''$.

(3) If $F \xrightarrow{vw} F''$, then there exists a unique F' such that $F \xrightarrow{v} F' \xrightarrow{w} F''$. If $F \xRightarrow{vw} F''$ then there exist unique F'_1, F'_2 such that $F \xrightarrow{v} F'_1 \xRightarrow{w} F''$ and $F \xRightarrow{v} F'_2 \xrightarrow{w} F''$.

Lemma 2.2. *Suppose $F \xrightarrow{s_i} F'$ and say $F = gU_+$. Then there exists a unique $t \in \mathbb{C}$ such that $F' = gz_i(t)U_+$. Similarly, if $F' = g'U_+$, there exists a unique $t' \in \mathbb{C}$ such that $F = g'\bar{z}_i(t')U_+$. The maps $(g,F') \mapsto t$ and $(g',F) \mapsto t'$ are regular on the appropriate subvarieties of $G \times G/U_+$.*

Lemma 2.3. *Suppose $F \xRightarrow{v} gU_+ \xrightarrow{s_i} gz_i(t)U_+$ and $F \xRightarrow{w} gz_i(t)U_+$. If $vs_i > v$, then $w = vs_i$ for all $t \in \mathbb{C}$. If $vs_i < v$, then there exists $t^* \in \mathbb{C}$ such that $w = vs_i$ for $t = t^*$ and $w = v$ for $t \in \mathbb{C} \setminus \{t^*\}$.*

2.3. **Subexpressions and the Deodhar torus.** Fix $\beta = i_1 i_2 \dots i_m \in (\pm I)^m$. We write $[m] := \{1,2,\dots,m\}$ and $[0,m] := \{0,1,\dots,m\}$. Recall the notation s_i^+ and s_i^- from (1.1).

Given two elements $u, v \in W$ with $u \le v$ in the Bruhat order, we write $\max(u,v) := v$ and $\min(u,v) := u$. We also define $u * s_i := \max(u, us_i)$. The *Demazure product* of β is defined by

$$\pi(\beta) := s^-_{i_m} * s^-_{i_{m-1}} * \cdots * s^-_{i_1} * s^+_{i_1} * s^+_{i_2} * \cdots * s^+_{i_m} \quad \in W. \tag{2.4}$$

From now on, we assume that $\pi(\beta) = w_\circ$.

Definition 2.4. A $w_\circ$*-subexpression* of β is a sequence $\mathbf{u} = (u_0, u_1, \dots, u_m) \in W^{m+1}$ such that $u_0 = \mathrm{id}$, $u_m = w_\circ$, and such that for each $c \in [m]$, we have either $u_{c-1} = u_c$ or $u_{c-1} = s^-_{i_c} u_c s^+_{i_c}$. Since $\pi(\beta) = w_\circ$ there exists a unique "rightmost" subexpression, called the *positive distinguished subexpression*. It is given by $u_m := w_\circ$ and $u_{c-1} := \min(u_c, s^-_{i_c} u_c s^+_{i_c})$ for all $c = m, m-1, \dots, 1$.

We also define $w_c := w_\circ u_c$ for $c \in [0,m]$ and $\mathbf{w} = w_\circ \mathbf{u} := (w_0, w_1, \dots, w_m)$. Note that $w_0 = w_\circ$. We set $J_\beta := \{c \in [m] \mid u_c = u_{c-1}\}$. We refer to the indices in J_β as *solid crossings* and to the indices in $[m] \setminus J_\beta$ as *hollow crossings*. We denote $d(\beta) := m - \ell(w_\circ) = |J_\beta|$.

Definition 2.5. The *Deodhar torus* $T_\beta \subset \mathring{R}_\beta$ consists of all tuples $(X_\bullet, Y_\bullet)$ satisfying

$$X_c \xLeftarrow{w_c} Y_c \quad \text{for } c \in [0,m]. \tag{2.5}$$

2.4. **Torus-valued functions.** Given $(X_\bullet, Y_\bullet) \in \mathring{R}_\beta$, let $Z_c := Y_c^{-1} X_c \in U_+ \backslash G / U_+$. Abusing notation, we use double cosets $Z_c \in U_+ \backslash G / U_+$ interchangeably with their representatives in G. For $(X_\bullet, Y_\bullet) \in T_\beta$, Z_c belongs to the Bruhat cell $\mathring{\mathcal{X}}_{w_c} := B_+ w_c B_+ = U_+ w_c H U_+$ of G. There exist unique elements $h_c^+, h_c^- \in H$ satisfying

$$Z_c \in U_+ \dot{w}_c h_c^+ U_+, \quad Z_c \in U_+ \overline{w_\circ} h_c^- \overline{\overline{u_c}} U_+, \quad \text{thus,} \quad h_c^- = u_c \cdot h_c^+. \tag{2.6}$$

The third statement follows from the first two since $\dot{w}_c = \overline{w_\circ} \cdot \overline{\overline{u_c}}$ and $u_c \cdot h_c^+ = \overline{\overline{u_c}} h_c^+ \overline{\overline{u_c}}^{-1}$. The map $(X_\bullet, Y_\bullet) \mapsto h_c^\pm$ is a rational H-valued function on $\mathring{R}_\beta$, regular on T_β.

Lemma 2.6. *There exist rational functions $(t_c)_{c \in J_\beta}$ on $\mathring{R}_\beta$ such that for $c \in [m]$,*

$$h^+_{c-1} = \begin{cases} s_{i_c} \cdot h_c^+, & \text{if } c \text{ is hollow, } i_c \in I; \\ \alpha^\vee_{i_c}(t_c) h_c^+, & \text{if } c \text{ is solid, } i_c \in I; \end{cases} \quad h^-_{c-1} = \begin{cases} s_{|i_c|} \cdot h_c^-, & \text{if } c \text{ is hollow, } i_c \in -I; \\ \alpha^\vee_{|i_c|}(t_c) h_c^-, & \text{if } c \text{ is solid, } i_c \in -I. \end{cases} \tag{2.7}$$

Proof. Fix $(X_\bullet, Y_\bullet) \in T_\beta$. For $c \in J_\beta$, define t_c to be such that if $Z_c = \dot{w}_c h_c^+ = \overline{w_\circ} h_c^- \overline{\overline{u_c}}$ then $Z_{c-1} = Z_c z_{i_c}(t_c)$ if $i_c \in I$ and $Z_{c-1} = \bar{z}_{|i_c|^*}(t_c)^{-1} Z_c$ if $i_c \in -I$; see Lemma 2.2. The following identities in G can be checked inside SL_2:

$$x_i(t)\dot{s}_i = y_i(1/t)\alpha_i^\vee(t)x_i(-1/t) \quad \text{and} \quad \dot{s}_i x_i(t) = x_i(-1/t)\alpha_i^\vee(1/t)y_i(1/t). \tag{2.8}$$

Suppose that $i_c \in I$. We have $Z_{c-1} = \dot{w}_c h_c^+ x_{i_c}(t_c) \dot{s}_{i_c}$. If c is hollow then $\dot{w}_c h_c^+ x_{i_c}(t_c) \in U_+ \dot{w}_c h_c^+$, and thus $Z_{c-1} \in U_+ \dot{w}_c \dot{s}_{i_c} (s_{i_c} \cdot h_c^+)$. This implies that $h^+_{c-1} = s_{i_c} \cdot h_c^+$. If c is solid then we use the first identity in (2.8) to write $Z_{c-1} = \dot{w}_c h_c^+ y_{i_c}(1/t_c) \alpha^\vee_{i_c}(t_c) x_{i_c}(-1/t_c)$. Since $\dot{w}_c h_c^+ y_{i_c}(t) \in U_+ \dot{w}_c h_c^+$, we see that $Z_{c-1} \in U_+ \dot{w}_c h_c^+ \alpha^\vee_{i_c}(t_c) U_+$, which implies that $h^+_{c-1} = h_c^+ \alpha^\vee_{i_c}(t_c)$.

The case when $i_c \in -I$ is handled similarly. When c is solid, we use the second identity in (2.8) together with $\alpha^\vee_{|i_c|^*}(1/t_c)\overline{w_\circ} = \overline{w_\circ}\alpha^\vee_{|i_c|}(t_c)$; see [FZ99, Equation (1.2)]. □

Corollary 2.7. *Suppose c is hollow. If $i_c\in I$ then $h^-_{c-1}=h^-_c$, and if $i_c\in -I$ then $h^+_{c-1}=h^+_c$.*

Corollary 2.8. *The Deodhar torus $T_\beta\subset \mathring{R}_\beta$ is isomorphic to an algebraic torus of dimension $d(\beta)$, and the functions $(t_c)_{c\in J_\beta}$ form a basis of the character lattice of T_β.*

Proof. For $c\in J_\beta$, the function t_c is regular on T_β by Lemma 2.2. With notation as in the proof of Lemma 2.6, we have $Z_c=\dot{w}_c h^+_c=\overline{\overline{w_\circ}} h^-_c \overline{\overline{u_c}}$ and $Z_{c-1}=Z_c z_{i_c}(t_c)$ if $i_c\in I$ and $Z_{c-1}=\bar{z}_{|i_c|^*}(t_c)^{-1}Z_c$ if $i_c\in -I$. It follows that $t_c\neq 0$ if and only if $Z_{c-1}\in\mathring{\mathcal{X}}_{w_{c-1}}=\mathring{\mathcal{X}}_{w_c}$ (thus, $t^*=0$ in the notation of Lemma 2.3). The fact that the map $T_\beta\to(\mathbb{C}^\times)^{J_\beta}$, $(X_\bullet,Y_\bullet)\mapsto(t_c)_{c\in J_\beta}$ is an isomorphism follows from (2.7). □

2.5. **Grid and chamber minors.** Recall that $A=(a_{ij})_{i,j\in I}$, $a_{ij}:=\langle\alpha_i,\alpha_j^\vee\rangle$ is the Cartan matrix, and that $d_ia_{ij}=d_ja_{ji}$. For $i,j\in\pm I$, we define $a_{ij}=0$ if i,j have different signs, and $a_{ij}=a_{(-i)(-j)}$ otherwise. Also set $d_{-i}:=d_i$ for $i\in I$.

Following [FZ99, Definition 1.4], for $i\in I$ and $v,w\in W$, we have a *generalized minor* $\Delta_{v\omega_i,w\omega_i}:G\to\mathbb{C}$. It is a regular function satisfying

$$\Delta_{\omega_i,\omega_i}(y_-xy_+)=\Delta_{\omega_i,\omega_i}(x)\quad\text{for all }(y_-,x,y_+)\in U_-\times G\times U_+;\tag{2.9}$$

$$\Delta_{v\omega_i,w\omega_i}(x)=\Delta_{\omega_i,\omega_i}\left(\overline{\overline{v^{-1}}}x\overline{w}\right)=\Delta_{\omega_i,\omega_i}\left(\dot{v}^{-1}x\dot{w}\right);\tag{2.10}$$

see [FZ99, Section 1.4]. For $h\in H$, we have $\Delta_{\omega_i,\omega_i}(h)=h^{\omega_i}$. For $x\in G$, we also have [FZ99, Equation (2.14)]

$$\Delta_{\omega_k,\omega_k}(xh)=\Delta_{\omega_k,\omega_k}(hx)=h^{\omega_k}\Delta_{\omega_k,\omega_k}(x).\tag{2.11}$$

Definition 2.9. For $c\in[0,m]$ and $k\in I$, we define the *grid minors*

$$\Delta_{c,k}(X_\bullet,Y_\bullet)=\Delta_{w_c\omega_k,\omega_k}(Z_c)\quad\text{and}\quad\Delta_{c,-k}(X_\bullet,Y_\bullet)=\Delta_{w_\circ\omega_k,u_c^{-1}\omega_k}(Z_c).\tag{2.12}$$

The *chamber minors* are defined as $\Delta_c:=\Delta_{c-1,i_c}$, for $c\in[m]$.

Recall that we also view $Z_c\in U_+\backslash G/U_+$ as an element of G, and that for $(X_\bullet,Y_\bullet)\in T_\beta$, we have $Z_c\in\mathring{\mathcal{X}}_{w_c}$ for all $c\in[0,m]$. Thus, for $(X_\bullet,Y_\bullet)\in\mathring{R}_\beta$, Z_c belongs to the closure $\mathcal{X}_{w_c}$ of $\mathring{\mathcal{X}}_{w_c}$ inside G. The next result implies that the grid minors restricted to $\mathring{\mathcal{X}}_{w_c}$ (and therefore to $\mathcal{X}_{w_c}$) are invariant under the $U_+\times U_+$-action and thus are well-defined on $\mathring{R}_\beta$.

Lemma 2.10. *For $c\in[0,m]$, $k\in I$, and $(X_\bullet,Y_\bullet)\in T_\beta$, we have*

$$\Delta_{c,k}(X_\bullet,Y_\bullet)=(h^+_c)^{\omega_k}\quad\textit{and}\quad\Delta_{c,-k}(X_\bullet,Y_\bullet)=(h^-_c)^{\omega_k}.\tag{2.13}$$

Proof. Write $Z_c=y'_+\dot{w}_ch^+_cy''_+$ for $y'_+,y''_+\in U_+$. We have

$$\Delta_{c,k}(X_\bullet,Y_\bullet)=\Delta_{w_c\omega_k,\omega_k}(Z_c)=\Delta_{\omega_k,\omega_k}(\overline{\overline{w_c^{-1}}}y'_+\dot{w}_ch^+_cy''_+)=\Delta_{\omega_k,\omega_k}(\dot{w}_c^{-1}y'_+\dot{w}_ch^+_c).$$

Factorizing $\dot{w}_c^{-1}y'_+\dot{w}_c=b_-b_+$ for $(b_-,b_+)\in U_-\times U_+$ using (2.3), and using $(h^+_c)^{-1}b_+h^+_c\in U_+$, we get the first identity in (2.13). The proof of the second identity is similar. □

Combining Lemma 2.10 with (2.7), we get the following.

Corollary 2.11. *If c is solid and $k\in\pm I$ has the same sign as i_c then*

$$\Delta_{c-1,k}=\begin{cases}t_c\Delta_{c,k}, & \text{if } k=i_c;\\ \Delta_{c,k}, & \text{if } k\neq i_c.\end{cases}\tag{2.14}$$

Proposition 2.12.

(1) *The grid minors are characters of T_β.*
(2) *The solid chamber minors $(\Delta_c)_{c\in J_\beta}$ form a basis of the character lattice of T_β.*

Proof. We relate the parameters $(t_c)_{c\in J_\beta}$ from Corollary 2.8 to the grid and chamber minors by combining (2.7) with (2.13). Suppose that $i_c\in I$ and let $k\in I$. If $k\neq i_c$ then $\Delta_{c-1,k}=\Delta_{c,k}$ by Corollary 2.11. If c is hollow then

$$\Delta_{c-1,i_c}=(h^+_{c-1})^{\omega_{i_c}}=(s_{i_c}\cdot h^+_c)^{\omega_{i_c}}=(h^+_c)^{s_{i_c}\omega_{i_c}}.$$

Expand $s_{i_c}\omega_{i_c}=\omega_{i_c}-\alpha_{i_c}$ in the basis of fundamental weights using $\alpha_{i_c}=\sum_{j\in I}a_{i_cj}\omega_j$. This gives

$$\Delta_{c-1,i}\Delta_{c,i}\prod_{j\neq i}\Delta_{c,j}^{a_{ij}}=1, \quad \text{if } c \text{ is hollow and } i:=i_c, \tag{2.15}$$

which holds for $i_c\in\pm I$. If c is solid, Corollary 2.11 yields $\Delta_{c-1,i_c}=t_c\Delta_{c,i_c}$. Thus

(i) For each solid $c\in J_\beta$, $t_c=\Delta_{c-1,i_c}/\Delta_{c,i_c}$ is a ratio of two grid minors.
(ii) For each solid $c\in J_\beta$, the grid minors $(\Delta_{c-1,j})_{j\in\pm I}$ are Laurent monomials in the grid minors $(\Delta_{c,k})_{k\in\pm I}$ and the chamber minor $\Delta_c=\Delta_{c-1,i_c}$.
(iii) For each hollow $c\in[m]\setminus J_\beta$, the grid minors $(\Delta_{c-1,j})_{j\in\pm I}$ are Laurent monomials in the grid minors $(\Delta_{c,k})_{k\in\pm I}$.
(iv) Every grid minor $\Delta_{c,j}$ is a Laurent monomial in the solid chamber minors $(\Delta_e)_{e\in J_\beta}$.

We have already shown (i)–(iii), and (iv) follows from (ii)–(iii). This implies the result. □

2.6. **Almost positive sequences and Deodhar hypersurfaces.** Recall that we have $u_{c-1}=\min(u_c,s_{i_c}^-u_cs_{i_c}^+)$ and $w_{c-1}=\max(w_c,s_{i_c^*}^-w_cs_{i_c}^+)$ for all $c\in[m]$.

Definition 2.13. Let $e\in J_\beta$. Let $u_m^{\langle e\rangle}:=w_\circ$, and for $c=m,m-1,\dots,1$, define

$$u_{c-1}^{\langle e\rangle}:=\begin{cases}\max(u_e^{\langle e\rangle},s_{i_e}^-u_e^{\langle e\rangle}s_{i_e}^+), & \text{if } c=e,\\ \min(u_c^{\langle e\rangle},s_{i_c}^-u_c^{\langle e\rangle}s_{i_c}^+), & \text{otherwise.}\end{cases}$$

We call the sequence $\mathbf{u}^{\langle e\rangle}:=(u_0^{\langle e\rangle},\dots,u_m^{\langle e\rangle})$ the $\langle e\rangle$*-almost positive sequence.* We set $w_c^{\langle e\rangle}:=w_\circ u_c^{\langle e\rangle}$ for all $c\in[0,m]$, and write $\mathbf{w}^{\langle e\rangle}=w_\circ\mathbf{u}^{\langle e\rangle}:=(w_\circ u_0^{\langle e\rangle},\dots,w_\circ u_m^{\langle e\rangle})$.

Definition 2.14. We say that $e\in J_\beta$ is *mutable* if $u_0^{\langle e\rangle}=\mathrm{id}$. Otherwise, e is *frozen*. We let J_β^{mut} (resp., J_β^{fro}) denote the set of mutable (resp., frozen) indices.

We define $\mathring{\mathcal{Y}}_\beta:=\{(X_\bullet,Y_\bullet) \text{ satisfying (1.2)}\}\subset(G/U_+)^{[0,m]}\times(G/U_+)^{[0,m]}$. Removing the condition $X_0\overset{w_\circ}{\Longleftarrow}Y_0$ yields a partial compactification $\mathcal{Y}_\beta$ of $\mathring{\mathcal{Y}}_\beta$.

Remark 2.15. We defined $\mathring{R}_\beta$ as the quotient of $\mathring{\mathcal{Y}}_\beta$ by the diagonal G-action. This action is free since the G-action on pairs (X_0,Y_0) which are weakly $w_\circ$-related is free; cf. [GLSBS22, Proposition 6.8]. The G-action on $\mathcal{Y}_\beta$ is no longer free.

Remark 2.16. We can parameterize the variety $\mathcal{Y}_\beta$ as follows. We choose an arbitrary weighted flag $X_m=Y_m$, and then for $c=m,m-1,\dots,1$, assuming $(X_c,Y_c)=(g_cU_+,g'_cU_+)$, we set (cf. Lemma 2.2)

$$(X_{c-1},Y_{c-1}):=\begin{cases}(g_cz_{i_c}(t'_c)U_+,g'_cU_+), & \text{if } i_c>0,\\ (g_cU_+,g'_c\bar{z}_{|i_c|^*}(t'_c)U_+), & \text{if } i_c<0,\end{cases} \tag{2.16}$$

for arbitrary parameters $\mathbf{t}':=(t'_1,t'_2,\dots,t'_m)\in\mathbb{C}^m$. For $(X_\bullet,Y_\bullet)$ to be a point in $\mathring{\mathcal{Y}}_{u,\beta}$, we require further that $X_0\overset{w_\circ}{\Longleftarrow}Y_0$, which is an extra open condition on the parameters $(\mathbf{t}',X_m=Y_m)$.

Lemma 2.17. *For each $c\in[0,m]$ and $k\in\pm I$, the grid minor $\Delta_{c,k}$ gives rise to a G-invariant regular function on $\mathcal{Y}_\beta$ and $\mathring{\mathcal{Y}}_\beta$. These regular functions are compatible with the quotient map $\mathring{\mathcal{Y}}_{u,\beta}\to\mathring{R}_{u,\beta}$ and the inclusion map $\mathring{\mathcal{Y}}_{u,\beta}\hookrightarrow\mathcal{Y}_{u,\beta}$.*

Definition 2.18. Let $e\in J_\beta$. Define the *Deodhar hypersurface* $\tilde{V}_e\subset\mathcal{Y}_\beta$ to be the closure of the locus satisfying

$$X_c\overset{w_c^{\langle e\rangle}}{\Longleftarrow}Y_c \quad \text{for all } c\in[0,m]. \tag{2.17}$$

It follows that an index $e\in J_\beta$ is mutable (resp., frozen) if and only if $\tilde{V}_e\subset\mathring{\mathcal{Y}}_\beta$ (resp., $\tilde{V}_e\cap\mathring{\mathcal{Y}}_\beta=\emptyset$). If e is mutable, then the G-action on $\tilde{V}_e$ is free and $V_e=\tilde{V}_e/G$ is a subvariety of $\mathring{R}_\beta$. We let $\tilde{T}_\beta$ denote the preimage of T_β under the quotient map $\mathring{\mathcal{Y}}_\beta\to\mathring{R}_\beta$.

Proposition 2.19. *The closed subset $\mathcal{Y}_\beta\setminus\tilde{T}_\beta$ is a union of the Deodhar hypersurfaces $\tilde{V}_e$ for $e\in J_\beta$. Each $\tilde{V}_e$ is irreducible and has codimension one in $\mathcal{Y}_\beta$.*

Proof. We first show that the closed subset $\mathcal{Y}_\beta\setminus\tilde{T}_\beta$ is a union of the Deodhar hypersurfaces $\tilde{V}_e$ for $e\in J_\beta$. Applying (2.16), we see that the conditions (2.17) cut out an iterated fiber bundle over $G/U_+\times G/U_+$, where each fiber is either $\mathbb{C}$, $\mathbb{C}^\times$, or (in the case of the crossing $c=e$) a point. It follows that $\tilde{V}_e$ is an irreducible subvariety of $\mathcal{Y}_\beta$ of codimension one.

Let $(X_\bullet,Y_\bullet)\in\mathcal{Y}_\beta\setminus\tilde{T}_\beta$. Then (2.5) must fail. Let $e\in[0,m]$ be the largest index such that we do not have $X_e\overset{w_e}{\Longleftarrow}Y_e$. Let $\tilde{V}_e'\subset\mathcal{Y}_\beta$ be the locus of points where (2.5) holds for $c>e$ and fails for $c=e$. By Lemma 2.3 and Definition 2.13, an open dense subset $\tilde{V}_e''$ of $\tilde{V}_e'$ consists of points satisfying (2.17). Thus, $\tilde{V}_e'\subset\tilde{V}_e$, and therefore $(X_\bullet,Y_\bullet)\in\tilde{V}_e$. We have shown that $\mathcal{Y}_\beta\setminus\tilde{T}_\beta=\bigcup_{e\in J_\beta}\tilde{V}_e$. □

2.7. **Cluster variables.** The irreducible components of $\mathcal{Y}_\beta\setminus\tilde{T}_\beta$ are the Deodhar hypersurfaces $\tilde{V}_e$, $e\in J_\beta$. For a grid minor $\Delta_{c,k}$ and $e\in J_\beta$, we denote by $\mathrm{ord}_{V_e}\Delta_{c,k}\in\mathbb{Z}$ the order of vanishing of $\Delta_{c,k}$ on the hypersurface $\tilde{V}_e$; cf. Lemma 2.17. Since $\Delta_{c,k}$ is regular on $\mathcal{Y}_\beta$, we have that $\mathrm{ord}_{V_e}\Delta_{c,k}\geq 0$.

We have the following basic unitriangularity property.

Proposition 2.20. *For $e\in J_\beta$ solid, $c\in[0,m]$, and $k\in\pm I$, we have*

$$\mathrm{ord}_{V_e}\Delta_{c,k}=\begin{cases}0, & \text{if } e\leq c;\\ 1, & \text{if } (c,k)=(e-1,i_e),\text{ i.e., } \Delta_{c,k}=\Delta_e.\end{cases}\tag{2.18}$$

Proof. Suppose that $e\leq c$. Let $(X_\bullet,Y_\bullet)$ be a generic point in $\tilde{V}_e$. Then we have $X_c\overset{w_c}{\Longleftarrow}Y_c$, and thus $\Delta_{c,k}(X_\bullet,Y_\bullet)\neq 0$. It follows that $\mathrm{ord}_{V_e}\Delta_{c,k}=0$ when $e\leq c$.

Suppose now that $(c,k)=(e-1,i_e)$. We may parameterize $\mathcal{Y}_\beta$ using parameters $(\mathbf{t}',X_m=Y_m)$ as in (2.16). The union $\tilde{T}_\beta\cup\tilde{V}_e$ contains an open dense subset of tuples $(X_\bullet,Y_\bullet)$ satisfying $X_{e'}\overset{w_{e'}}{\Longleftarrow}Y_{e'}$ for all $e'\geq e$. Choose one such tuple $(X_\bullet,Y_\bullet)$ and note that $X_e\overset{w_e}{\Longleftarrow}Y_e$.

Assume that $i_e\in I$. Thus, we have $Z_e\in U_+\dot{w}_e hU_+$ for some $h\in H$. The proof of Lemma 2.10 implies that $\dot{w}_e^{-1}Z_e\in U_-hU_+$. Let us write $\dot{w}_e^{-1}Z_e=y_-hy_+$ for $(y_-,y_+)\in U_-\times U_+$. Since $e\in J_\beta$ is solid, we have $w_{e-1}=w_e$. Setting $t:=t_e'$, we find

$$\Delta_{e-1,k}(X_\bullet,Y_\bullet)=\Delta_{w_{e-1}\omega_k,\omega_k}(Z_ez_k(t))=\Delta_{\omega_k,\omega_k}(\dot{w}_e^{-1}Z_ez_k(t))=\Delta_{\omega_k,\omega_k}(y_-hy_+z_k(t)).$$

Recall that $z_k(t)=x_k(t)\dot{s}_k$. Let $\Psi:=\Phi^+\setminus\{\alpha_k\}$ and let $U_+(\Psi):=(\dot{s}_k^{-1}U_+\dot{s}_k)\cap U_+$ be the corresponding *root subgroup*; see [Hum75, Theorem 26.3]. We have $x_k(-t)U_+(\Psi)x_k(t)\subset U_+(\Psi)$ by [Hum75, Lemma 32.5]. Next, we have $\dot{s}_k^{-1}U_+(\Psi)\dot{s}_k\subset U_+(\Psi)$, since s_k permutes Ψ. Using (2.3), we can factorize $y_+=x_k(p)y_+'$ for some $p\in\mathbb{C}$ and $y_+'\in U_+(\Psi)$. We therefore get $y_+'x_k(t)\dot{s}_k\in x_k(t)\dot{s}_kU_+$. Using (2.9), we get

$$\Delta_{e-1,k}(X_\bullet,Y_\bullet)=\Delta_{\omega_k,\omega_k}(y_-hx_k(p)y_+'x_k(t)\dot{s}_k)=\Delta_{\omega_k,\omega_k}(hx_k(p+t)\dot{s}_k).$$

It is clear that if $p+t=0$ then $\Delta_{e-1,k}(X_\bullet,Y_\bullet)=0$. If $p+t\neq 0$, applying the first identity in (2.8) to $x_k(p+t)\dot{s}_k$ and using (2.11), we find

$$\Delta_{e-1,k}(X_\bullet,Y_\bullet)=(p+t)\Delta_{\omega_k,\omega_k}(h).\tag{2.19}$$

Thus (2.19) holds regardless of whether $p+t=0$, and $p+t=0$ if and only if the condition $X_{e-1}\overset{w_{e-1}}{\Longleftarrow}Y_{e-1}$ fails, i.e., $(X_\bullet,Y_\bullet)\in\tilde{V}_e$. By (2.19), since $\Delta_{\omega_k,\omega_k}(h)\neq 0$, we have $p+t=0$ if and only if $\Delta_{e-1,k}(X_\bullet,Y_\bullet)=0$. Since $\Delta_{e-1,k}$ is of degree 1 in t, we find that $\mathrm{ord}_{V_e}\Delta_{e-1,k}\leq 1$. On the other hand, we have shown that $\Delta_{e-1,k}$ vanishes on $\tilde{V}_e$, so $\mathrm{ord}_{V_e}\Delta_{e-1,k}\geq 1$. □

The integers $\mathrm{ord}_{V_e}\Delta_{c,k}$ are nonnegative. Our next result shows that whether $\mathrm{ord}_{V_e}\Delta_{c,k}$ is zero or positive is determined by the almost positive subexpression $\mathbf{u}^{\langle e\rangle}$. The stronger result that $\mathrm{ord}_{V_e}\Delta_{c,k}\in\{0,1\}$ holds when $G=\mathrm{SL}_n$ [GLSBS22, Proposition 7.10]. The precise value of $\mathrm{ord}_{V_e}\Delta_{c,k}$ is for G of arbitrary type is given in Section 7.

Proposition 2.21. *For all $c\in[0,m]$, $e\in J_\beta$, and $k\in I$, we have*

$$\mathrm{ord}_{V_e}\Delta_{c,k}=0 \iff u_c\omega_k=u_c^{\langle e\rangle}\omega_k \quad \text{and} \quad \mathrm{ord}_{V_e}\Delta_{c,-k}=0 \iff u_c^{-1}\omega_k=(u_c^{\langle e\rangle})^{-1}\omega_k.$$

Proof. Let $\leq$ denote the *Bruhat order* on W. Comparing Definitions 2.4 and 2.13, we see that $u_c\leq u_c^{\langle e\rangle}$ and $w_c\geq w_c^{\langle e\rangle}$ for all $c\in[0,m]$. Thus $w_c\omega_k\geq w_c^{\langle e\rangle}\omega_k$ for all $c\in[0,m]$ and $k\in I$.

Let $\tilde{V}_e'\subset\tilde{V}_e$ be the open dense subset of points satisfying (2.17). For $(X_\bullet,Y_\bullet)\in\tilde{V}_e'$, we have $Z_c\in\mathring{\mathcal{X}}_{w_c^{\langle e\rangle}}\subset\mathcal{X}_{w_c}$ for all $c\in[0,m]$, because $w_c^{\langle e\rangle}\leq w_c$. Recall that $\Delta_{c,k}(X_\bullet,Y_\bullet)=\Delta_{w_c\omega_k,\omega_k}(Z_c)$. It is well known that the function $\Delta_{w_c\omega_k,\omega_k}(Z_c)$ when restricted to $Z_c\in\mathcal{X}_{w_c}$ does not vanish at $Z_c\in\mathring{\mathcal{X}}_{w_c^{\langle e\rangle}}$ if and only if $w_c^{\langle e\rangle}\omega_k=w_c\omega_k$; this can be shown by an analog of the proof of [FZ99, Proposition 2.4]. Similarly, we consider $\Delta_{c,-k}(X_\bullet,Y_\bullet)=\Delta_{w_\circ\omega_k,u_c^{-1}\omega_k}(Z_c)$ and observe that this function does not vanish at $Z_c\in\mathring{\mathcal{X}}_{w_\circ u_c^{\langle e\rangle}}\subset\mathcal{X}_{w_\circ u_c}$ if and only if $u_c^{-1}\omega_k=(u_c^{\langle e\rangle})^{-1}\omega_k$. □

Corollary 2.22. *The $J_\beta\times J_\beta$ matrix $M_\beta=(\mathrm{ord}_{V_e}\Delta_c)_{c,e\in J_\beta}$ is upper unitriangular.*

Inverting the matrix M_β, we arrive at the following definition, which is crucial for our analysis; cf. Proposition-Definition 1.3. Recall from Proposition 2.12 that a character on T_β is just a Laurent monomial in the solid chamber minors $\{\Delta_c\}_{c\in J_\beta}$.

Definition 2.23. For $c\in J_\beta$, the *cluster variable* x_c is the unique character of T_β satisfying

$$\mathrm{ord}_{V_e}x_c=\begin{cases}1, & \text{if } c=e,\\ 0, & \text{otherwise,}\end{cases}\qquad \text{for all } e\in J_\beta.$$

We denote the *cluster* by $\mathbf{x}_\beta=\{x_c\}_{c\in J_\beta}$.

Corollary 2.24.

(1) For $c\in J_\beta$, the character $x_c\in X^*(T_\beta)$ extends to a regular function on $\mathring{R}_\beta$, $\mathring{\mathcal{Y}}_\beta$, $\mathcal{Y}_\beta$.
(2) For $c\in J_\beta^{\mathrm{fro}}$, the frozen cluster variable x_c is invertible in $\mathbb{C}[\mathring{R}_\beta]$ and $\mathbb{C}[\mathring{\mathcal{Y}}_\beta]$.
(3) The cluster variables in $\mathbf{x}_\beta$ are irreducible and algebraically independent.

Proof. For $c\in J_\beta$, x_c is a rational function on $\mathcal{Y}_\beta$ which does not have a pole on $\tilde{T}_\beta$ or on $\tilde{V}_e$ for all $e\in J_\beta$. This implies that x_c is regular. By Proposition 2.19, x_c is irreducible. Since $\mathbf{x}_\beta=\{x_c\}_{c\in J_\beta}$ is a basis of the character lattice of T_β, we see that the cluster variables are algebraically independent. Finally, for $c\in J_\beta^{\mathrm{fro}}$, the function $1/x_c$ is regular on $\mathcal{Y}_\beta\setminus\tilde{V}_c$, and as we mentioned after Definition 2.18, we have $\tilde{V}_c\cap\mathring{\mathcal{Y}}_\beta=\emptyset$. □

Let us denote $\mathrm{ord}(\beta;e,c,k):=\mathrm{ord}_{V_e}\Delta_{c,k}$. Recall from Proposition 2.20 that $\mathrm{ord}(\beta;e,c,k)$ can only be nonzero when $e>c$. The next result follows from the parametrization (2.16) and will be used later in the proof.

Lemma 2.25. *The integer $\mathrm{ord}_{V_e}\Delta_{c,k}$ only depends on $i_{c+1},\dots,i_m$. That is, suppose that $\beta=i_1i_2\cdots i_m$ and $\beta'=i_1'i_2'\cdots i_{m'}'$ are two double braid words in the alphabet $\pm I$ such that for some $c\in[m]$ and $c'\in[m']$ (with $m-c=m'-c'$), we have $i_{c+1}\cdots i_m=i_{c'+1}'\cdots i_{m'}'$. Then we have*

$$\mathrm{ord}(\beta;e,c,k)=\mathrm{ord}(\beta';e',c',k)$$

for all $k\in\pm I$, $e>c$, and $e'>c'$ such that $m-e=m'-e'$.

2.8. **A two-form on the braid variety.** We start by introducing a family of 1-forms on T_β. For $i,j\in\pm I$, recall that $a_{ij}=0$ if i,j have different signs, and $a_{ij}=a_{(-i)(-j)}$ otherwise, and that $d_i:=d_{|i|}$. For each $c\in[0,m]$ and $i\in\pm I$, we set

$$L_{c,i}:=\frac{1}{2}\sum_{k\in\pm I}a_{ik}\,\mathrm{dlog}\Delta_{c,k}. \tag{2.20}$$

Consider the following 2-forms on T_β:

$$\omega_{\beta,c} := \mathrm{sign}(i) d_i L_{c-1,i} \wedge L_{c,i} \quad \text{for } c \in [m],\ i := i_c, \qquad \text{and} \qquad \omega_\beta := \sum_{c\in[m]} \omega_{\beta,c}. \tag{2.21}$$

Since T_β is open dense in $\mathring{R}_\beta$, the forms ω_β and $\omega_{\beta,c}$ are rational 2-forms on $\mathring{R}_\beta$. Though it is not apparent from the above formula, it will follow from our main result that ω_β is actually a regular 2-form on $\mathring{R}_\beta$.

Recall that (2.15) holds for $i_c \in \pm I$. Taking dlog of both sides of (2.15), we get

$$L_{c-1,i_c} + L_{c,i_c} = 0 \quad \text{if } c \text{ is hollow.} \tag{2.22}$$

Thus, $\omega_{\beta,c} = 0$ for all $c \in [m] \setminus J_\beta$, which implies the following result.

Corollary 2.26. *We have* $\omega_\beta = \sum_{c\in J_\beta} \omega_{\beta,c}$.

3. Cluster algebras

Cluster algebras were discovered by Fomin and Zelevinsky [FZ02]. We consider skew-symmetrizable cluster algebras of geometric type, relying on formalism similar to [FG09].

3.1. Background.

Definition 3.1. A rank n and dimension $n+m$ *(abstract) seed* is a quadruple $\Sigma = (T, \mathbf{x}, \mathbf{d}, \omega)$, where

1. T is a complex algebraic torus of dimension $n+m$,
2. $\mathbf{x} = (x_1, \dots, x_{n+m})$ is an ordered basis of $X^*(T)$, where $x_1, \dots, x_n$ (resp., $x_{n+1}, \dots, x_{n+m}$) are mutable (resp., frozen) variables,
3. $\mathbf{d} = (d_1, \dots, d_{n+m})$ is a collection of positive integers,
4. ω is a 2-form on T of the form

$$\omega = \sum_{i\le j} d_j \tilde{B}_{ij} \mathrm{dlog} x_i \wedge \mathrm{dlog} x_j = \sum_{i\le j} d_i \tilde{B}_{ji} \mathrm{dlog} x_j \wedge \mathrm{dlog} x_i, \tag{3.1}$$

where $\tilde{B}_{ii} = 0$ for $i \in [n+m]$ and $\tilde{B}_{ij} \in \mathbb{Q}$ for all $i,j \in [n+m]$.

We say that Σ is *integral* if $\tilde{B}_{ij} \in \mathbb{Z}$ for all $i \in [n+m]$ and $j \in [n]$.

The matrix $\tilde{B} = (\tilde{B}_{ij})_{(i,j)\in[n+m]\times[n]}$ is the usual $(n+m)\times n$ *extended exchange matrix* in the theory of cluster algebras. We have $d_j \tilde{B}_{ij} = -d_i \tilde{B}_{ji}$ for $i,j \in [n+m]$; in particular, the top $n \times n$ *principal part* B of $\tilde{B}$ is skew-symmetrizable.

Definition 3.2. Let $\Sigma = (T, \mathbf{x}, \mathbf{d}, \omega)$ be a seed and k a mutable index. We say that Σ is *integral at* k if $\tilde{B}_{jk} \in \mathbb{Z}$ for all $j \in [n+m]$. In this case, we define

$$x_k' := \frac{\prod_{\tilde{B}_{jk}>0} x_j^{\tilde{B}_{jk}} + \prod_{\tilde{B}_{jk}<0} x_j^{-\tilde{B}_{jk}}}{x_k}. \tag{3.2}$$

The *mutation* of Σ in the direction k is the seed $\mu_k(\Sigma) = (T', \mathbf{x}', \mathbf{d}, \omega')$ where T' is the algebraic torus with basis of characters $\mathbf{x}' = (x_1, \dots, x_k', \dots, x_{n+m})$ and the 2-form ω' on T' is the pullback of ω via the natural rational map $T' \to T$; see [FG09].

For the rest of this subsection, we assume that all seeds are integral. Following [LS16, Section 5.1], a seed Σ is *really full rank* if the columns of $\tilde{B}$ span $\mathbb{Z}^n$ over $\mathbb{Z}$. We will prove this for the seeds Σ_β from Section 2 in Corollary 6.7.

Let X be an irreducible complex algebraic variety of dimension $n+m$. A *seed on* X is an abstract seed $\Sigma = (T, \mathbf{x}, \mathbf{d}, \omega)$ together with an identification $T \subset X$ of T with an open dense subset of X. The inclusion $T \hookrightarrow X$ induces an identification of the field $\mathbb{C}(X)$ of rational functions on X with the field of rational functions $\mathbb{C}(\mathbf{x}) := \mathbb{C}(x_1, \dots, x_{n+m})$ in the initial cluster variables $\mathbf{x}$. In practice, we abuse notation and write $\mathbf{x}$ for a tuple of elements in $\mathbb{C}(X)$.

The *cluster algebra* $\mathcal{A}(\Sigma)$ is the subring of $\mathbb{C}(\mathbf{x})$ generated by all cluster variables together with inverses of frozen variables. We let $\mathcal{X}(\Sigma) := \mathrm{Spec}(\mathcal{A}(\Sigma))$ denote the *cluster variety*. We say that (X,Σ) *is a cluster variety* if X is an affine variety and the coordinate ring $\mathbb{C}[\mathcal{X}]$ is identified with $\mathcal{A}(\Sigma)$ under the identification $\mathbb{C}(X)\cong\mathbb{C}(\mathbf{x})$.

We will need the following property of cluster variables.

Proposition 3.3 ([GLS13, Theorem 3.1]). *Each cluster variable is an irreducible element of* $\mathcal{A}(\Sigma)$.

Definition 3.4. Let Σ be an abstract seed of rank n and dimension $n+m$, and let $I\subset[n]$. The *freezing of* Σ *at* I, denoted $\Sigma^{\backslash I}$, is the seed obtained from Σ by declaring the variables $\{x_c\}_{c\in I}$ to be frozen. It is a seed of rank $n-|I|$ and dimension $n+m$. For $k\in[n]$, we denote $\Sigma^{\backslash k}:=\Sigma^{\backslash\{k\}}$.

To a seed Σ we associate a directed graph $\tilde{\Gamma}(\Sigma)$ with vertex set $[n+m]$ and an arrow $i\to j$ whenever $\tilde{B}_{ij}>0$. We let $\Gamma:=\Gamma(\Sigma)$ be the *mutable part* of $\tilde{\Gamma}(\Sigma)$, i.e., the induced subgraph of $\tilde{\Gamma}(\Sigma)$ with vertex set $[n]$. We say that a mutable index $s\in[n]$ is a *sink* if it has no outgoing arrows in Γ. Let $N_s^{\mathrm{in}}(\Gamma)$ denote the set of vertices of Γ having an arrow to s, and denote $\widehat{N}_s^{\mathrm{in}}(\Gamma):=N_s^{\mathrm{in}}(\Gamma)\cup\{s\}$. The following definition is a variation of locally-acyclic seeds [Mul13] and Louise seeds [LS16]; see also [GL22, Section 5.4 and Remark 5.14].

Definition 3.5. The class of *sink-recurrent seeds* is defined recursively as follows.

- Any seed Σ such that $\Gamma(\Sigma)$ has no arrows is sink-recurrent.
- Any seed that is mutation equivalent to a sink-recurrent seed is sink-recurrent.
- Suppose that Σ is a seed with a sink $s\in[n]$ such that the seeds $\Sigma^{\backslash s}$ and $\Sigma^{\backslash\widehat{N}_s^{\mathrm{in}}(\Gamma(\Sigma))}$ are sink-recurrent. Then Σ is sink-recurrent.

The *upper cluster algebra* [BFZ05] $\mathcal{U}(\Sigma)\subset\mathbb{C}[\mathbf{x}^{\pm1}]$ is the intersection $\mathbb{C}[\mathbf{x}^{\pm1}]\cap\bigcap_{k\in[n]}\mathbb{C}[\mu_k(\mathbf{x})^{\pm1}]$.

Proposition 3.6. *Suppose that* Σ *is a sink-recurrent seed. Then* $\mathcal{A}(\Sigma)=\mathcal{U}(\Sigma)$.

Proof. It follows from induction and [Mul13, Lemma 5.3] that sink-recurrent seeds are locally acyclic in the sense of [Mul13, Mul14]. By [Mul14, Theorem 2], we have $\mathcal{A}(\Sigma)=\mathcal{U}(\Sigma)$. □

3.2. Quasi-equivalence.

Definition 3.7. Two seeds $\Sigma=(T,\mathbf{x},\mathbf{d},\omega)$ and $\tilde{\Sigma}=(\tilde{T},\tilde{\mathbf{x}},\tilde{\mathbf{d}},\tilde{\omega})$ of rank n and dimension $n+m$ are *quasi-equivalent*, denoted $\Sigma\sim\tilde{\Sigma}$, if the following conditions are satisfied:

- $T=\tilde{T}$, $\mathbf{d}=\tilde{\mathbf{d}}$, $\omega=\tilde{\omega}$;
- the sublattice of $X^*(T)$ spanned by the frozen variables $x_{n+1},\dots,x_{n+m}$ coincides with the sublattice spanned by $\tilde{x}_{n+1},\dots,\tilde{x}_{n+m}$;
- for each $k\in[n]$, we have $\tilde{x}_k=x_kM_k$, where M_k is a Laurent monomial in $x_{n+1},\dots,x_{n+m}$.

It is easy to see that if Σ is integral and $\Sigma\sim\tilde{\Sigma}$ then $\tilde{\Sigma}$ is integral. The following is also straightforward to check.

Lemma 3.8. *If* Σ *and* $\tilde{\Sigma}$ *are quasi-equivalent seeds then* $\mu_k(\Sigma)\sim\mu_k(\tilde{\Sigma})$ *for all mutable* k.

Corollary 3.9. *Suppose two seeds* $\Sigma,\tilde{\Sigma}$ *are quasi-equivalent. Then they define the same cluster algebra* $\mathcal{A}(\Sigma)=\mathcal{A}(\Sigma')\subset\mathbb{C}(T)$.

Proof. It follows from Lemma 3.8 that each cluster variable in $\mathcal{A}(\Sigma)$ differs from the corresponding cluster variable in $\mathcal{A}(\tilde{\Sigma})$ by a factor equal to a Laurent monomial in the frozen variables. □

3.3. **Deletion-contraction.** We give an inductive criterion for a pair (X,Σ) to be a sink-recurrent cluster variety. In Section 4.8, we will apply this criterion to the seeds Σ_β constructed in Section 2. See [GL22, Corollary 5.15] for a different application suggesting our nomenclature.

Assumption 3.10. Throughout this section, we let $\Sigma=(T,\mathbf{x},\mathbf{d},\omega)$ be an abstract seed of rank n and dimension $n+m$. Let $\Gamma:=\Gamma(\Sigma)$. We assume that Σ is sink-recurrent, with sink s in Γ such that Σ is integral at s. Further, we assume that there exists a frozen index f such that $\tilde{B}_{fs}=\pm 1$ and $\tilde{B}_{fj}=0$ for $j\in[n]\setminus\{s\}$. Suppose that the exchange relation for x_s in Σ is given by $x_sx'_s=M_1+x_fM_2$ for some monomials M_1,M_2 in $\{x_j\}_{j\in[n+m]\setminus\{s,f\}}$.

Definition 3.11. Suppose that s has $q:=|N_s^{\mathrm{in}}(\Gamma)|$ mutable neighbors. The *contraction* $\Sigma^{/s}=(T^{/s},\mathbf{x}^{/s},\mathbf{d}^{/s},\omega^{/s})$ is a seed of rank $n-q-1$ and dimension $n+m-2$ defined as follows.

(1) $\mathbf{x}^{/s}$ is obtained from $\mathbf{x}$ by omitting x_s and x_f and declaring the indices in $N_s^{\mathrm{in}}(\Gamma)$ to be frozen.
(2) $T^{/s}$ is an algebraic torus with character lattice generated by $\mathbf{x}^{/s}$.
(3) $\mathbf{d}^{/s}$ is obtained by restricting the sequence $\mathbf{d}$ to the set $[n+m]\setminus\{s,f\}$.
(4) $\omega^{/s}$ is obtained from ω by writing it in the form (3.1) and substituting $\mathrm{dlog}x_s:=0$ and $\mathrm{dlog}x_f:=\mathrm{dlog}M_1-\mathrm{dlog}M_2$.

The *deletion* $\Sigma^{\setminus s}$ is the seed of rank $n-1$ and dimension $n+m$ obtained by declaring x_s to be frozen (cf. Definition 3.4).

Theorem 3.12 (Deletion-contraction recurrence)**.** *Let X be an affine, normal, irreducible, complex algebraic variety, and let $\Sigma=(T,\mathbf{x},\mathbf{d},\omega)$ be a seed on X with a sink s satisfying Assumption 3.10. Assume that all cluster variables in $\mathbf{x}\sqcup\{x'_s\}$ are regular on X. Define subvarieties $W:=\{x_s\neq 0\}$ and $V:=\{x_s=0\}$ of X. Suppose we have isomorphisms $W\cong\mathcal{X}(\Sigma^{\setminus s})$ and $V\cong\mathrm{Spec}(\mathbb{C}[x'_s])\times\mathcal{X}(\Sigma^{/s})\cong\mathbb{C}\times\mathcal{X}(\Sigma^{/s})$ such that each cluster variable of $\Sigma^{\setminus s}$, resp. $\Sigma^{/s}$, is the pullback of the same-named cluster variable of Σ under the inclusion $\iota_W:W\hookrightarrow X$, resp. $\iota_V:V\hookrightarrow X$, and the pullback of x'_s under ι_V is the same-named function on the first factor of $\mathrm{Spec}(\mathbb{C}[x'_s])\times\mathcal{X}(\Sigma^{/s})\cong V$. Then (X,Σ) is a cluster variety.*

Proof. First, $\Sigma^{\setminus s}$ is integral by assumption. Since Σ is integral at s, this implies Σ is also integral.

Let $j\in[n]\setminus\{s\}$ be a mutable index. Clearly, the (pullback under ι_W of the) exchange relation for x_j in Σ coincides with the exchange relation for x_j in $\Sigma^{\setminus s}$. Thus, the mutated variable x'_j is regular on W. Next, assume that $j\notin N_s^{\mathrm{in}}(\Gamma)$. By assumption, j is not connected to s,f in Γ, and thus the terms involving $\mathrm{dlog}x_j$ are unchanged when passing from ω to $\omega^{/s}$. Thus, the pullback of the exchange relation for x_j under ι_V is still the exchange relation for x_j in $\Sigma^{/s}$, and therefore the mutated variable x'_j is regular on V. For $j\in N_s^{\mathrm{in}}(\Gamma)$, x'_j must also be regular on V since the pullback of x_j is a frozen variable. It follows that for all $j\in[n]\setminus\{s\}$, the mutated variable x'_j is a regular function on X since it is regular on both V and W. For $j=s$, x'_s is regular on X by assumption.

Next, we show that $\mathbb{C}[X]\subset\mathcal{U}(\Sigma)$. This is equivalent to constructing inclusions $T\hookrightarrow X$ and $\mu_j(T)\hookrightarrow X$ for all $j\in[n]$. Since Σ is a seed on X, we have $T\subset X$. For the tori $\mu_j(T)$, we show that the subset $X_j\subset X$ where the regular functions in $\mu_j(\mathbf{x})$ are all non-vanishing is isomorphic to an algebraic torus $\mu_j(T)\cong(\mathbb{C}^\times)^{n+m}$ via the map $\varphi_j:X_j\to\mu_j(T)$ sending $y\in X_j$ to $z:=(x_1(y),\dots,x'_j(y),\dots,x_{n+m}(y))$. If $j\in[n]\setminus s$ then we have $X_j\subset W$, and thus the statement follows since W is a cluster variety. So let $j=s$. Consider the torus $\mu_s(T)\cong(\mathbb{C}^\times)^{n+m}$. Let $p:=M_1+x_fM_2\in\mathbb{C}[X]$ be the exchange binomial for x_s (cf. Assumption 3.10). Since p does not involve x_s and x'_s, we can also view p as a regular function on $\mu_s(T)$ compatible with pullback under φ_s. Let $z=(z_1,\dots,z_{n+m})\in\mu_s(T)$. Our goal is to show that z has a unique preimage under φ_s. Suppose first that $p(z)\neq 0$. Then $\varphi_s^{-1}(z)\subset T$, and the result follows. Suppose now that $p(z)=0$. Then $\varphi_s^{-1}(z)\subset V$. Recall that $V\cong\mathrm{Spec}(\mathbb{C}[x'_s])\times\mathcal{X}(\Sigma^{/s})$. Since $x'_s=z_s$, the $\mathrm{Spec}(\mathbb{C}[x'_s])$-coordinate of the preimage is uniquely determined by z. The $\mathcal{X}(\Sigma^{/s})$-coordinate of the preimage is uniquely determined by $(z_i)_{i\in[n+m]\setminus\{s,f\}}$. We have shown that z has a unique preimage under φ_s, which completes the proof of the inclusion $\mathbb{C}[X]\subset\mathcal{U}(\Sigma)$. The statement of the theorem now follows from Proposition 3.13 below. □

Proposition 3.13. *Let X be an affine, normal, irreducible, complex algebraic variety, and let $\Sigma = (T, \mathbf{x}, \mathbf{d}, \omega)$ be an integral sink-recurrent seed on X. Suppose that $\mathbb{C}[X] \subset \mathcal{U}(\Sigma)$. Then (X, Σ) is a cluster variety.*

Proof. The inclusions $\mathbb{C}[X] \subset \mathbb{C}[x_1^{\pm 1}, \dots, (x_j')^{\pm 1}, \dots, x_{n+m}^{\pm 1}]$ give tori $\mu_j(T) \cong X_j \subset X$ as in the proof of Theorem 3.12. By a standard argument, this implies that the complement of $T \cup \bigcup_{j \in [n]} X_j$ has codimension greater than or equal to two in X; see [Zel00, Section 3], [BFZ05, Proof of Theorem 2.10], or [GLSBS22, Lemmas 9.5–9.8]. Since X is normal, we have $\mathbb{C}[X] = \mathbb{C}[T \cup \bigcup_{j \in [n]} X_j] = \mathcal{U}(\Sigma)$. By assumption, Σ is sink-recurrent, and we are done by Proposition 3.6. □

Remark 3.14. If the seed $\Sigma^{\setminus s}$ is really full rank then it follows from Assumption 3.10 that Σ is really full rank. Indeed, row f of the exchange matrix of Σ contains a single nonzero entry equal to ± 1 in column s. The exchange matrix of $\Sigma^{\setminus s}$ is obtained from that of Σ by removing column s. This implies that if $\Sigma^{\setminus s}$ is really full rank then so is Σ.

Remark 3.15. The statement of Theorem 3.12 remains true if "sink-recurrent" is replaced with "locally acyclic."

4. Double braid moves

In this section, we study natural isomorphisms between braid varieties corresponding to *double braid moves*, and determine the effect of these isomorphisms on seeds. Double braid moves are defined as follows:

(B1) $ij \leftrightarrow ji$ if $i, j \in \pm I$ have different signs;
(B2) $ij \leftrightarrow ji$ if $i, j \in \pm I$ have the same sign and $(s_{|i|} s_{|j|})^2 = 1$;
(B3) $\underbrace{iji\ldots}_{m_{ij}\text{ letters}} \leftrightarrow \underbrace{jij\ldots}_{m_{ij}\text{ letters}}$ if $i, j \in \pm I$ have the same sign and $(s_{|i|} s_{|j|})^{m_{ij}} = 1$ with $m_{ij} \geq 3$;
(B4) $\beta_0 i \leftrightarrow \beta_0(-i^*)$ for $i \in \pm I$ and $\beta_0 \in (\pm I)^{m-1}$;
(B5) $i\beta_0 \leftrightarrow (-i)\beta_0$ for $i \in \pm I$ and $\beta_0 \in (\pm I)^{m-1}$.

If double braid words β and β' are related by one of the moves (B1)–(B5), there is a natural isomorphism $\phi : \mathring{R}_\beta \xrightarrow{\sim} \mathring{R}_{\beta'}$.

Definition 4.1. Suppose that β and β' are related by one of the moves (B1)–(B3). If this move involves indices $l, l+1, \dots, r$, the isomorphism ϕ sends $(X_\bullet, Y_\bullet) \in \mathring{R}_\beta$ to the unique tuple $(X'_\bullet, Y'_\bullet) \in \mathring{R}_{\beta'}$ such that $X'_c = X_c$ and $Y'_c = Y_c$ for $0 \leq c < l$ or $r \leq c \leq m$. The remaining weighted flags $X'_l, \dots, X'_{r-1}, Y'_l, \dots, Y'_{r-1}$ are uniquely determined using Lemma 2.1.

For the moves (B4) and (B5), the isomorphism ϕ is described in Sections 4.6 and 4.7, respectively. The main result of this section is the following.

Theorem 4.2. *Suppose that β and β' are related by one of the moves* (B1)–(B5)*. If $(\mathring{R}_\beta, \Sigma_\beta)$ is a cluster variety then so is $(\mathring{R}_{\beta'}, \Sigma_{\beta'})$.*

We then use Theorem 4.2 and Theorem 3.12 to prove Theorem 1.1; see Theorem 4.10 and Sections 4.9 and 6.3.

The proof of Theorem 4.2 will occupy Sections 4–6. Along the way, we will construct a seed $\Sigma' = (T', \mathbf{x}', \mathbf{d}', \omega')$ obtained from $\Sigma_\beta = (T, \mathbf{x}, \mathbf{d}, \omega)$ by one or several mutations, followed by a relabeling. We will show the following for moves (B1)–(B5):

(F) The 2-form is invariant: $\phi^* \omega_{\beta'} = \omega_\beta$.
(Q) Suppose that $(\mathring{R}_\beta, \Sigma_\beta)$ is a cluster variety. Then the seeds Σ' and $\phi^* \Sigma_{\beta'}$ are quasi-equivalent.

Here, for a seed $\Sigma_{\beta'} = (T_{\beta'}, \mathbf{x}_{\beta'}, \mathbf{d}_{\beta'}, \omega_{\beta'})$, $\phi^* \Sigma_{\beta'} = (T^*, \mathbf{x}^*, \mathbf{d}^*, \omega^*)$ is an abstract seed on $\mathring{R}_\beta$ defined by $T^* := \phi^{-1}(T_{\beta'})$, $\mathbf{x}^* := \phi^* \mathbf{x}_{\beta'}$, $\mathbf{d}^* := \mathbf{d}_{\beta'}$, and $\omega^* := \phi^* \omega_{\beta'}$. Note that (Q) immediately implies Theorem 4.2.

Definition 4.3. A move (B1)–(B3) is *solid* if all indices involved are solid. For $i,j\in I$, the (B1) move $(-i)j\leftrightarrow j(-i)$ on indices $c,c+1$ is *special* if $u_c s_i = s_j u_c$ and *solid-special* if it is both solid and special. A (B3) move with $m_{ij}>3$ is *long*; all other moves are *short*. Finally, a move (B1)–(B5) is a *mutation move* if it involves at least one cluster mutation; otherwise it is a *non-mutation move*.

Remark 4.4. As we will show in Section 4.1, a solid-special (B1) move corresponds to a single mutation, at the rightmost index involved in the move. The move (B3) involving q solid indices corresponds to a sequence of $\binom{q-1}{2}$ mutations on the rightmost $m_{ij}-2$ indices involved in the move.

We will show (F), (Q) for short moves directly. This will complete the proof of Theorem 4.2 in simply-laced types. We then use this and folding to show (F), (Q) for long moves in Sections 5 and 6.

Throughout this section, we fix β,β' related by a short move and thus an isomorphism $\phi:\mathring{R}_\beta\xrightarrow{\sim}\mathring{R}_{\beta'}$. For a rational function or a form f on $\mathring{R}_{\beta'}$, we use the shorthand $f^*:=\phi^* f$.

Remark 4.5. If all indices involved in a move (B1)–(B3) are hollow, then the statements (F), (Q) follow trivially; cf. Corollary 2.26.

4.1. **Mutation move: (B1), solid-special.** Consider the case of a solid-special move (B1) on indices $c,c+1$. Since both indices are solid, we denote $u:=u_{c-1}=u_c=u_{c+1}$ and $w:=w_{c-1}=w_c=w_{c+1}$. The indices $i,j\in\pm I$ are of opposite signs; we assume that $i\in -I$ and $j\in I$ as the other case is similar. The solid-special condition yields

$$u<s_{|i|}u=us_j \quad\text{and}\quad s_{|i|^*}w=ws_j<w. \tag{4.1}$$

Proposition 4.6 ([FZ99, Theorem 1.17])**.** *We have*[1]

$$\Delta_{c,j}\Delta^*_{c,j}=\Delta_{c+1,j}\Delta_{c-1,j}+\prod_{k\neq j}\Delta_{c,k}^{-a_{jk}}. \tag{4.2}$$

Proof. We may choose $t,t'\in\mathbb{C}$ such that $Z_c=Z_{c+1}z_j(t)$, $Z^*_c=\bar z_{|i|^*}(t')^{-1}Z_{c+1}$, and $Z_{c-1}=Z^*_{c-1}=\bar z_{|i|^*}(t')^{-1}Z_{c+1}z_j(t)=\dot s_{|i|^*}x_{|i|^*}(t')Z_{c+1}x_j(t)\dot s_j$. Let $Z:=x_{|i|^*}(t')Z_{c+1}x_j(t)$. By [FZ99, Theorem 1.17], we have

$$\Delta_{w\omega_j,s_j\omega_j}(Z)\Delta_{ws_j\omega_j,\omega_j}(Z)=\Delta_{ws_j\omega_j,s_j\omega_j}(Z)\Delta_{w\omega_j,\omega_j}(Z)+\prod_{k\neq j}\Delta_{ws_j\omega_k,\omega_k}(Z)^{-a_{jk}}. \tag{4.3}$$

Using properties of generalized minors from Section 2.5, one can check that each term of (4.2) equals the corresponding term of (4.3). For example, we have

$$\Delta_{c,j}=\Delta_{w\omega_j,\omega_j}(Z_{c+1}x_j(t)\dot s_j)=\Delta_{\omega_j,\omega_j}(\dot w^{-1}Z_{c+1}x_j(t)\dot s_j)=\Delta_{\omega_j,\omega_j}(\dot w^{-1}Z\dot s_j)=\Delta_{w\omega_j,s_j\omega_j}(Z),$$

where we have used $\dot w^{-1}x_{|i|^*}(t')\in U_-\dot w^{-1}$; cf. (2.9) and (4.1). For $\Delta_{c,k}^{-a_{jk}}$, $k\neq j$, we additionally used that $s_j\omega_k=\omega_k$. ☐

We shall use the following analog of [GLSBS22, Lemma 8.10].

Lemma 4.7. *For $e\in[0,m]$ and $-i,j\in I$ such that $u_e s_j=s_{|i|}u_e$, we have*

$$\prod_{k\in\pm I}\Delta_{e,k}^{a_{ik}}=\prod_{k\in\pm I}\Delta_{e,k}^{\epsilon a_{jk}}\quad\textit{and}\quad L_{e,i}=\epsilon L_{e,j},\quad\textit{where}\quad \epsilon:=\begin{cases}1, & \textit{if } u_e<u_e s_j,\\ -1, & \textit{if } u_e>u_e s_j.\end{cases} \tag{4.4}$$

Proof. We have $\alpha_j=\sum_{k\in I}a_{jk}\omega_k$ and similarly for $\alpha_{|i|}$. The first identity in (4.4) therefore becomes $(h_e^-)^{\alpha_{|i|}}=(h_e^+)^{\epsilon\alpha_j}$, which follows from the assumption $u_e\alpha_j=\epsilon\alpha_{|i|}$ together with $h_e^-=u_e\cdot h_e^+$; cf. (2.6). Taking dlog of both sides, we obtain the second identity. ☐

Remark 4.8. Equations (4.2) and (4.4) are true as stated in the case $i,-j\in I$ as well.

[1]Our Cartan matrix is the transpose of that of [FZ99]; see [FZ99, Equation (2.27)].

Proof of (F) *for* (B1)*, solid-special.* Only the terms $\omega_{\beta,c}$ and $\omega_{\beta,c+1}$ change when doing the move (B1). Noting that we must have $d_i=d_j$ and applying (4.4), we get

$$\begin{aligned}\frac{1}{d_j}(\omega_\beta-\omega^*_{\beta'})&=\frac{1}{d_j}\left(\omega_{\beta,c}+\omega_{\beta,c+1}-\omega^*_{\beta',c}-\omega^*_{\beta',c+1}\right)\\&=-L_{c-1,i}\wedge L_{c,i}+L_{c,j}\wedge L_{c+1,j}-L^*_{c-1,j}\wedge L^*_{c,j}+L^*_{c,i}\wedge L^*_{c+1,i}\\&=-L_{c-1,j}\wedge L_{c,j}+L_{c,j}\wedge L_{c+1,j}-L_{c-1,j}\wedge L^*_{c,j}+L^*_{c,j}\wedge L_{c+1,j}\\&=(L_{c,j}+L^*_{c,j})\wedge(L_{c-1,j}+L_{c+1,j}).\end{aligned}$$

For $e\in\{c-1,c,c+1\}$, let $M_e:=\prod_{k\neq j}\Delta_{e,k}^{-a_{jk}}$. Thus, M_c is the third term in (4.2). By (2.7), we have $h_c^+=\alpha_j^\vee(t_{c+1})h^+_{c+1}$ and $h^-_{c-1}=\alpha^\vee_{|i|}(t_c)h^-_c$. This implies that $h^+_{c-1}=\alpha^\vee_j(t_ct_{c+1})h^+_{c+1}$ since $h^-_c=u\cdot h^+_c$. Thus, we have $M:=M_{c-1}=M_c=M_{c+1}$. Since $M^*_{c+1}=M_{c+1}$, we get that $M=M^*_{c-1}=M^*_c=M^*_{c+1}$. Set $A:=\frac{\Delta_{c,j}\Delta^*_{c,j}}{M}$ and $B:=\frac{\Delta_{c-1,j}\Delta_{c+1,j}}{M}+1$. Then (4.2) gives $A=B+1$. Thus $dA=dB$ and $\mathrm{dlog}A\wedge\mathrm{dlog}B=0$. It remains to note that $\mathrm{dlog}A=L_{c,j}+L^*_{c,j}$ while $\mathrm{dlog}B=L_{c-1,j}+L_{c+1,j}$. □

Proof of (Q) *for* (B1)*, solid-special.* We do not use the assumption that $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety until the last paragraph of this proof. Let $x:=x_{c+1}$ and $V:=V_{c+1}$. Applying Propositions 2.20 and 2.21, we see that x is mutable,

$$\mathrm{ord}_V\Delta_{c,j}=\mathrm{ord}_V\Delta_{c,i}=1,\quad\text{and}\quad\mathrm{ord}_V\Delta_{e,k}=0\quad\text{for }(e,k)\notin\{(c,j),(c,i)\}. \tag{4.5}$$

In particular, $\mathrm{dlog}x$ appears in ω_β only in the terms $L_{c,j}$ and $L_{c,i}$ in $\omega_{\beta,c+1}=d_jL_{c,j}\wedge L_{c+1,j}$ and $\omega_{\beta,c}=-d_iL_{c-1,i}\wedge L_{c,i}$, respectively. Recall from (4.4) that we actually have $L_{c,j}=L_{c,i}$. We see from (4.5) that the coefficient of $\mathrm{dlog}x$ in $L_{c,j}=L_{c,i}$ is equal to 1. Collecting the terms of ω_β involving $\mathrm{dlog}x$ and using $d_i=d_j$, we get

$$d_j\mathrm{dlog}x\wedge(L_{c+1,j}+L_{c-1,i})=d_j\mathrm{dlog}x\wedge\left(\mathrm{dlog}(\Delta_{c+1,j}\Delta_{c-1,j})-\mathrm{dlog}\prod_{k\neq j}\Delta_{c,k}^{-a_{jk}}\right). \tag{4.6}$$

By Proposition 2.20, a cluster variable x_e for $e\in J_\beta$ may appear on the right-hand side of (4.6) only for $e\geq c$. Moreover, we have already observed that $d_c=d_{|i|}=d_j=d_{c+1}$. Let us denote $p_e:=\mathrm{ord}_{V_e}(\Delta_{c+1,j}\Delta_{c-1,j})$ and $q_e:=\mathrm{ord}_{V_e}\prod_{k\neq j}\Delta_{c,k}^{-a_{jk}}$. Clearly, $p_e,q_e\geq 0$. We therefore see from (3.1) that for all $e\in J_\beta$, we have $\tilde{B}_{e,c+1}=q_e-p_e$. By Definition 2.14, the cluster variable x is mutable. Thus, the mutated variable $x':=x'_{c+1}$ satisfies

$$xx'=\prod_{e\in J_\beta:p_e>q_e}x_e^{p_e-q_e}+\prod_{e\in J_\beta:q_e>p_e}x_e^{q_e-p_e}. \tag{4.7}$$

We have $V_e=V^*_e$ and $x_e=x^*_e$ for all $e\in J_\beta\setminus\{c+1\}$. Let $V^*:=V^*_{c+1}$ and $x^*:=x^*_{c+1}$. A generic point $(X_\bullet,Y_\bullet)\in V$ satisfies $X_{c-1}\overset{ws_j}{\Longleftarrow}Y_{c+1}$ and $X_{c+1}\overset{w}{\Longleftarrow}Y_{c-1}$, while a generic point $(X_\bullet,Y_\bullet)\in V^*$ satisfies $X_{c+1}\overset{ws_j}{\Longleftarrow}Y_{c-1}$ and $X_{c-1}\overset{w}{\Longleftarrow}Y_{c+1}$. Thus, $V\neq V^*$.

For $e\in J_\beta$, applying ord_{V_e} to both sides of (4.2), we get

$$\mathrm{ord}_{V_e}\Delta_{c,j}+\mathrm{ord}_{V_e}\Delta^*_{c,j}\geq\min(p_e,q_e). \tag{4.8}$$

For $e=c+1$, we have $\mathrm{ord}_V\Delta_{c,j}=1$, $\mathrm{ord}_V\Delta^*_{c,j}=0$ (since $V\neq V^*$), and $p_{c+1}=q_{c+1}=0$. Similarly, $\mathrm{ord}_{V^*}\Delta^*_{c,j}=1$, $\mathrm{ord}_{V^*}\Delta_{c,j}=0$, and the order of vanishing of $\Delta^*_{c+1,j}\Delta^*_{c-1,j}$ and $\prod_{k\neq j}(\Delta^*_{c,k})^{-a_{jk}}$ at V^* is zero.

Dividing both sides of (4.2) by $\prod_{e\in J_\beta\setminus\{c+1\}}x_e^{\min(p_e,q_e)}$, we get

$$xx^*\prod_{e\in J_\beta\setminus\{c+1\}}x_e^{r_e}=\prod_{e\in J_\beta:p_e>q_e}x_e^{p_e-q_e}+\prod_{e\in J_\beta:q_e>p_e}x_e^{q_e-p_e},$$

where $r_e:=\mathrm{ord}_{V_e}\Delta_{c,j}+\mathrm{ord}_{V_e}\Delta^*_{c,j}-\min(p_e,q_e)\geq 0$. By (4.7), we get

$$x'=x^*\prod_{e\in J_\beta\setminus\{c+1\}}x_e^{r_e}. \tag{4.9}$$

Now, assume that $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety. We get from Proposition 3.3 that the mutated cluster variable x' is irreducible in $\mathbb{C}[\mathring{R}_\beta]$. The function x^* vanishes on $V^*\subset\mathring{R}_\beta$ and therefore is not a unit in $\mathbb{C}[\mathring{R}_\beta]$. It follows that $r_e=0$ for all mutable e, and thus the mutated seed $\Sigma':=\mu_{c+1}\Sigma_\beta$ is quasi-equivalent to the pulled back seed $\Sigma^*_{\beta'}$. □

4.2. **Non-mutation move: (B1), not solid-special.** We continue to assume that the move involves indices $c,c+1$, and that $i\in -I$, $j\in I$.

4.2.1. (B1), *special, non-solid.* Suppose that at least one of the indices is hollow, and that the move is special. Then it follows that $c+1$ is hollow and c is solid in both β and β'. By (2.22), $L_{c,j}=-L_{c+1,j}$ and $L^*_{c,i}=-L^*_{c+1,i}$. Applying (4.4) with $\epsilon=1$ for $e=c-1,c$ and $\epsilon=-1$ for $e=c+1$ and using $d_i=d_j$, we obtain

$$\frac{\omega_{\beta,c}}{d_j}=-L_{c-1,i}\wedge L_{c,i}=L_{c-1,i}\wedge L_{c+1,i}=L^*_{c-1,i}\wedge L^*_{c+1,i}=-L^*_{c-1,j}\wedge L^*_{c+1,j}=L^*_{c-1,j}\wedge L^*_{c,j}=\frac{\omega^*_{\beta',c}}{d_i},$$

which proves (F). The clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ are identical, which proves (Q).

4.2.2. (B1), *non-special.* We start by introducing a formalism for working with the forms $L_{e,k}$. Let $\lambda:=\sum_{k\in I}b_k\omega_k$ with $b_k\in\mathbb{Q}$, and let h be an H-valued rational function on $\mathring{R}_\beta$. We introduce a rational 1-form

$$\mathrm{dlog}h^\lambda:=\sum_{k\in I}b_k\mathrm{dlog}(h^{\omega_k}).$$

It is clear that

$$\mathrm{dlog}h^{\lambda_1+\lambda_2}=\mathrm{dlog}h^{\lambda_1}+\mathrm{dlog}h^{\lambda_2}\quad\text{and}\quad \mathrm{dlog}(h_1h_2)^\lambda=\mathrm{dlog}h_1^\lambda+\mathrm{dlog}h_2^\lambda. \tag{4.10}$$

For $e\in[0,m]$ and $k\in I$, Lemma 2.10 gives

$$L_{e,k}=\mathrm{dlog}(h_e^+)^{\alpha_k/2}=\mathrm{dlog}(h_e^-)^{u_e\alpha_k/2},\quad L_{e,-k}=\mathrm{dlog}(h_e^-)^{\alpha_k/2}=\mathrm{dlog}(h_e^+)^{u_e^{-1}\alpha_k/2}. \tag{4.11}$$

Finally, suppose that $h_1=h_2\alpha_k^\vee(t)$. Then we have

$$\mathrm{dlog}h_1^\lambda=\mathrm{dlog}h_2^\lambda+\langle\lambda,\alpha_k^\vee\rangle\mathrm{dlog}t. \tag{4.12}$$

Proof of (F) *and* (Q) *for* (B1)*, non-special.* Suppose as before that the move involves indices $c,c+1$, and that $i\in -I$, $j\in I$. Assume first that both $c,c+1$ are solid, and let $u:=u_{c-1}=u_c=u_{c+1}$. Let $a:=\langle u^{-1}\alpha_{|i|}/2,\alpha_j^\vee\rangle$ and $a':=\langle u\alpha_j/2,\alpha_{|i|}^\vee\rangle$. Using (4.11)–(4.12) and (2.7), we get

$$L_{c,i}=L_{c+1,i}+a\mathrm{dlog}t_{c+1},\qquad L_{c-1,i}=L_{c,i}+\mathrm{dlog}t_c,\qquad L_{c,j}=L_{c+1,j}+\mathrm{dlog}t_{c+1}; \tag{4.13}$$

$$L^*_{c,j}=L^*_{c+1,j}+a'\mathrm{dlog}t^*_{c+1},\qquad L^*_{c-1,j}=L^*_{c,j}+\mathrm{dlog}t^*_c,\qquad L^*_{c,i}=L^*_{c+1,i}+\mathrm{dlog}t^*_{c+1}. \tag{4.14}$$

Since the move is non-special, the coroots $\alpha_j^\vee$ and $u^{-1}\alpha_{|i|}^\vee$ are linearly independent, which implies $t^*_c=t_{c+1}$ and $t^*_{c+1}=t_c$. Note also that we have $L^*_{c+1,i}=L_{c+1,i}$ and $L^*_{c+1,j}=L_{c+1,j}$. Using (4.13)–(4.14) to express each 1-form $L_{e,k}$ in terms of $L_{c+1,i}$, $L_{c+1,j}$, $\mathrm{dlog}t_c$, and $\mathrm{dlog}t_{c+1}$, we find

$$\omega_{\beta,c}+\omega_{\beta,c+1}-\omega^*_{\beta',c}-\omega^*_{\beta',c+1}=(d_ja'-d_ia)\mathrm{dlog}t_c\wedge\mathrm{dlog}t_{c+1}.$$

Since $d_ja'=d_ia$, we get that $\omega_\beta=\omega^*_{\beta'}$. The clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ differ by a relabeling $c\leftrightarrow c+1$.

Suppose now that one of $c,c+1$ is hollow. For instance, let $c\notin J_\beta$ and $c+1\in J_\beta$. By Corollary 2.7, we have $h_c^+=h_{c-1}^+$, and thus $L_{c,j}=L_{c-1,j}$. Similarly, $L^*_{c,j}=L^*_{c+1,j}$. Recall that $L^*_{c\pm1,j}=L_{c\pm1,j}$. Thus, $\omega_{\beta,c+1}=\omega^*_{\beta',c}$, and so $\omega_\beta=\omega^*_{\beta'}$. The case where $c\in J_\beta$ and $c+1\notin J_\beta$ is similar. The clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ differ by a relabeling $c\leftrightarrow c+1$. For the case $c,c+1\notin J_\beta$, see Remark 4.5. □

4.3. **Non-mutation move: (B2).** Suppose that the move involves indices $c,c+1$. We have $\omega_{\beta,c}=\omega^*_{\beta',c+1}$ and $\omega_{\beta,c+1}=\omega^*_{\beta',c}$, so $\omega_\beta=\omega^*_{\beta'}$. The chamber minors satisfy $\Delta_c=\Delta^*_{c+1}$ and $\Delta_{c+1}=\Delta^*_c$. Thus, the clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ differ by a relabeling $c\leftrightarrow c+1$. This shows (F) and (Q).

4.4. **Mutation move: (B3), solid, short.** We proceed analogously to the case of solid-special (B1) in Section 4.1. Suppose that the move $\beta\to\beta'$, $iji\to jij$, involves indices $c-1,c,c+1$, and that all three indices are solid. Suppose in addition that $i,j\in I$; the case $i,j\in -I$ is similar.

Proposition 4.9 ([FZ99, Theorem 1.16(1)])**.** *We have*

$$\Delta_{c,i}\Delta^*_{c,j}=\Delta_{c+1,i}\Delta_{c-2,j}+\Delta_{c-2,i}\Delta_{c+1,j}. \tag{4.15}$$

Proof. We have $Z_{c-2}=Z_{c+1}z_i(t_1)z_j(t_2)z_i(t_3)$ for some $t_1,t_2,t_3\in\mathbb{C}$. We have $z_i(t_1)z_j(t_2)z_i(t_3)=z_j(t_3)z_i(t_2')z_j(t_1)$ for $t_2':=t_1t_3-t_2$, which can be checked inside SL_3. Thus, $Z_{c-1}=Z_{c+1}z_i(t_1)z_j(t_2)$, $Z_c=Z_{c+1}z_i(t_1)$, and $Z^*_c=Z_{c+1}z_j(t_3)$. Let $Z:=Z_{c-2}(\dot s_i\dot s_j\dot s_i)^{-1}$. Let $w:=w_{c-1}=w_c=w_{c+1}$. By [FZ99, Theorem 1.16(1)],

$$\Delta_{w\omega_i,s_i\omega_i}(Z)\Delta_{w\omega_j,s_j\omega_j}(Z)=\Delta_{w\omega_i,\omega_i}(Z)\Delta_{w\omega_j,s_is_j\omega_j}(Z)+\Delta_{w\omega_i,s_js_i\omega_i}(Z)\Delta_{w\omega_j,\omega_j}(Z). \tag{4.16}$$

Similarly to the proof of Proposition 4.6, we observe that each term in (4.15) equals the corresponding term in (4.16). □

Proof of (F) *for* (B3)*, solid, short.* Let $\tilde\beta=\beta w_\circ$ and $\tilde\beta'=\beta' w_\circ$. By definition, $\sum_{j\in J_{\tilde\beta}}\omega_j(\tilde\beta)$ and $\sum_{j\in[m]}\omega_j(\beta)$ are identical when expressed in terms of the symbols $\Delta_{c,i}$. It is known ([SW21, Proposition 3.25] or [BFZ05]) that Proposition 4.9 implies $\omega_{\tilde\beta}=\omega_{\tilde\beta'}$. Since the same identity for grid minors in Proposition 4.9 holds on $\mathring{R}_\beta$, we deduce that $\omega_\beta=\omega_{\beta'}$. □

Proof of (Q) *for* (B3)*, solid, short.* Let $x:=x_{c+1}$ and $V:=V_{c+1}$. By Propositions 2.20 and 2.21, x is mutable, $\mathrm{ord}_V\Delta_{c-1,i}=\mathrm{ord}_V\Delta_{c,i}=1$, and $\mathrm{ord}_V\Delta_{e,k}=0$ when $e\notin\{c-1,c\}$ or $k\in I\setminus\{i\}$.

Note that $d_i=d_j$. Collecting the terms of $\omega_{\beta,c-1}+\omega_{\beta,c}+\omega_{\beta,c+1}$ involving $\mathrm{dlog}x$, we get

$$d_i\mathrm{dlog}x\wedge\left(L_{c+1,i}-L_{c-2,i}+\frac{1}{2}(L_{c-1,j}-L_{c,j})\right). \tag{4.17}$$

Applying (2.20) and using Corollary 2.11, we get

$$L_{c+1,i}-L_{c-2,i}=\mathrm{dlog}(\Delta_{c+1,i})-\mathrm{dlog}(\Delta_{c-2,i})+\frac{1}{2}\mathrm{dlog}(\Delta_{c-2,j})-\frac{1}{2}\mathrm{dlog}(\Delta_{c+1,j});$$
$$L_{c-1,j}-L_{c,j}=\mathrm{dlog}(\Delta_{c-1,j})-\mathrm{dlog}\Delta_{c,j}=\mathrm{dlog}(\Delta_{c-2,j})-\mathrm{dlog}\Delta_{c+1,j}.$$

Thus, (4.17) becomes $d_i\mathrm{dlog}x\wedge(\mathrm{dlog}(\Delta_{c+1,i}\Delta_{c-2,j})-\mathrm{dlog}(\Delta_{c-2,i}\Delta_{c+1,j}))$. The rest of the proof is entirely analogous to the argument for solid-special (B1) given at the end of Section 4.1, using (4.15) in place of (4.2). □

4.5. **Non-mutation move: (B3), non-solid, short.** Suppose that at least one of the indices $c-1,c,c+1$ is hollow. By Remark 4.5, we may assume that there are either one or two hollow indices in $\{c-1,c,c+1\}$. Explicitly, underlining the hollow crossings, the possible moves are $i\underline{j}i\leftrightarrow ji\underline{j}$ and $i\underline{j}\underline{i}\leftrightarrow \underline{j}\underline{i}j$ (or the moves obtained from these by swapping the roles of i and j).

For $l\in\{i,j\}$ and $e\in J_\beta$, let us denote

$$A_l:=\mathrm{dlog}\prod_{k\neq i,j}\Delta^{a_{lk}}_{c+1,k},\quad B_l:=\mathrm{dlog}\Delta_{c+1,l},\quad\text{and}\quad T_e:=\mathrm{dlog}t_e. \tag{4.18}$$

Using (2.14)–(2.15), we can express the dlogs of grid minors $\Delta_{e,l}$ for $l\in\{i,j\}$ and $e\in\{c-1,c,c+1\}$ in the symbols (4.18). Using $\mathrm{dlog}\Delta^*_{c-2,l}=\mathrm{dlog}\Delta_{c-2,l}$ for $l\in\{i,j\}$, we express T^*_e in terms of $T_{e'}$ for all indices $e\in\{c-1,c,c+1\}$ which are solid in β'. Thus, we can express the forms $\omega_{\beta,e}$, $\omega^*_{\beta',e}$, $e\in\{c-1,c,c+1\}$ in terms of the symbols (4.18). Using a straightforward computation, we check $\omega_\beta=\omega^*_{\beta'}$.

We observe using Corollary 2.11 that the clusters $\mathbf{x}_\beta$ and $\mathbf{x}^*_{\beta'}$ differ by a relabeling, which shows (Q).

4.6. **Non-mutation move: (B4).** Suppose that $i\in I$. The isomorphism $\phi:\mathring{R}_\beta\xrightarrow{\sim}\mathring{R}_{\beta'}$ sending $(X_\bullet,Y_\bullet)\mapsto(X'_\bullet,Y'_\bullet)$ is given by $X'_{m-1}=X'_m=Y'_m:=X_{m-1}$, $Y'_{m-1}:=Y_{m-1}$, and $(X'_c,Y'_c):=(X_c,Y_c)$ for all $0\le c<m-1$. The last crossing in $\beta_0 i$ and $\beta_0(-i^*)$ is always hollow, and thus the statements (F) and (Q) follow trivially.

4.7. **Non-mutation move: (B5).** Suppose that $i \in I$. The isomorphism $\phi : \mathring{R}_\beta \xrightarrow{\sim} \mathring{R}_{\beta'}$ sending $(X_\bullet, Y_\bullet) \mapsto (X'_\bullet, Y'_\bullet)$ is defined as follows. For $c \in [m]$, we set $(X'_c, Y'_c) := (X_c, Y_c)$ and $X'_0 := X'_1$. Note that $Y_0 = Y_1 = Y'_1$ and recall $X_0 \overset{w_\circ}{\Longrightarrow} Y_0$. We let Y'_0 be the unique weighted flag satisfying $X_0 \overset{w_\circ s_{i^*}}{\Longrightarrow} Y'_0 \xrightarrow{s_{i^*}} Y_0$. It follows that $X'_0 \overset{w_\circ}{\Longleftarrow} Y'_0$ and $Y'_0 \xrightarrow{s_{i^*}} Y'_1$, so $(X'_\bullet, Y'_\bullet) \in \mathring{R}_{\beta'}$. The inverse map is defined similarly: X_0 is the unique weighted flag satisfying $X'_0 \xrightarrow{s_i} X_0 \overset{s_i w_\circ}{\Longrightarrow} Y'_0$.

The statement (F) is trivial if the first crossing of β is hollow. If the first crossing of β is solid, we have

$$X'_0 = X'_1 = X_1 \xrightarrow{s_i} X_0 \overset{s_i w_\circ}{\Longrightarrow} Y'_0 \xrightarrow{s_{i^*}} Y_0 = Y_1 = Y'_1.$$

It follows that after acting on all these flags by some $g \in G$, we can find $t,t' \in \mathbb{C}$ and $h \in H$ such that

$$X'_0 = X'_1 = X_1 = \dot{w}_\circ \dot{s}_i h \bar{z}_i(t) U_+, \quad X_0 = \dot{w}_\circ \dot{s}_i h U_+, \quad Y'_0 = U_+, \quad Y_0 = Y_1 = Y'_1 = z_{i^*}(t') U_+.$$

Here, we have $X_0 \overset{s_i w_\circ}{\Longrightarrow} Y'_0$ and thus $X_0 \overset{w_\circ s_i}{\Longleftarrow} Y'_0$, and we have used a representative $\dot{w}_\circ \dot{s}_i$ of $w_\circ s_i$ in $N_G(H)$. Let us denote $h_0 := h_0^+ = h_0^-$ and $h_0^* := (h_0^+)^* = (h_0^-)^*$. We have $Z_0 = Y_0^{-1} X_0 = z_{i^*}(t')^{-1} \dot{w}_\circ \dot{s}_i h$, and thus, proceeding as in the proof of Lemma 2.6, we get $h_0 = h$. Similarly, $Z'_0 = (Y'_0)^{-1} X'_0 = \dot{w}_\circ \dot{s}_i h \bar{z}_i(t)$, so $h_0^* = s_i \cdot h$. Applying (4.11), we find

$$L^*_{0,-i} = \mathrm{dlog}(s_i \cdot h_0)^{\alpha_i/2} = \mathrm{dlog}(h_0)^{-\alpha_i/2} = -L_{0,i}.$$

Applying (4.4) for $e = 0,1$, we obtain $L_{0,i} = L_{0,-i}$ and $L_{1,i} = L_{1,-i}$. Recall that $L_{1,i} = L^*_{1,i}$. Thus, we get $\omega_{\beta,1} = \omega^*_{\beta',1}$, and therefore $\omega_\beta = \omega^*_{\beta'}$, finishing the proof of (F).

We now prove (Q). Let $\beta = i\beta_0$ and $\beta' = (-i)\beta_0$. If the first crossing is hollow, the claim is trivial. Suppose that the first crossing is solid. We have $\Delta_{c,k} = \Delta^*_{c,k}$ for all $c \geq 1$ and $k \in \pm I$. Thus, $x_c = x^*_c$ for all $c \in J_\beta$ such that $c > 1$. Since $h_0^* = s_i \cdot h_0$, Lemma 2.10 implies that $x_1^* = x_1^{-1} M$, where M is a Laurent monomial in the grid minors $\Delta_{0,k}$ for $k \neq i$ of the same sign as i. It follows from Propositions 2.20 and 2.21 that M is a Laurent monomial in the frozen variables other than x_1. This shows (Q).

4.8. **Deletion-contraction for double braid varieties.** We apply the cluster algebraic results from Section 3.3 to the seeds Σ_β.

Theorem 4.10. *Let $i \in I$ and consider a double braid word $\beta = ii\beta'$ on positive letters. If $(\mathring{R}_{i\beta'}, \Sigma_{i\beta'})$ and $(\mathring{R}_{\beta'}, \Sigma_{\beta'})$ are sink-recurrent cluster varieties, then $(\mathring{R}_\beta, \Sigma_\beta)$ is a sink-recurrent cluster variety.*

Proof. Suppose first that at least one of the first two crossings in β is hollow, in which case 1 must be solid and 2 must be hollow. Consider an arbitrary point $(X_\bullet, Y_\bullet) \in \mathring{R}_\beta$. Since the letters in β are positive, we have $Y_0 = Y_1 = \cdots = Y_m = X_m$. Since $w_2 \leq w_\circ s_i$ and $w_0 = w_\circ$, we must have $X_1 \overset{w_\circ}{\Longleftarrow} X_m$ and $X_2 \overset{w_\circ s_i}{\Longleftarrow} X_m$. It follows that $h_1^\pm$ and $h_2^\pm$ are regular functions on $\mathring{R}_\beta$. Choose a representative $Z_2 = \dot{w}_\circ h_2^- \dot{s}_i^{-1}$ as in (2.6), and let $t,t' \in \mathbb{C}$ be such that $Z_1 = Z_2 z_i(t)$ and $Z_0 = Z_2 z_i(t) z_i(t')$. Thus, t,t' are regular functions on $\mathring{R}_\beta$. Proceeding as in the proof of Lemma 2.6, we find $h_1^+ = h_2^-$ and $h_0^+ = h_2^- \alpha_i^\vee(t')$, where h_0^+, h_1^+ are regular on $\mathring{R}_\beta$. It follows that $\Delta_{0,i} = t' \Delta_{1,i}$. For any $e \in J_\beta$ such that $e > 1$, the function x_e depends on $Z_2, Z_3, \ldots, Z_m$ but does not depend on t,t'. By Proposition 2.20, we have $\Delta_{0,i} = x_1 M$ for some monomial M in $\{x_e\}_{e>1}$. The Deodhar hypersurface V_1 is clearly given by the equation $t' = 0$. We conclude that $x_1 = t'$. We thus have an isomorphism

$$r : \mathring{R}_\beta \xrightarrow{\sim} \mathring{R}_{i\beta'} \times \mathbb{C}^\times, \quad (X_\bullet, Y_\bullet) \mapsto ((X_1, \ldots, X_m, Y_1, \ldots, Y_m), x_1). \tag{4.19}$$

Moreover, since $\Delta_{1,i} = M$ involves only frozen variables, we see that 1 is connected to only frozen indices in $\tilde{\Gamma}(\Sigma_\beta)$. It follows that the principal parts of Σ_β and $\Sigma_{i\beta'}$ agree, and therefore $(\mathring{R}_\beta, \Sigma_\beta)$ is a sink-recurrent cluster variety. Moreover, if $\Sigma_{i\beta'}$ is really full rank then so is Σ_β.

Suppose now that the first two crossings are both solid. Our goal is to apply Theorem 3.12. First, we show that Σ_β is sink-recurrent. Let $\Gamma := \Gamma(\Sigma_\beta)$. The seed $\Sigma_\beta^{\setminus 2}$ is obtained from $\Sigma_{i\beta'}$ by adding an isolated frozen variable x_1, and $\Sigma_\beta^{\setminus \widehat{N}_s^{\mathrm{in}}(\Gamma)}$ is obtained from $\Sigma_{\beta'}$ by adding isolated frozen variables x_1 and x_2, so both of these seeds are sink-recurrent. Further, in Σ_β, the variable x_1 is frozen, $\tilde{B}_{12} = -1$, and $\tilde{B}_{1c} = 0$ for mutable $c > 2$. Next, by Corollary 2.11, the sum of terms of ω_β involving x_2 is clearly of the form $d_i \mathrm{dlog} x_2 \wedge \mathrm{dlog} M$ for a Laurent monomial M in $\mathbf{x}$, and thus Σ_β is integral at 2. We have shown that Σ_β satisfies Assumption 3.10.

Next, we show that the mutated cluster variable x_2' is regular on $\mathring{R}_\beta$. We apply the moves $ii\beta' \xrightarrow{\text{(B5)}} (-i)i\beta' \xrightarrow{\text{(B1)}} i(-i)\beta'$. Denote $\dot{\mathbf{x}} := \mathbf{x}_{(-i)i\beta'}$ and $\ddot{\mathbf{x}} := \mathbf{x}_{i(-i)\beta'}$. It follows from the argument in Section 4.1 that $\dot{x}_2$ is mutable in $\Sigma_{(-i)i\beta'}$, and by (4.9), its mutation $\dot{x}_2'$ is regular on $\mathring{R}_{(-i)i\beta'}$, as it equals the pullback $\ddot{x}_2^*$ times a monomial in the other cluster variables in $\dot{\mathbf{x}}$ with nonnegative exponents. As explained in Section 4.7, the seeds Σ_β and $\Sigma_{(-i)i\beta'}$ are quasi-equivalent. By Lemma 3.8, we find that the mutation x_2' differs from $\dot{x}_2'$ by a unit (cf. part (2) of Corollary 2.24), and thus x_2' is regular on $\mathring{R}_\beta$.

Let $W := \{x_2 \neq 0\}$ $V := V_2 = \{x_2 = 0\}$ be the open-closed covering of $\mathring{R}_\beta$ coming from x_2. Our final goal is to construct isomorphisms

$$W \cong \mathring{R}_{i\beta'} \times \mathbb{C}^\times \cong \mathcal{X}(\Sigma_\beta^{\setminus 2}) \quad \text{and} \quad V \cong \mathrm{Spec}(\mathbb{C}[x_2']) \times \mathring{R}_{\beta'} \cong \mathbb{C} \times \mathcal{X}(\Sigma_\beta^{/2})$$

satisfying the conditions of Theorem 3.12. By Proposition 2.21, for $e \in J_\beta$, $e > 2$, we have $\mathrm{ord}_{V_e} \Delta_{1,i} = 0$ if and only if $\mathrm{ord}_{V_e} \Delta_{0,i} = 0$. Moreover, the same proposition implies $\mathrm{ord}_{V_2} \Delta_{0,i} = 0$. It follows by Proposition 2.20 that $\Delta_{1,i}$ is equal to x_2 times a monomial in the frozen variables, and that $\Delta_{0,i}$ is equal to x_1 times a monomial in the same set of frozen variables. Since W is the complement of V_2, we see that $(X_\bullet, Y_\bullet) \in W$ if and only if $X_1 \overset{w_\circ}{\Longleftarrow} Y_1 = X_m$. Thus, h_1^+ is a regular function on W. We choose a representative $Z_1 = \dot{w}_\circ h_1^+$ and let $t \in \mathbb{C}$ be such that $Z_0 = Z_1 z_i(t')$. Then we get $t' = \Delta_{0,i}/\Delta_{1,i} = M x_1/x_2$, where M is a Laurent monomial in the frozen variables other than x_1. Similarly to (4.19), we let $r : W \to \mathring{R}_{i\beta'} \times \mathbb{C}^\times$ be the map sending $(X_\bullet, Y_\bullet)$ to $((X_1, \dots, X_m, Y_1, \dots, Y_m), M x_1/x_2)$. By assumption, we have $\mathring{R}_{i\beta'} \cong \mathcal{X}(\Sigma_{i\beta'})$. The frozen index 1 is only connected to other frozen indices in $\tilde{\Gamma}(\Sigma_\beta)$. Thus, the seed $\Sigma_\beta^{\setminus 2}$ is obtained from $\Sigma_{i\beta'}$ by adding an isolated frozen vertex, and therefore $\mathcal{X}(\Sigma_\beta^{\setminus 2}) \cong \mathcal{X}(\Sigma_{i\beta'}) \times \mathbb{C}$. Adjusting the isolated frozen variable by a Laurent monomial in the other frozen variables, we see that the pullbacks of $x_1, \dots, x_{n+m}$ under the inclusion $\mathcal{X}(\Sigma_\beta^{\setminus 2}) \cong W \hookrightarrow X$ are indeed the same-named cluster variables in $\Sigma_\beta^{\setminus 2}$. This verifies the assumptions on ι_W in Theorem 3.12.

Now suppose that $(X_\bullet, Y_\bullet) \in V$. We have $X_0 \overset{w_\circ}{\Longleftarrow} X_m$ but not $X_1 \overset{w_\circ}{\Longleftarrow} X_m$, so we must have $X_1 \overset{w_\circ s_i}{\Longleftarrow} X_m$, and therefore $X_2 \overset{w_\circ}{\Longleftarrow} X_m$. Consider the map $r : V \to \mathring{R}_{\beta'} \times \mathbb{C}$ sending $(X_\bullet, Y_\bullet)$ to $((X_2, \dots, X_m, Y_2, \dots, Y_m), x_2')$. We claim that this map is an isomorphism. To construct an inverse, we need to show how to recover X_0, X_1, Y_0, Y_1 from the image of r. We have $Y_0 = Y_1 = X_m$. Also, X_1 is uniquely determined by $Y_1 = Y_2$ and X_2, since $Y_1 \overset{s_i w_\circ}{\Longleftarrow} X_1 \xleftarrow{s_i} X_2$. It remains to recover X_0. Note also that we can recover the cluster variables x_e, $e > 2$, as well as the mutated cluster variable x_2', from the image of r. Since $(X_\bullet, Y_\bullet) \in V$, the frozen variable x_1 is also recovered from the exchange relation $0 = x_2 x_2' = M_1 + x_1 M_2$ for x_2.

In order to recover X_1, we apply moves $ii\beta' \xrightarrow{\text{(B5)}} (-i)i\beta' \xrightarrow{\text{(B1)}} i(-i)\beta'$ as we did above. Let $(\ddot{X}_\bullet, \ddot{Y}_\bullet)$ denote the image of $(X_\bullet, Y_\bullet)$ in $\mathring{R}_{i(-i)\beta'}$ under this isomorphism ϕ, and let $\ddot{\mathbf{x}} := \mathbf{x}_{i(-i)\beta'}$. As in Section 4.7, let Y_0' be the unique weighted flag satisfying $X_0 \overset{w_\circ s_{i^*}}{\Longrightarrow} Y_0' \xrightarrow{s_{i^*}} Y_0$. Then $\ddot{X}_2 = X_2$, and $\ddot{Y}_2 = Y_2$, and $(\ddot{X}_1, \ddot{X}_0, \ddot{Y}_0, \ddot{Y}_1) = (X_2, X_1, Y_0', Y_0')$. We have that Y_0' is uniquely determined by X_2, Y_2, and $\ddot{\mathbf{x}}$: if $\ddot{x}_2 = 0$ then Y_0' is uniquely determined by $Y_2 \xleftarrow{s_{i^*}} Y_0' \overset{w_\circ s_{i^*}}{\Longleftarrow} X_2$; otherwise, we have $X_2 \overset{w_\circ}{\Longleftarrow} Y_0'$, and the values of $\ddot{\mathbf{x}}$ uniquely fixes the $U_+ \times U_+$-double coset $\ddot{Z}_1 := (Y_0')^{-1} X_2$ which determines Y_0'. The weighted flag X_0 is then uniquely determined by $X_1 \xrightarrow{s_i} X_0 \overset{s_i w_\circ}{\Longrightarrow} Y_0'$. It thus suffices to show that $\ddot{\mathbf{x}}$ is uniquely determined by the image of r. For $e \in J_\beta$, $e > 2$, we have $\ddot{x}_e = x_e$. Moreover, $\ddot{x}_1 = M/x_1$ for some monomial M in the frozen variables x_e other than x_1 (all of which must satisfy $e > 2$ since x_2 is mutable). Finally, by (4.9), $\ddot{x}_2$ differs from x_2' by a monomial in the cluster variables other than x_2. We are done with verifying the assumptions on ι_V in Theorem 3.12.

We have verified all conditions in Assumption 3.10 and Theorem 3.12. Thus, $(\mathring{R}_\beta, \Sigma_\beta)$ is a cluster variety. We have already shown that it is sink-recurrent. $\square$

4.9. **Proof of Theorem 1.1 for G simply-laced.** We proceed by induction on the number m of indices in β. Recall that we always assume $\pi(\beta)=w_\circ$. The base case is $m=\ell(w_\circ)$, where all indices are hollow. The cluster algebra is $\mathcal{A}_\beta=\mathbb{C}$ and the braid variety $\mathring{R}_\beta$ is a point.

Suppose now that $m>\ell(w_\circ)$. Applying (B1) and (B4), we can assume that all letters of β belong to I. Since G is simply-laced, all braid moves are automatically short. Applying (B2)–(B3), we may therefore transform β into a braid word of the form $\beta_1 ii\beta_2$ for some braid words β_1,β_2 and $i\in I$. We can also apply *conjugation moves* to β: if $\beta=j\beta_0$, the conjugation move consists of the moves

$$\beta=j\beta_0\xrightarrow{\text{(B5)}}(-j)\beta_0\xrightarrow{\text{(B1)}}\cdots\xrightarrow{\text{(B1)}}\beta_0(-j)\xrightarrow{\text{(B4)}}\beta_0 j^*. \tag{4.20}$$

Applying conjugation moves, we may further transform β into the form $\beta':=ii\beta_2\beta_1^*$, where β_1^* is obtained from β_1 by applying the map $j\mapsto j^*$ to each letter. Applying Theorem 4.10 to β', we find that $(\mathring{R}_{\beta'},\Sigma_{\beta'})$ is a cluster variety. It follows from Theorem 4.2 (for short moves) and Corollary 3.9 that $(\mathring{R}_\beta,\Sigma_\beta)$ is therefore also a cluster variety. ☐

Remark 4.11. It follows from our proof that the seed Σ_β is really full rank when G is simply-laced. Indeed, this property is preserved under moves (B1)–(B5), and by Remark 3.14 is compatible with deletion-contraction.

Finally, we show that for G simply-laced, double braid moves correspond to mutation equivalence.

Proposition 4.12. *Suppose that G is simply-laced and β,β' are related by a braid move* (B1)–(B4)*. The seeds $\Sigma_\beta,\Sigma^*_{\beta'}$ are mutation equivalent (up to relabeling cluster variables).*

Proof. By Theorem 1.1 for simply-laced G, $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety. By (Q), there is a seed Σ', which differs from Σ_β by mutation and possibly relabeling, such that $\Sigma'\sim\Sigma^*_{\beta'}$. We claim that these seeds are actually identical. Indeed, choose a double braid word β_0 such that all cluster variables of β,β' become mutable in $\tilde\beta:=\beta_0\beta$, $\tilde\beta':=\beta_0\beta'$; cf. Lemma 2.25. Let $\tilde\Sigma'$ be obtained from $\Sigma_{\tilde\beta}$ using the same mutations and relabeling by which Σ' was produced from Σ_β. Now, $(\mathring{R}_{\tilde\beta},\Sigma_{\tilde\beta})$ is also a cluster variety, so by (Q), $\tilde\Sigma'\sim\Sigma^*_{\tilde\beta'}$. Since all frozen variables of Σ' are mutable in $\tilde\Sigma'$, it follows that the seeds Σ' and $\Sigma^*_{\beta'}$ are identical. ☐

5. Folding

We review some background on folding before completing the proof in Section 6.

5.1. **Pinnings.** Let G be a complex, simple, simply-connected algebraic group. Choose a pinning $(H,B_+,B_-,x_i,y_i;i\in I)$. Then there exists an algebraic group $\dot G$ of simply-laced type with pinning $(\dot H,\dot B_+,\dot B_-,\dot x_{i'},\dot y_{i'};i'\in\dot I)$; see [Lus94, §1.6]. We have a bijection $\sigma:\dot I\to\dot I$ which extends to an automorphism $\sigma:\dot G\to\dot G$, and a map $\iota:G\to\dot G$ which yields algebraic group isomorphisms

$$\iota:G\xrightarrow{\sim}\dot G^\sigma,\quad H\xrightarrow{\sim}\dot H^\sigma,\quad B_\pm\xrightarrow{\sim}(\dot B_\pm)^\sigma,\quad U_\pm\xrightarrow{\sim}(\dot U_\pm)^\sigma.$$

The maps $gB_+\mapsto\iota(g)\dot B_+$ and $gU_+\mapsto\iota(g)\dot U_+$ induce isomorphisms of varieties:

$$\iota:G/B_+\xrightarrow{\sim}(\dot G/\dot B_+)^\sigma\quad\text{and}\quad G/U_+\xrightarrow{\sim}(\dot G/\dot U_+)^\sigma. \tag{5.1}$$

For the first isomorphism, see [Lus94, §8.8]. The surjectivity and injectivity of the second map follow from that of the first by a straightforward computation.

For an element $i\in I$, we denote by $\boldsymbol{i}\subset\dot I$ the associated σ-orbit, i.e., the orbit under the cyclic group generated by σ. We also let $\boldsymbol{-i}:=\{-i'\mid i'\in\boldsymbol{i}\}\subset-\dot I$. We let $\{\dot\alpha_{i'}\mid i'\in\dot I\}$, $\{\dot\alpha^\vee_{i'}\mid i'\in\dot I\}$, and $\{\dot\omega_{i'}\mid i'\in\dot I\}$ be the simple roots, simple coroots, and fundamental weights of the root system of $\dot G$. Letting $\dot a_{i'j'}:=\langle\dot\alpha_{i'},\dot\alpha^\vee_{j'}\rangle$ be the entries of the associated Cartan matrix (and setting $\dot a_{(-i')(-j')}:=\dot a_{i'j'}$ and $\dot a_{(\pm i')(\mp j')}:=0$ as before), we have

$$d_i=|\boldsymbol{i}|\quad\text{and}\quad a_{ij}=\sum_{j'\in\boldsymbol{j}}\dot a_{i'j'}\quad\text{for all } i,j\in\pm I\text{ and } i'\in\boldsymbol{i}. \tag{5.2}$$

The Coxeter generators of the Weyl group $\dot W$ of $\dot G$ are denoted by $\{\tilde s_{i'} \mid i' \in \dot I\}$. Restricting ι to the normalizer of H, we get a group isomorphism

$$\iota : W \xrightarrow{\sim} \dot W^\sigma, \quad s_i \mapsto \prod_{i' \in \boldsymbol{i}} \tilde s_{i'}. \tag{5.3}$$

Here the order inside $\boldsymbol{i}$ is immaterial since the corresponding elements $\tilde s_{i'}$ commute. It follows that the longest element $w_\circ \in W$ gets mapped under (5.3) to the longest element $\tilde w_\circ$ of $\dot W$, because $\sigma : \dot W \to \dot W$ preserves Coxeter length and therefore $\tilde w_\circ \in \dot W^\sigma$. The following result is immediate.

Lemma 5.1. *Let $B_1, B_2 \in G/U_+$. If $B_1 \xrightarrow{w} B_2$ then $\iota(B_1) \xrightarrow{\iota(w)} \iota(B_2)$. If $B_1 \xRightarrow{w} B_2$ then $\iota(B_1) \xRightarrow{\iota(w)} \iota(B_2)$. In particular, if $B_1 \xRightarrow{w_\circ} B_2$ then $\iota(B_1) \xRightarrow{\tilde w_\circ} \iota(B_2)$.*

5.2. **Braid varieties.** Let $\beta = i_1 i_2 \dots i_m \in (\pm I)^m$ be a double braid word. Let $\tilde\beta = i'_1 i'_2 \dots i'_{\tilde m} \in (\pm \dot I)^{\tilde m}$ be obtained by concatenating the letters in $\boldsymbol{i_1}, \boldsymbol{i_2}, \dots, \boldsymbol{i_m}$ (choosing the order inside each $\boldsymbol{i_c}$ arbitrarily), where $\tilde m := |\boldsymbol{i_1}| + |\boldsymbol{i_2}| + \dots + |\boldsymbol{i_m}|$. We let $\lambda_\beta : [\tilde m] \to [m]$ denote the unique order-preserving map satisfying $|\lambda_\beta^{-1}(c)| = |\boldsymbol{i_c}|$ for all $c \in [m]$. It is clear that an index $c \in [m]$ is solid (resp., hollow) if and only if all indices in $\lambda_\beta^{-1}(c)$ are solid (resp., hollow). In other words, the set $\dot J_{\tilde\beta}$ of solid crossings for $\tilde\beta$ is given by

$$\dot J_{\tilde\beta} = \lambda_\beta^{-1}(J_\beta). \tag{5.4}$$

Let $\mathring{\mathcal{Y}}'_\beta$ be the variety of tuples $(\dot X_\bullet, \dot Y_\bullet)$ of weighted flags in $\dot G/\dot U_+$ satisfying

$$\begin{array}{ccccccc}
\dot X_0 & \xleftarrow{\iota(s^+_{i_1})} & \dot X_1 & \xleftarrow{\iota(s^+_{i_2})} & \cdots & \xleftarrow{\iota(s^+_{i_m})} & \dot X_m \\
\Big\Uparrow{\scriptstyle \tilde w_\circ} & & & & & & \Big\| \\
\dot Y_0 & \xrightarrow[\iota(s^-_{i^*_1})]{} & \dot Y_1 & \xrightarrow[\iota(s^-_{i^*_2})]{} & \cdots & \xrightarrow[\iota(s^-_{i^*_m})]{} & \dot Y_m.
\end{array}$$

Let $\mathcal{Y}'_\beta$ be obtained by omitting the condition $\dot X_0 \xLeftarrow{\tilde w_\circ} \dot Y_0$. Lemma 2.1 yields isomorphisms $\mathring{\mathcal{Y}}'_\beta \cong \mathring{\mathcal{Y}}_{\tilde\beta}$ and $\mathcal{Y}'_\beta \cong \mathcal{Y}_{\tilde\beta}$. Let $\mathring{R}'_\beta$ be the quotient of $\mathring{\mathcal{Y}}'_\beta$ by the free $\dot G$-action. Then $\mathring{R}'_\beta \cong \mathring{R}_{\tilde\beta}$.

The map σ acts on the varieties $\mathring{\mathcal{Y}}'_\beta$, $\mathcal{Y}'_\beta$, and $\mathring{R}'_\beta$ termwise by acting on each $\dot X_c$ and $\dot Y_c$. Let $T'_\beta \subset \mathring{R}'_\beta$ be the image of the Deodhar torus $T_{\tilde\beta} \subset \mathring{R}_{\tilde\beta}$ under the isomorphism $\mathring{R}_{\tilde\beta} \cong \mathring{R}'_\beta$. We have the following straightforward result.

Proposition 5.2. *Applying ι termwise yields isomorphisms*

$$\mathring{\mathcal{Y}}_\beta \xrightarrow{\sim} (\mathring{\mathcal{Y}}'_\beta)^\sigma, \quad \mathcal{Y}_\beta \xrightarrow{\sim} (\mathcal{Y}'_\beta)^\sigma, \quad \mathring{R}_\beta \xrightarrow{\sim} (\mathring{R}'_\beta)^\sigma, \quad \textit{and} \quad T_\beta \xrightarrow{\sim} (T'_\beta)^\sigma. \tag{5.5}$$

5.3. **Grid minors.** Recall that we have the character and cocharacter lattices $X^*(H) := \mathrm{Hom}(H, \mathbb{C}^\times)$, $X_*(H) := \mathrm{Hom}(\mathbb{C}^\times, H)$. The map $\iota : H \to \dot H$ induces a map $\iota_* : X_*(H) \to X_*(\dot H)$ sending $\alpha_i^\vee \mapsto \sum_{i' \in \boldsymbol{i}} \dot\alpha_{i'}^\vee$ for $i \in I$, so that $\iota(\alpha_i^\vee(t)) = \prod_{i' \in \boldsymbol{i}} \dot\alpha_{i'}^\vee(t)$ for $t \in \mathbb{C}^\times$. It also induces a map $\iota^* : X^*(\dot H) \to X^*(H)$ sending $\dot\omega_{i'} \mapsto \omega_i$ for all $i \in I$ and $i' \in \boldsymbol{i}$, so that $\iota(h)^{\dot\omega_{i'}} = h^{\omega_i}$ for $h \in H$. It follows that for all $g \in G$, $v, w \in W$, $i \in I$, and $i' \in \boldsymbol{i}$, we have

$$\Delta_{v\omega_i, w\omega_i}(g) = \Delta_{\iota(v)\dot\omega_{i'}, \iota(w)\dot\omega_{i'}}(\iota(g)). \tag{5.6}$$

Let $(X_\bullet, Y_\bullet) \in \mathring{R}'_\beta$. As usual, for $c = 0, 1, \dots, m$, we denote $Z_c := Y_c^{-1} X_c$. Let $\tilde u_c := \iota(u_c)$ and $\tilde w_c := \iota(w_c)$. For $i' \in \dot I$, consider analogs of grid minors for $\mathring{R}'_\beta$:

$$\Delta'_{c,i'}(X_\bullet, Y_\bullet) = \Delta_{\tilde w_c \dot\omega_{i'}, \dot\omega_{i'}}(Z_c), \qquad \Delta'_{c,-i'}(X_\bullet, Y_\bullet) = \Delta_{\tilde w_\circ \dot\omega_{i'}, \tilde u_c^{-1} \dot\omega_{i'}}(Z_c). \tag{5.7}$$

Comparing (5.6)–(5.7) to Definition 2.9, we find that the grid minors on $\mathring{R}_\beta$ are pullbacks of the minors defined in (5.7): for $c = 0, 1, \dots, m$, $i \in \pm I$, and $i' \in \boldsymbol{i}$, we have

$$\iota^* \Delta'_{c,i'} = \Delta_{c,i}. \tag{5.8}$$

Using Corollary 2.11, we obtain the following description of chamber minors on $\mathring{R}_{\tilde\beta}$, which we denote by $\tilde\Delta_{c'}$, $c' \in \dot J_{\tilde\beta}$; cf. (5.4).

Lemma 5.3. *Let $c' \in \dot{J}_{\tilde{\beta}}$ be a solid crossing for $\tilde{\beta}$. Set $i' := i'_{c'}$ and $c := \lambda_\beta(c') \in J$. Then the isomorphism $\mathring{R}_{\tilde{\beta}} \cong \mathring{R}'_\beta$ sends the chamber minor $\tilde{\Delta}_{c'}$ to $\Delta'_{c-1,i'}$, and we have $\iota^*\Delta'_{c-1,i'} = \Delta_c$.*

5.4. **2-form.** Our next goal is to show the following result.

Lemma 5.4. *Let ω'_β be the pullback of the 2-form $\omega_{\tilde{\beta}}$ on $\mathring{R}_{\tilde{\beta}}$ under the isomorphism $\mathring{R}'_\beta \cong \mathring{R}_{\tilde{\beta}}$. We have $\iota^*\omega'_\beta = \omega_\beta$.*

Proof. Recall from (2.20)–(2.21) that we have 1-forms $L_{c,i} = \frac{1}{2}\sum_{k\in\pm I} a_{ik}\mathrm{dlog}\Delta_{c,k}$ on $\mathring{R}_\beta$ for $(c,i) \in [m]\times(\pm I)$, and that for $c\in[m]$ and $i := i_c$, we set $\omega_c(\beta) := \mathrm{sign}(i) d_i L_{c-1,i} \wedge L_{c,i}$. For $i' \in \pm\dot{I}$, introduce a 1-form $L'_{c,i'} := \frac{1}{2}\sum_{j'\in\pm\dot{I}} \dot{a}_{i'j'}\mathrm{dlog}\Delta'_{c,j'}$. By Corollary 2.11, we have

$$\omega'_\beta = \sum_{c\in J_\beta} \mathrm{sign}(i_c) \sum_{i'\in \boldsymbol{i_c}} L'_{c-1,i'} \wedge L'_{c,i'}. \tag{5.9}$$

Next, applying (5.8) and (5.2), we see that for all $c\in[m]$, $i\in\pm I$, and $i'\in\boldsymbol{i}$, we have

$$\iota^* L'_{c,i'} = \iota^*\left(\frac{1}{2}\sum_{j'\in\pm\dot{I}} \dot{a}_{i'j'}\mathrm{dlog}\Delta'_{c,j'}\right) = \frac{1}{2}\sum_{j\in\pm I}\left(\sum_{j'\in\boldsymbol{j}} \dot{a}_{i'j'}\right)\mathrm{dlog}\Delta_{c,j} = L_{c,i}. \tag{5.10}$$

The result follows by combining (5.9)–(5.10) with (5.2). ☐

5.5. **Folding seeds.** We briefly review the notion of folding seeds, following [FWZ16, Section 4.4], though translating into our conventions.

Definition 5.5. Let $\dot{\Sigma} = (\dot{T}, \dot{x}, \dot{\mathbf{d}}, \dot{\omega})$ be a seed with $\dot{\mathbf{d}} = (1,\dots,1)$, with mutable indices $\dot{J}^{\mathrm{mut}}$ and frozen indices $\dot{J}^{\mathrm{fro}}$. Let σ be a bijection acting on $\dot{J} := \dot{J}^{\mathrm{mut}} \sqcup \dot{J}^{\mathrm{fro}}$. Let J be the set of σ-orbits, and for $j \in J$, we denote the corresponding orbit by $\boldsymbol{j}$. An orbit is *mutable* (resp., *frozen*) if it consists entirely of mutable (resp., frozen) indices. The bijection σ also acts on the set of cluster variables by $\sigma(\dot{x}_{j'}) = \dot{x}_{\sigma(j')}$. We call $\dot{\Sigma}$ *weakly σ-admissible*[2] if:

1. Every orbit is either mutable or frozen.
2. The 2-form ω is invariant under the σ-action.
3. For all $a', a'' \in \dot{J}^{\mathrm{mut}}$ in the same σ-orbit, $\dot{\tilde{B}}_{a'a''} = 0$, where $\dot{\tilde{B}}$ is the exchange matrix of $\dot{\Sigma}$.

Part (1) implies a natural decomposition $J = J^{\mathrm{mut}} \sqcup J^{\mathrm{fro}}$. The map σ also acts on the torus $\dot{T}$ by permuting coordinates. Notice that $\dot{T}^\sigma$ is isomorphic to $(\mathbb{C}^\times)^{|J|}$. We denote by $\iota : \dot{T}^\sigma \hookrightarrow \dot{T}$ the inclusion map.

Definition 5.6. Suppose $\dot{\Sigma}$ is weakly σ-admissible, with notation as in Definition 5.5. The *folded seed* is a seed with index set $J = J^{\mathrm{mut}} \sqcup J^{\mathrm{fro}}$, defined as $\iota^*\dot{\Sigma} = \Sigma := (T, \mathbf{x}, \mathbf{d}, \omega)$ where

- $T = \dot{T}^\sigma$;
- $\mathbf{x} = (x_j)_{j\in J}$, where for $j \in J$, $x_j := \iota^*\dot{x}_{j'}$ for any $j' \in \boldsymbol{j}$;
- $d_j = |\boldsymbol{j}|$ for $j \in J$;
- $\omega = \iota^*\dot{\omega}$.

Note that x_j is well-defined, since $\iota^*\dot{x}_{j'} = \iota^*\dot{x}_{\sigma(j')}$ for all $j' \in \boldsymbol{j}$. The exchange matrix $\tilde{B}$ of Σ is therefore written in terms of the exchange matrix $\dot{\tilde{B}}$ of $\dot{\Sigma}$ as

$$\tilde{B}_{ab} = \sum_{a'\in\boldsymbol{a}} \dot{\tilde{B}}_{a'b'}, \quad \text{where } b' \in \boldsymbol{b} \text{ is arbitrary.} \tag{5.11}$$

In particular, if $\dot{\Sigma}$ is integral then so is Σ. For the rest of this subsection, we assume that $\dot{\Sigma}$ and Σ are integral.

[2]Our notion of weak σ-admissibility differs from the notion of admissibility in [FWZ16, Definition 4.4.1] in that we do not require that for any a', a'' in the same orbit and any k' mutable, $\dot{\tilde{B}}_{a'k'}\tilde{B}_{a''k'} \geq 0$.

For weakly σ-admissible $\dot{\Sigma}$ and $j \in J^{\mathrm{mut}}$, we denote by $\mu_{\boldsymbol{j}}\dot{\Sigma} = \mu_{\boldsymbol{j}}(\dot{\Sigma})$ the result of mutating $\dot{\Sigma}$ once at each index in $\boldsymbol{j}$, and call $\mu_{\boldsymbol{j}}$ an *orbit-mutation*. Note that $\mu_{\boldsymbol{j}}\dot{\Sigma}$ does not depend on the order of mutation. The seed $\mu_{\boldsymbol{j}}\dot{\Sigma}$ may not be weakly σ-admissible; we introduce the following notion to avoid such mutations.

Definition 5.7. Let $\dot{\Sigma}$ be a weakly σ-admissible seed and $j \in J^{\mathrm{mut}}$. We call $\mu_{\boldsymbol{j}}$ *quasi-admissible* if for all $k \in J^{\mathrm{mut}}$, we have $\tilde{B}_{k'j'}\tilde{B}_{k''j'} \geq 0$ for all $k',k'' \in \boldsymbol{k}$ and $j' \in \boldsymbol{j}$.

The name "quasi-admissible" is justified by the following proposition.

Proposition 5.8. *Let $\dot{\Sigma}$ be a weakly σ-admissible seed and $j \in J^{\mathrm{mut}}$. If $\mu_{\boldsymbol{j}}$ is quasi-admissible, then $\mu_{\boldsymbol{j}}(\dot{\Sigma})$ is weakly σ-admissible and*

$$\mu_j(\iota^*\dot{\Sigma}) \sim \iota_j^*(\mu_{\boldsymbol{j}}\dot{\Sigma}),$$

where ι_j is an inclusion of the associated tori.

Proof. We use the notation of Definitions 5.5 and 5.6. In particular, let $\Sigma := \iota^*\dot{\Sigma}$ be the folded seed on index set J. Since the exchange matrix $\dot{\tilde{B}}$ of $\dot{\Sigma}$ is skew-symmetric, it is equivalent to a quiver $\dot{Q}$; we will use the two interchangeably.

It is clear that $\mu_{\boldsymbol{j}}\dot{\Sigma}$ satisfies condition (1) of Definition 5.5. Because there are no arrows between vertices in $\boldsymbol{j}$, mutating at all vertices of $\boldsymbol{j}$ shows that the exchange matrix of $\mu_{\boldsymbol{j}}\dot{\Sigma}$ satisfies $(\mu_{\boldsymbol{j}}\dot{\tilde{B}})_{a'b'} = (\mu_{\boldsymbol{j}}\dot{\tilde{B}})_{\sigma(a')\sigma(b')}$ for $a',b' \in \dot{J}$. The assumption that $\mu_{\boldsymbol{j}}$ is quasi-admissible implies that for $k \in J^{\mathrm{mut}}$, $(\mu_{\boldsymbol{j}}\dot{\tilde{B}})_{k'k''} = 0$ for all $k',k'' \in \boldsymbol{k}$. Thus, $\mu_{\boldsymbol{j}}\dot{\Sigma}$ is weakly σ-admissible.

Let $\Sigma_1 := \mu_j(\Sigma)$ and $\Sigma_2 := \iota_j^*(\mu_{\boldsymbol{j}}\dot{\Sigma})$. Let $\mathbf{y}$ and $\mathbf{z}$ be the clusters in Σ_1 and Σ_2, respectively. For $k \neq j$, $y_k = z_k$ because both are equal to the cluster variable x_k of Σ.

To analyze the relationship between y_j and z_j, we need the following notions. Let $a,k \in J$ and choose $a' \in \boldsymbol{a}$, $k' \in \boldsymbol{k}$. We call a path $a' \to k' \to a''$ in $\dot{Q}$ a *bad path* if a',a'' are in the same orbit; condition (2) of Definition 5.7 implies that no bad path in $\dot{Q}$ begins or ends in a mutable orbit. Let $P_{k'}$ be a maximal (by inclusion) collection of arrow-disjoint bad paths with middle vertex k'.

In $\dot{\Sigma}$, for $j' \in \boldsymbol{j}$, the mutation $\dot{x}'_{j'}$ of $\dot{x}_{j'}$ is defined by the exchange relation

$$\dot{x}_{j'}\dot{x}'_{j'} = M'N' + M''N'', \quad \text{where}$$

$$N' := \prod_{(a'\to j'\to a'')\in P_{j'}} \dot{x}_{a'} \qquad \text{and} \qquad N'' := \prod_{(a'\to j'\to a'')\in P_{j'}} \dot{x}_{a''},$$

and M',M'' are the appropriate monomials in the cluster variables of $\dot{\Sigma}$. Notice that if $\dot{x}_{a'}$ appears in M' and $\dot{x}_{b'}$ appears in M'' for $a' \in \boldsymbol{a}$, $b' \in \boldsymbol{b}$, then $\boldsymbol{a} \neq \boldsymbol{b}$, by the maximality of $P_{j'}$. Notice also that by assumption, N' and N'' are monomials in the frozen variables. We set $N := \iota^*(N') = \iota^*(N'')$. Using (5.11), we have

$$\iota^*(\dot{x}'_{j'}) = N\frac{\iota^*M' + \iota^*M''}{x_j}, \qquad \text{and} \qquad \mu_j(x_j) = \frac{\iota^*M' + \iota^*M''}{x_j}. \tag{5.12}$$

This shows that the tori and the lattices spanned by the frozens of Σ_1,Σ_2 agree, and that cluster variables differ by Laurent monomials in frozens. The multipliers $\mathbf{d}$ of both seeds are the same by definition. The 2-forms of the two seeds agree by the functoriality of pullbacks. $\square$

6. Proof of theorem 4.2 for long braid moves

Fix multiply-laced braid words β,β' related by a long braid move (B3), so that

$$\beta = \beta_1 \underbrace{iji\ldots}_{m_{ij}\text{ letters}} \beta_2 = \beta_1\delta\beta_2 \quad \text{and} \quad \beta' = \beta_1 \underbrace{jij\ldots}_{m_{ij}\text{ letters}} \beta_2 = \beta_1\delta'\beta_2.$$

By Definition 4.1, we have an isomorphism $\phi : \mathring{R}_\beta \xrightarrow{\sim} \mathring{R}_{\beta'}$. The goal of this section is to show (F), (Q), and thus Theorem 4.2, for the geometrically defined seeds $\Sigma_\beta, \phi^*\Sigma_{\beta'}$ and a particular mutation Σ' of

Σ_β (defined in (6.1)). Note that we already have shown Theorems 1.1 and 4.2 for simply-laced braid varieties.

6.1. **Proof of (F) for long braid moves.** Let $\tilde\beta_1,\tilde\beta_2,\tilde\delta,\tilde\delta'$ be lifts of $\beta_1,\beta_2,\delta,\delta'$, respectively, following the conventions of Section 5.2. Define $\tilde\beta := \tilde\beta_1\tilde\delta\tilde\beta_2$, $\tilde\beta' := \tilde\beta_1\tilde\delta'\tilde\beta_2$ which are lifts of β and β', respectively. There is at least one sequence of short braid moves (B3) relating $\tilde\beta$ and $\tilde\beta'$; fix such a sequence of braid moves and denote the corresponding isomorphism of braid varieties by $\tilde\phi : \mathring{R}_{\tilde\beta} \xrightarrow{\sim} \mathring{R}_{\tilde\beta'}$.

Throughout this section, we abuse notation and use ι to denote the compositions

$$\iota : \mathring{R}_\beta \xrightarrow{\sim} (\mathring{R}'_\beta)^\sigma \hookrightarrow \mathring{R}'_\beta \xrightarrow{\sim} \mathring{R}_{\tilde\beta} \qquad \text{and} \qquad \iota : \mathring{R}_{\beta'} \xrightarrow{\sim} (\mathring{R}'_{\beta'})^\sigma \hookrightarrow \mathring{R}'_{\beta'} \xrightarrow{\sim} \mathring{R}_{\tilde\beta'}.$$

Lemma 6.1. *We have the equality* $\iota\circ\phi = \tilde\phi\circ\iota$.

Proof. Observe that the words $\tilde\delta,\tilde\delta'$ are reduced. Thus, the sequence of moves (B3) from $\tilde\delta$ to $\tilde\delta'$ fixes the weighted flags to the left and right of the indices involved in $\tilde\delta,\tilde\delta'$, and all weighted flags in between are uniquely determined; cf. Definition 4.1. □

Lemma 6.2. *We have* $\omega_\beta = \phi^*\omega_{\beta'}$*; that is,* (F) *holds for* β,β'.

Proof. We have $\omega_\beta = \iota^*\omega_{\tilde\beta} = \iota^*\tilde\phi^*\omega_{\tilde\beta'} = \phi^*\iota^*\omega_{\tilde\beta'} = \phi^*\omega_{\beta'}$, where we have used Lemma 5.4, (F) for $\tilde\beta,\tilde\beta'$, Lemma 6.1, and Lemma 5.4 again (in that order). □

6.2. **Proof of (Q) for long braid moves.** We continue to use the notation established earlier in this section. Without loss of generality, we assume that either $\delta = ijij$ (in the case when α_i,α_j form a root subsystem of type B_2 or C_2, where $|\boldsymbol{i}| = 2$ and $|\boldsymbol{j}| = 1$), or $\delta = 121212$ (in the case of $G = G_2$, where $|\mathbf{1}| = 3$ and $|\mathbf{2}| = 1$).

The words δ,δ' involve indices $r+1,\dots,r+p$, and for convenience, we decrease all indices by r so that δ,δ' are supported on $1,\dots,p$. Similarly, we assume that $\tilde\delta,\tilde\delta'$ involve indices $1,\dots,\tilde p$. We define the seed

$$\Sigma' := \pi_{\text{fold}}\circ\mu_{\text{fold}}(\Sigma), \tag{6.1}$$

where π_{fold} is a permutation and μ_{fold} is a sequence of mutations involving $1,\dots,p$. We list $\pi_{\text{fold}},\mu_{\text{fold}}$ in Table 1(a–b). In our tables, we only list the restriction of π_{fold} to the solid crossings in $1,\dots,p$. We would like to show that Σ' and $\phi^*\Sigma_{\beta'}$ are quasi-equivalent. To do so, we will eventually fold the seeds $\Sigma_{\tilde\beta}$ and $\Sigma_{\tilde\beta'}$, and then establish a chain of quasi-equivalences involving the folded seeds, Σ' and $\phi^*\Sigma_{\beta'}$.

As a first step, we fix a particular sequence S of braid moves (B3) between the lifts $\tilde\beta,\tilde\beta'$, and thus also fix $\tilde\phi$. By Theorems 1.1 and 4.2, there is a corresponding mutation sequence μ_{braid} and relabeling π_{braid} such that

$$\pi_{\text{braid}}\circ\mu_{\text{braid}}(\dot\Sigma_{\tilde\beta}) = \tilde\phi^*\dot\Sigma_{\tilde\beta'}, \tag{6.2}$$

where the seeds $\dot\Sigma_{\tilde\beta},\dot\Sigma_{\tilde\beta'}$ are the seeds denoted $\Sigma_{\tilde\beta},\Sigma_{\tilde\beta'}$ in Section 2. The sequence S of braid moves is chosen so that μ_{braid} and π_{braid} are as in Table 1(c–d).

We have the relabeling maps $\lambda_\beta : [\tilde m] \to [m]$ and $\lambda'_\beta : [\tilde m] \to [m]$ as in Section 5.2, where β,β' (resp., $\tilde\beta,\tilde\beta'$) are on m (resp., on $\tilde m$) letters. By construction, we can extend the action of σ from I to $\dot J_{\tilde\beta}$ and $\dot J_{\tilde\beta'}$. Specifically, for each letter i_c in β, σ permutes the letters $\boldsymbol{i_c}$ in the corresponding consecutive subword $\lambda_\beta^{-1}(c)$ of $\tilde\beta$, and similarly for $\tilde\beta'$.

Proposition 6.3. *Let* $C \subset J_\beta$ *be the set of indices* c *such that none of* $c' \in \boldsymbol{c}$ *is used in* μ_{braid}*, and let* $\boldsymbol{C} := \lambda_\beta^{-1}(C)$*. Let* $C' \subset J_{\beta'}$ *and* $\boldsymbol{C'} \subset \dot J_{\tilde\beta'}$ *be defined similarly. Then*

$$\iota^*(\dot\Sigma_{\tilde\beta}^{\setminus \boldsymbol{C}}) \sim \Sigma_\beta^{\setminus C} \qquad \text{and} \qquad \iota^*(\dot\Sigma_{\tilde\beta'}^{\setminus \boldsymbol{C'}}) \sim \Sigma_{\beta'}^{\setminus C'}.$$

Proof. We focus on the first quasi-equivalence. By Lemma 6.2, it suffices to show that for each $e \in J_\beta \setminus C$ and $e' \in \lambda_\beta^{-1}(e)$, we have $\iota^*(\dot x_{e'}) = x_e$. Let us fix such e,e'. Choose also $c \in [0,p]$, $k \in I$, and $k' \in \boldsymbol{k}$. It is enough to show the statement

$$\operatorname{ord}_{V_e}\Delta_{c,k} = \operatorname{ord}_{V'_{e'}}\Delta'_{c,k'}, \tag{6.3}$$

where $V'_{e'} \subset \mathcal{Y}'_\beta$ is the Deodhar hypersurface corresponding to $\dot{x}_{e'} \in \mathbb{C}[\mathring{R}_{\tilde\beta}] \cong \mathbb{C}[\mathring{R}'_\beta]$ and $\Delta'_{c,k'}$ was defined in Section 5.3.

We observe that the hollow crossings in δ,δ' (and thus in $\tilde\delta,\tilde\delta'$) have a very special form: one of δ,δ' has hollow crossings in positions $[r+1,p]$, while the other one has hollow crossings in positions $[r,p-1]$, for some r; cf. Table 1(a–b). In this case, computing $\mathrm{ord}_{V_e}\Delta_{c,k}$ is straightforward. First, suppose that $e \le r$. Then all crossings in $[p]$ to the left of e are solid. It follows from Propositions 2.20 and 2.21 and Corollary 2.11 that for $c\in[0,p]$ and $k\in I$, we have $\mathrm{ord}_{V_e}\Delta_{c,k}=1$ if $(c,k)\in\{(e-1,i_e),(e-2,i_e)\}$ and $\mathrm{ord}_{V_e}\Delta_{c,k}=0$ otherwise. Applying the same argument to compute $\mathrm{ord}_{V'_{e'}}\Delta'_{c,k'}$, we obtain (6.3). It remains to consider the case $e=p$ when the hollow crossings are in positions $[r,p-1]$. The crossings $r-1$ and $r-2$ are solid, so Proposition 2.21 implies that $\mathrm{ord}_{V_e}\Delta_{c,k}=0$ for $k=i_{r-1}$, $c<r-1$ or $k=i_{r-2}$, $c<r-2$. Here $\{i_{r-1},i_{r-2}\}=\{i,j\}$. For $c=p-1$, we have $\mathrm{ord}_{V_e}\Delta_{c,k}=\langle\omega_k,\alpha^\vee_{i_p}\rangle$ by Propositions 2.20 and 2.21. Thus, for $c\in[r,p-1]$, Lemma 2.6 implies that $\mathrm{ord}_{V_e}\Delta_{c,k}=\langle\omega_k,s_{i_{c+1}}\cdots s_{i_{p-1}}\alpha^\vee_{i_p}\rangle$. We have thus determined the values $\mathrm{ord}_{V_e}\Delta_{c,k}$ for all $(c,k)\in[0,p]\times\{i,j\}$ except for $(c,k)=(r-1,i_{r-2})$. By Corollary 2.11, we have $\mathrm{ord}_{V_e}\Delta_{r-1,i_{r-2}}=\mathrm{ord}_{V_e}\Delta_{r,i_{r-2}}$. It is clear that we have $\mathrm{ord}_{V_e}\Delta_{c,k}=0$ for $k\in I\setminus\{i,j\}$. Computing $\mathrm{ord}_{V'_{e'}}\Delta'_{c,k'}$ via a similar argument, we obtain (6.3). □

For the remainder of the section, let $C,\boldsymbol{C},C',\boldsymbol{C'}$ be as in Proposition 6.3.

The sequence μ_{braid} is ill-adapted to folding, so we find another mutation sequence μ_{lift} relating $\dot\Sigma_{\tilde\beta}$ and a relabeling π_{lift} of $\tilde\phi^*\dot\Sigma_{\tilde\beta'}$. Explicitly, μ_{lift} is a sequence of orbit-mutations lifting the sequence μ_{fold} from Table 1(a–b), and π_{lift} is given in Table 1(e–f). Part (1) of the next result generalizes [FG06, Theorem 3.5], which concerns the "all solid" case.

Proposition 6.4. *Let μ_{lift} and π_{lift} be as listed in Table 1(e–f). Then*

(1) $\pi_{\mathrm{braid}}\circ\mu_{\mathrm{braid}}(\dot\Sigma_{\tilde\beta})=\pi_{\mathrm{lift}}\circ\mu_{\mathrm{lift}}(\dot\Sigma_{\tilde\beta})$.

(2) μ_{lift} *is a sequence of quasi-admissible mutations of* $\dot\Sigma^{\setminus\boldsymbol{C}}_{\tilde\beta}$.

We delay the proof of Proposition 6.4 to the end of the section. Proposition 5.8 and part (2) of Proposition 6.4 together imply the following result.

Corollary 6.5. *We have* $\iota^*(\mu_{\mathrm{lift}}\dot\Sigma^{\setminus\boldsymbol{C}}_{\tilde\beta})\sim\mu_{\mathrm{fold}}(\iota^*\dot\Sigma^{\setminus\boldsymbol{C}}_{\tilde\beta})$.

Proof of (Q) *for long braid moves.* We have a string of quasi-equivalences:

$$\iota^*(\mu_{\mathrm{lift}}\dot\Sigma^{\setminus\boldsymbol{C}}_{\tilde\beta})\sim\mu_{\mathrm{fold}}(\iota^*\dot\Sigma^{\setminus\boldsymbol{C}}_{\tilde\beta})\sim\mu_{\mathrm{fold}}\Sigma^{\setminus C}_{\beta},$$

where the first quasi-equivalence is Corollary 6.5 and the second follows from Proposition 6.3 and Lemma 3.8. On the other hand,

$$\iota^*(\mu_{\mathrm{lift}}\dot\Sigma^{\setminus\boldsymbol{C}}_{\tilde\beta})=\iota^*(\pi^{-1}_{\mathrm{lift}}\circ\tilde\phi^*\dot\Sigma^{\setminus\boldsymbol{C'}}_{\tilde\beta'})=\pi^{-1}_{\mathrm{fold}}\circ\iota^*(\tilde\phi^*\dot\Sigma^{\setminus\boldsymbol{C'}}_{\tilde\beta'})=\pi^{-1}_{\mathrm{fold}}\circ\phi^*(\iota^*\dot\Sigma^{\setminus\boldsymbol{C'}}_{\tilde\beta'})\sim\pi^{-1}_{\mathrm{fold}}\circ\phi^*\Sigma^{\setminus C'}_{\beta'},$$

where the first equality holds by Proposition 6.4 and (6.2), the second holds by direct computation (cf. Table 1(c–f)), the third holds by Lemma 6.1, and the final quasi-equivalence follows from Proposition 6.3 and the fact that ϕ^* preserves quasi-equivalence.

Summarizing, we have that $\mu_{\mathrm{fold}}\Sigma^{\setminus C}_{\beta}$ is quasi-equivalent to (a relabeling of) $\phi^*\Sigma^{\setminus C'}_{\beta'}$. Notice that the cluster variables of $\mu_{\mathrm{fold}}\Sigma^{\setminus C}_{\beta}$, resp., $\phi^*\Sigma^{\setminus C'}_{\beta'}$ are equal to those of $\mu_{\mathrm{fold}}\Sigma_\beta$, resp., $\phi^*\Sigma_{\beta'}$. By assumption $(\mathring{R}_\beta,\Sigma_\beta)$ is a cluster variety, so Proposition 3.3 implies the cluster variables of $\mu_{\mathrm{fold}}\Sigma_\beta$ are irreducible elements of $\mathbb{C}[\mathring{R}_\beta]$. On the other hand, by Corollary 2.24, the cluster variables in $\phi^*\Sigma_{\beta'}$ are irreducible elements of $\mathbb{C}[\mathring{R}_\beta]$. Thus, the cluster variables in $\mu_{\mathrm{fold}}\Sigma_\beta$ and $\phi^*\Sigma_{\beta'}$ can differ only by units and $\mu_{\mathrm{fold}}\Sigma_\beta$ is quasi-equivalent to (a relabeling of) $\phi^*\Sigma_{\beta'}$. □

Proof of Proposition 6.4. Recall that $\tilde\beta=\tilde\beta_1\tilde\delta\tilde\beta_2$ and $\tilde\beta'=\tilde\beta_1\tilde\delta'\tilde\beta_2$, and that we index the crossings of $\tilde\delta$ by $1,\dots,\tilde p$. Let $J:=J_\beta\setminus C$ be the set of indices which are mutated in μ_{fold}.

B_2/C_2:

$\delta \to \delta'$	μ_{fold}	π_{fold}
$ijij \to jiji$	$\mu_{(4,3,4)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
$ij\underline{i}j \to jij\underline{i}$	μ_4	$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \end{pmatrix}$
$iji\underline{j} \to ji\underline{j}i$	μ_3	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$
$ij\underline{ij} \to j\underline{ij}i$	id	$\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$
$i\underline{ji}j \to ji\underline{ji}$	id	$\begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}$
$\underline{iji}j \to j\underline{iji}$	id	$\begin{pmatrix} 4 \\ 1 \end{pmatrix}$
$i\underline{jij} \to \underline{jij}i$	id	$\begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(a)

G_2:

$\delta \to \delta'$	μ_{fold}	π_{fold}
$121212 \to 212121$	$\mu_{(6,3,4,6,5,6,3,4,5,6)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$
$12121\underline{2} \to 2121\underline{2}1$	$\mu_{(3,4,5,3,4,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 6 \end{pmatrix}$
$1212\underline{1}2 \to 21212\underline{1}$	$\mu_{(6,3,4,6,3,4)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 6 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}$
$1212\underline{12} \to 212\underline{12}1$	$\mu_{(4,3,4)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 6 & 3 \end{pmatrix}$
$121\underline{21}2 \to 2121\underline{21}$	$\mu_{(6,3,6)}$	$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 2 & 1 & 4 & 3 \end{pmatrix}$
$121\underline{212} \to 21\underline{212}1$	μ_3	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 6 \end{pmatrix}$
$12\underline{121}2 \to 212\underline{121}$	μ_6	$\begin{pmatrix} 1 & 2 & 6 \\ 2 & 1 & 3 \end{pmatrix}$
$12\underline{1212} \to 2\underline{1212}1$	id	$\begin{pmatrix} 1 & 2 \\ 6 & 1 \end{pmatrix}$
$1\underline{2121}2 \to 21\underline{2121}$	id	$\begin{pmatrix} 1 & 6 \\ 2 & 1 \end{pmatrix}$
$\underline{12121}2 \to 2\underline{12121}$	id	$\begin{pmatrix} 6 \\ 1 \end{pmatrix}$
$1\underline{21212} \to \underline{21212}1$	id	$\begin{pmatrix} 1 \\ 6 \end{pmatrix}$

(b)

$[p] \setminus \dot{J}_{\tilde{\beta}}$	μ_{braid}	π_{braid}
$\emptyset$	$\mu_{(4,5,6,4)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 5 & 4 & 6 \end{pmatrix}$
$\{4,5\}$	μ_6	$\begin{pmatrix} 1 & 2 & 3 & 6 \\ 3 & 1 & 2 & 4 \end{pmatrix}$
$\{6\}$	$\mu_{(4,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 6 & 5 \end{pmatrix}$
$\{4,5,6\}$	id	$\begin{pmatrix} 1 & 2 & 3 \\ 6 & 5 & 1 \end{pmatrix}$
$\{3,4,5\}$	id	$\begin{pmatrix} 1 & 2 & 6 \\ 2 & 3 & 1 \end{pmatrix}$
$\{1,2,3,4,5\}$	id	$\begin{pmatrix} 6 \\ 1 \end{pmatrix}$
$\{3,4,5,6\}$	id	$\begin{pmatrix} 1 & 2 \\ 6 & 5 \end{pmatrix}$

(c)

$[p] \setminus \dot{J}_{\tilde{\beta}}$	μ_{braid}	π_{braid}
$\emptyset$	$\mu_{(8,9,5,6,7,8,11,10,9,5,12,6,10,8,5,11)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 9 & 6 & 5 & 7 & 11 & 12 & 8 & 10 \end{pmatrix}$
$\{12\}$	$\mu_{(8,5,6,7,8,11,9,5,6,10,8,5,11)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 1 & 12 & 6 & 5 & 7 & 11 & 10 & 8 \end{pmatrix}$
$[9,11]$	$\mu_{(8,5,6,7,8,12,5,6,8,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 12 \\ 2 & 3 & 4 & 1 & 9 & 6 & 5 & 7 & 8 \end{pmatrix}$
$[9,12]$	$\mu_{(6,7,5,6,8,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 4 & 2 & 1 & 11 & 12 & 5 & 10 \end{pmatrix}$
$[8,11]$	$\mu_{(6,5,7,12,6,5)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 5 & 8 \end{pmatrix}$
$[8,12]$	$\mu_{(5,6,7)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 10 & 11 & 12 \end{pmatrix}$
$[5,11]$	μ_{12}	$\begin{pmatrix} 1 & 2 & 3 & 4 & 12 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix}$
$[5,12]$	id	$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 11 & 12 & 1 \end{pmatrix}$
$[4,11]$	id	$\begin{pmatrix} 1 & 2 & 3 & 12 \\ 2 & 3 & 4 & 1 \end{pmatrix}$
$[2,12]$	id	$\begin{pmatrix} 1 \\ 12 \end{pmatrix}$
$[1,11]$	id	$\begin{pmatrix} 12 \\ 1 \end{pmatrix}$

(d)

$[p] \setminus \dot{J}_{\tilde{\beta}}$	μ_{lift}	π_{lift}
$\emptyset$	$\mu_{(6,4,5,6)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 6 & 5 & 4 \end{pmatrix}$

(e)

$[p] \setminus \dot{J}_{\tilde{\beta}}$	μ_{lift}	π_{lift}
$\emptyset$	$\mu_{(12,5,6,7,8,12,9,10,11,12,5,6,7,8,9,10,11,12)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 & 10 & 11 & 12 & 9 \end{pmatrix}$
$\{12\}$	$\mu_{(5,6,7,8,9,10,11,5,6,7,8,9,10,11)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 & 10 & 11 & 12 \end{pmatrix}$
$[9,11]$	$\mu_{(12,5,6,7,8,12,5,6,7,8)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 & 9 \end{pmatrix}$
$[9,12]$	$\mu_{(8,5,6,7,8)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 1 & 10 & 11 & 12 & 5 \end{pmatrix}$
$[8,11]$	$\mu_{(5,6,7,12,5,6,7)}$	$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 12 \\ 2 & 3 & 4 & 1 & 6 & 7 & 8 & 5 \end{pmatrix}$

(f)

TABLE 1. The mutation sequences μ_{fold}, μ_{braid}, μ_{lift}, and the relabelings π_{fold}, π_{braid}, π_{lift} used in Section 6. Hollow crossings are underlined. We denote $\mu_{(a_1,\dots,a_r)} := \mu_{a_1} \circ \cdots \circ \mu_{a_r}$. For the case B_2/C_2, we denote $\boldsymbol{i} = \{i', i''\}$, $\boldsymbol{j} = \{j'\}$, $\tilde{\delta} = i'i''j'i'i''j'$, $\tilde{\delta}' = j'i'i''j'i'i''$; for G_2, we denote $\mathbf{1} = \{\dot{1}, \dot{3}, \dot{4}\}$, $\mathbf{2} = \{\dot{2}\}$, $\tilde{\delta} = \dot{1}\dot{3}\dot{4}\dot{2}\dot{1}\dot{3}\dot{4}\dot{2}\dot{1}\dot{3}\dot{4}\dot{2}$, $\tilde{\delta}' = \dot{2}\dot{1}\dot{3}\dot{4}\dot{2}\dot{1}\dot{3}\dot{4}\dot{2}\dot{1}\dot{3}\dot{4}$. In (e) and (f), the cases where μ_{lift} and μ_{braid} coincide are omitted; we define $\pi_{\text{lift}} := \pi_{\text{braid}}$ in those cases.

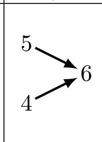
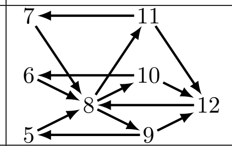
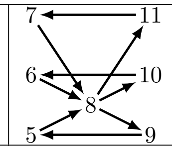
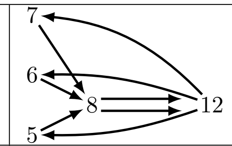
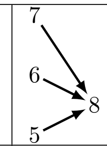
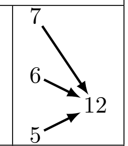
Type	B_2/C_2	G_2				
$\dot{Q}_{\mathrm{res}}$						
$[p]\setminus\dot{J}_{\tilde{\beta}}$	$\emptyset$	$\emptyset$	$\{12\}$	$[9,11]$	$[9,12]$	$[8,11]$

TABLE 2. The quivers $\dot{Q}_{\mathrm{res}}$ from Proposition 6.4 listed in the same order as in Table 1(e–f). In the cases where $\mu_{\mathrm{braid}}=\mu_{\mathrm{lift}}$, $\dot{Q}_{\mathrm{res}}$ has no arrows.

We show part (1). By [GSV08, Theorem 4], a seed in $\mathcal{A}(\dot{\Sigma}_{\tilde{\beta}})$ is uniquely determined by its cluster, so we need only check (1) at the level of cluster variables. This is easy to check for the cluster variables $\{x_c : c\in C\}$ which are not touched by either mutation sequence.

Let $\dot{\Sigma}_{\tilde{\beta}}=(\dot{\mathbf{x}},\dot{Q})$, and let $\dot{Q}_{\mathrm{res}}$ be the induced subquiver of $\dot{Q}$ on $\dot{J}:=\lambda_{\tilde{\beta}}^{-1}(J)$. Let $\dot{Q}^{\mathrm{fr}}_{\mathrm{res}}$ be the *framing* of $\dot{Q}_{\mathrm{res}}$; the extended exchange matrix of $\dot{Q}^{\mathrm{fr}}_{\mathrm{res}}$ is thus of size $2|\dot{J}|\times|\dot{J}|$ and the bottom $|\dot{J}|\times|\dot{J}|$ submatrix is the identity. We denote by $\dot{\Sigma}_{\mathrm{res}}$ the seed $(\dot{\mathbf{y}},\dot{Q}^{\mathrm{fr}}_{\mathrm{res}})$ for some cluster $\dot{\mathbf{y}}$. By [FZ07, Theorem 3.7], to show (1), it suffices to check that

$$\pi_{\mathrm{braid}}\circ\mu_{\mathrm{braid}}(\dot{\Sigma}_{\mathrm{res}})=\pi_{\mathrm{lift}}\circ\mu_{\mathrm{lift}}(\dot{\Sigma}_{\mathrm{res}}). \tag{6.4}$$

The relevant cluster variables in $\mu_{\mathrm{braid}}(\dot{\Sigma}_{\tilde{\beta}})$ and $\mu_{\mathrm{fold}}(\dot{\Sigma})$ can then be obtained from those in (6.4) by specialization.

To check (6.4), recall the description of the orders of vanishing of the cluster variables in $\dot{J}$ from the proof of Proposition 6.3. This description only depends on which crossings in $\tilde{\delta},\tilde{\delta}'$ are hollow, which in turn is determined by which crossings in $\tilde{\beta}_2$ are hollow. This implies that to compute $\dot{Q}_{\mathrm{res}}$, we may assume $\tilde{\beta}_2$ is a type A_3 braid word (in the B_2/C_2 case) or a type D_4 braid word (in the G_2 case) consisting entirely of hollow crossings. Applying the algorithm from Section 7 (to the simply-laced braids $\tilde{\beta},\tilde{\beta}'$; cf. Remark 7.3), we get that $\dot{Q}_{\mathrm{res}}$ is as displayed in Table 2. Equation (6.4) may then be verified by computer.

Part (2) is also established by direct computation in $\dot{Q}_{\mathrm{res}}$. □

This completes the proof of Theorem 4.2 for long braid moves.

6.3. **Finishing the proof.** We now have shown Theorem 4.2 for all braid moves. Theorem 1.1 for multiply-laced G follows by the argument in Section 4.9. Repeating the proof of Proposition 4.12, we have the following.

Proposition 6.6. *Suppose β,β' are related by a braid move* (B1)–(B4). *The seeds $\Sigma_\beta,\Sigma^*_{\beta'}$ are mutation equivalent (up to relabeling cluster variables).*

Continuing Remark 4.11, we obtain the following.

Corollary 6.7. *The seeds Σ_β are really full rank for all β.*

Combining Corollary 6.7 with the proof of Proposition 3.6 and [LS16] (see also [GLSBS22, Section 10]), we obtain the following. (While [LS16] work in the skew-symmetric setting, the curious Lefschetz theorem therein generalizes to the skew-symmetrizable case.)

Theorem 6.8. *Even-dimensional double braid varieties $\mathring{R}_\beta$ satisfy the curious Lefschetz property and thus curious Poincaré symmetry. Odd-dimensional $\mathring{R}_\beta$ satisfy curious Poincaré symmetry.*

7. Combinatorial algorithm

The exchange matrices for our seeds Σ_β are defined via Deodhar geometry. Let $e\in J_\beta$ and $c\leq e$. We give an algorithm to compute the order of vanishing of Δ_c on V_e, which determines our cluster algebras.

The function $h_c^{\pm}$ of Section 2.4 is an H-valued character on T_β. We may thus write $h_c^{\pm} = \prod_{e\in J_\beta}\gamma^{\pm}_{\beta,c,e}(x_e)$, where $\gamma^{\pm}_{\beta,c,e}$ are cocharacters of H satisfying $\gamma^{-}_{\beta,c,e} = u_c\cdot\gamma^{+}_{\beta,c,e}$.

Lemma 7.1.

1. *Suppose β' is obtained from β by removing the first $c-1$ letters. Then $\gamma^{\pm}_{\beta,c,e} = \gamma^{\pm}_{\beta',1,e-c+1}$.*
2. *Suppose β' is obtained from β by doing non-mutation moves* (B1)–(B4) *involving indices greater than c, and let e' be the image of e under the resulting identification of cluster seeds. Then we have $\gamma^{\pm}_{\beta,c,e} = \gamma^{\pm}_{\beta',c,e'}$.*
3. *Suppose β' is obtained from β by removing solid crossings greater than e. Then $\gamma^{\pm}_{\beta,c,e} = \gamma^{\pm}_{\beta',c,e}$.*

Proof. Part (1) follows from Lemma 2.25, and part (2) is immediate from the definitions. We prove (3). Suppose β has a solid crossing $e' > e$. Using (1) and (2), we may assume that $e\in J_\beta^{\mathrm{mut}}$, that e' is the largest solid crossing, and that $i_{e'} = i_{e'+1}\in I$. Let β'' be obtained from β by removing the letter $i_{e'}$ from β. Let $W\subset \mathring{R}_\beta$ be the open subset obtained by removing the Deodhar hypersurface $V_{e'}$, if e' is mutable; otherwise, let $W := \mathring{R}_\beta$. The projection $\pi : W\to \mathring{R}_{\beta''}$ given by forgetting the flags $(X_{e'},Y_{e'})$ is a fiber bundle with fiber $\mathbb{C}^\times$. We have $\pi^*(\Delta_c^{\beta''}) = \Delta_c^{\beta}$ and π maps $V_e^\beta\cap W$ surjectively onto $V_e^{\beta''}$. (Here we use the superscript to refer to the braid variety on which Δ_c and V_e are defined.) Since both $V_e^\beta\subset\mathring{R}_\beta$ and $V_e^{\beta''}\subset\mathring{R}_{\beta''}$ are hypersurfaces, it follows that the order of vanishing of Δ_c^β on V_e^β is equal to that of $\Delta_c^{\beta''}$ on $V_e^{\beta''}$. Repeating this argument, we obtain (3). □

Using Lemma 7.1, we may assume that $c = 1$, $e = m-1$ and $\beta = (-\mathbf{b}^{*\mathrm{rev}})\mathbf{a}kk$, where $\mathbf{a},\mathbf{b}$ are words in I, and $(-\mathbf{b}^{*\mathrm{rev}})$ is obtained by reversing $\mathbf{b}$ and applying the map $i\mapsto -i^*$ to each letter, and $k = i_e = i_m\in I$. Define

$$\gamma(\mathbf{a},k,\mathbf{b}) := \gamma^{+}_{(-\mathbf{b}^{*\mathrm{rev}})\mathbf{a}kk,1,m-1},$$

and let a and b denote the Demazure product of $\mathbf{a}$ and $\mathbf{b}$, respectively.

Proposition 7.2. *Suppose that Theorems 1.1 and 4.2 hold for G. Then the cocharacter $\gamma(\mathbf{a},k,\mathbf{b})$ satisfies, and is recursively defined by the following properties.*

- I) *We have $\gamma(\mathbf{a},k,\mathbf{b}) = 0$ if $a = w_\circ$ or $b = w_\circ$.*
- II) *The cocharacter $\gamma(\mathbf{a},k,\mathbf{b}) = \gamma(a,k,b)$ only depends on the Demazure products a,b.*
- III) *We have $\gamma(\mathbf{a},k,\emptyset) = a\alpha_k^\vee$ if $as_k > a$ and $\gamma(\mathbf{a},k,\emptyset) = 0$ if $as_k < a$.*
- IV) *Suppose that $\mathbf{a},\mathbf{b}$ are reduced. We let $\mathbf{a}' = i\mathbf{a}$ and $\mathbf{b} = \mathbf{b}'j$, where the Demazure products satisfy $a' = s_i a > a$ and $b = b's_j > b'$.*
  1. *If $a' * s_k * b > a * s_k * b$, then $\gamma(\mathbf{a},k,\mathbf{b}) = s_i\cdot\gamma(\mathbf{a}',k,\mathbf{b})$.*
  2. *If $a * s_k * b > a * s_k * b'$, then $\gamma(\mathbf{a},k,\mathbf{b}) = \gamma(\mathbf{a},k,\mathbf{b}')$.*
  3. *If $w := a' * s_k * b = a * s_k * b = a * s_k * b'$, write $\alpha^\vee = \alpha_i^\vee$ and $\beta^\vee = -w^{-1}\cdot\alpha_j^\vee$.*
     - (3a) *Suppose that $\alpha^\vee \neq \beta^\vee$. Then $\gamma(\mathbf{a}',k,\mathbf{b}) = \gamma(\mathbf{a},k,\mathbf{b}') + x\alpha^\vee + y\beta^\vee$ for $x,y\in\mathbb{Z}$, and we have $\gamma(\mathbf{a},k,\mathbf{b}) = \gamma(\mathbf{a},k,\mathbf{b}') + y\beta^\vee$.*
     - (3b) *Suppose that $\alpha^\vee = \alpha_i^\vee = \beta^\vee$. Then $\gamma(\mathbf{a},k,\mathbf{b}) - \gamma(\mathbf{a}',k,\mathbf{b}')\in\mathbb{Z}\alpha_i^\vee$, and*

$$\langle\omega_i,\gamma(\mathbf{a},k,\mathbf{b})\rangle = -\langle\omega_i,\gamma(\mathbf{a}',k,\mathbf{b}')\rangle + \min\Big(\langle\omega_i,\gamma(\mathbf{a}',k,\mathbf{b})\rangle + \langle\omega_i,\gamma(\mathbf{a},k,\mathbf{b}')\rangle, -\sum_{l\neq i} a_{il}\langle\omega_l,\gamma(\mathbf{a}',k,\mathbf{b}')\rangle\Big).$$

Proof. We first argue that the stated properties determine $\gamma(\mathbf{a},k,\mathbf{b})$. By I) and III), we know $\gamma(\mathbf{a},k,\mathbf{b})$ when $a = w_\circ$ or $b = \mathrm{id}$. If $a\neq w_\circ$ and $b\neq\mathrm{id}$, property IV) allows us to express $\gamma(\mathbf{a},k,\mathbf{b})$ in terms of $\gamma(\mathbf{a}',k,\mathbf{b}),\gamma(\mathbf{a},k,\mathbf{b}'),\gamma(\mathbf{a}',k,\mathbf{b}')$ where $a' > a$ and $b' < b$. Thus, all values of $\gamma(\mathbf{a},k,\mathbf{b})$ are determined. We now prove I)–IV).

Suppose that $a = w_\circ$ or $b = w_\circ$. Then a generic point $(X_\bullet,Y_\bullet)$ in V_e satisfies $Y_0 \xRightarrow{w_\circ} X_0$. It follows that Δ_1 does not vanish on V_e, establishing I).

We show II). It is clear that if $as_k < a$ then $\gamma(\mathbf{a},k,\mathbf{b}) = 0$. Suppose that $as_k > a$. We apply the moves $\beta\xrightarrow{(B4)}(-\mathbf{b}^{*\mathrm{rev}})\mathbf{a}k(-k^*)\xrightarrow{(B1)}(-\mathbf{b}^{*\mathrm{rev}})\mathbf{a}(-k^*)k\xrightarrow{(B1)}\cdots\xrightarrow{(B1)}(-\mathbf{b}^{*\mathrm{rev}})(-k^*)\mathbf{a}k$. Since $as_k > a$, these are non-mutation moves, and thus part (2) of Lemma 7.1 applies. We may now remove the solid crossings from $\mathbf{a}$ using part (3) of Lemma 7.1, and then reverse the procedure to put β back into its original form with solid crossings removed from $\mathbf{a}$.

We prove III). If $as_k < a$, we have already shown that $\gamma(\mathbf{a},k,\emptyset) = 0$. Assume $as_k > a$. When $a = \mathrm{id}$, the result follows from Proposition 2.20. For $a \neq \mathrm{id}$, we apply induction, II), and the hollow case of Lemma 2.6.

We prove IV). For (1), adding the letter i in front of $(-\mathbf{b}^{*\mathrm{rev}})\mathbf{a}kk$ produces a new hollow crossing, and the claim follows from Lemmas 2.6 and 7.1. Similarly, for (2), the letter $-j^*$ is a hollow crossing in $(-\mathbf{b}^{*\mathrm{rev}})\mathbf{a}kk$. Case (3) holds if both i and $-j^*$ are solid crossings in the word $i(-j^*)(-(\mathbf{b}')^{*\mathrm{rev}})\mathbf{a}kk$. If swapping the order of i and $-j^*$ is a non-mutation move then we are in Case (3a), and the claim follows from Lemma 2.6 and the linear independence of $\alpha^\vee$ and $\beta^\vee$. If swapping the order of i and $-j^*$ is a mutation, then we are in Case (3b), and the claim follows from (4.8) and the assumption that Theorems 1.1 and 4.2 have been shown for G. □

The algorithm has been implemented at [Gal23], where some examples can be found.

Remark 7.3. The logical dependencies in our proof are summarized as follows. In Section 4, we give a complete proof of Theorems 1.1 and 4.2 for the case when G is simply-laced. Thus, Proposition 7.2 applies in this case. The proof for the case when G is multiply-laced is given in Section 6; it depends on Proposition 7.2, but only invokes it for the simply-laced group $\dot{G}$.

The following result follows from our algorithm, but we have been unable to show it directly from Deodhar geometry.

Corollary 7.4. *Let $\iota : \mathring{R}_\beta \hookrightarrow \mathring{R}_{\tilde{\beta}}$ and $\lambda_\beta : [\tilde{m}] \to [m]$ be as in Sections 5.2 and 6. Then for each $e \in J_\beta$ and $e' \in \lambda_\beta^{-1}(e)$, we have $\iota^*(\dot{x}_{e'}) = x_e$.*

Proof. With notation as in the proof of Proposition 6.3, it suffices to show that $\mathrm{ord}_{V_e} \Delta_{c,k} = \mathrm{ord}_{V'_{e'}} \Delta'_{c,k'}$ for $c \leq e$, $k \in I$, and $k' \in \boldsymbol{k}$. This follows from applying Proposition 7.2 to $\mathring{R}_\beta$ and $\mathring{R}_{\tilde{\beta}}$ separately. □

References

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[BIRS09] A. B. Buan, O. Iyama, I. Reiten, and J. Scott. Cluster structures for 2-Calabi-Yau categories and unipotent groups. *Compos. Math.*, 145(4):1035–1079, 2009.

[CGG+22] Roger Casals, Eugene Gorsky, Mikhail Gorsky, Ian Le, Linhui Shen, and José Simental. Cluster structures on braid varieties. `arXiv:2207.11607v1`, 2022.

[CGGS20] Roger Casals, Eugene Gorsky, Mikhail Gorsky, and José Simental. Algebraic Weaves and Braid Varieties. `arXiv:2012.06931v1`, 2020.

[CGGS21] Roger Casals, Eugene Gorsky, Mikhail Gorsky, and José Simental. Positroid Links and Braid varieties. `arXiv:2105.13948v1`, 2021.

[CK22] Peigen Cao and Bernhard Keller. On Leclerc's conjectural cluster structures for open Richardson varieties. `arXiv:2207.10184v1`, 2022.

[Deo85] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.*, 79(3):499–511, 1985.

[FG06] V. V. Fock and A. B. Goncharov. *Cluster χ-varieties, amalgamation, and Poisson—Lie groups*, pages 27–68. Birkhäuser Boston, Boston, MA, 2006.

[FG09] Vladimir V. Fock and Alexander B. Goncharov. Cluster ensembles, quantization and the dilogarithm. *Ann. Sci. Éc. Norm. Supér. (4)*, 42(6):865–930, 2009.

[FWZ16] Sergey Fomin, Lauren Williams, and Andrei Zelevinsky. Introduction to Cluster Algebras. Chapters 1-3. `arXiv:1608.05735v4`, 2016.

[FZ99] Sergey Fomin and Andrei Zelevinsky. Double Bruhat cells and total positivity. *J. Amer. Math. Soc.*, 12(2):335–380, 1999.

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529 (electronic), 2002.

[FZ07] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. IV. Coefficients. *Compos. Math.*, 143(1):112–164, 2007.

[Gal23] Pavel Galashin. Braid variety cluster structures program. `https://www.math.ucla.edu/~galashin/DoubleBraidCluster/tutorial.html`, 2023.

[GL19] Pavel Galashin and Thomas Lam. Positroid varieties and cluster algebras. *Ann. Sci. Éc. Norm. Supér.,* to appear. `arXiv:1906.03501v2`, 2019.

[GL20] Pavel Galashin and Thomas Lam. Positroids, knots, and q,t-Catalan numbers. `arXiv:2012.09745v2`, 2020.

[GL22] Pavel Galashin and Thomas Lam. Plabic links, quivers, and skein relations. `arXiv:2208.01175v1`, 2022.

[GLS06] Christof Geiß, Bernard Leclerc, and Jan Schröer. Rigid modules over preprojective algebras. *Invent. Math.*, 165(3):589–632, 2006.

[GLS13] Christof Geiss, Bernard Leclerc, and Jan Schröer. Factorial cluster algebras. *Doc. Math.*, 18:249–274, 2013.

[GLSBS22] Pavel Galashin, Thomas Lam, Melissa Sherman-Bennett, and David Speyer. Braid variety cluster structures, I: 3D plabic graphs. `arXiv:2210.04778v1`, 2022.

[GSV08] Michael Gekhtman, Michael Shapiro, and Alek Vainshtein. On the properties of the exchange graph of a cluster algebra. *Math. Res. Lett.*, 15(2):321–330, 2008.

[HRV08] Tamás Hausel and Fernando Rodriguez-Villegas. Mixed Hodge polynomials of character varieties. *Invent. Math.*, 174(3):555–624, 2008. With an appendix by Nicholas M. Katz.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Springer–Verlag, New York–Heidelberg, 1975. Graduate Texts in Mathematics, No. 21.

[Ing19] Grace Ingermanson. *Cluster Algebras of Open Richardson Varieties*. ProQuest LLC, Ann Arbor, MI, 2019. Thesis (Ph.D.)–University of Michigan.

[KLS13] Allen Knutson, Thomas Lam, and David E. Speyer. Positroid varieties: juggling and geometry. *Compos. Math.*, 149(10):1710–1752, 2013.

[KLS14] Allen Knutson, Thomas Lam, and David E. Speyer. Projections of richardson varieties. *Crelle's journal*, 687:133–157, 2014.

[Lec16] B. Leclerc. Cluster structures on strata of flag varieties. *Adv. Math.*, 300:190–228, 2016.

[LS16] Thomas Lam and David E. Speyer. Cohomology of cluster varieties. I. Locally acyclic case. `arXiv:1604.06843v1`, 2016.

[Lus94] G. Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pages 531–568. Birkhäuser Boston, Boston, MA, 1994.

[Mel19] Anton Mellit. Cell decompositions of character varieties. `arXiv:1905.10685v1`, 2019.

[Mén22] Etienne Ménard. Cluster algebras associated with open Richardson varieties: an algorithm to compute initial seeds. `arXiv:2201.10292v1`, 2022.

[MR04] R. J. Marsh and K. Rietsch. Parametrizations of flag varieties. *Represent. Theory*, 8:212–242, 2004.

[Mul13] Greg Muller. Locally acyclic cluster algebras. *Adv. Math.*, 233:207–247, 2013.

[Mul14] Greg Muller. $\mathcal{A} = \mathcal{U}$ for locally acyclic cluster algebras. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 10:Paper 094, 8, 2014.

[Rie98] Konstanze Christina Rietsch. *Total Positivity and Real Flag Varieties*. Ph.D. thesis, Massachusetts Institute of Technology, 1998.

[SSB22] Khrystyna Serhiyenko and Melissa Sherman-Bennett. Leclerc's conjecture on a cluster structure for type A Richardson varieties. `arXiv:2210.13302v1`, 2022.

[SW21] Linhui Shen and Daping Weng. Cluster structures on double Bott-Samelson cells. *Forum Math. Sigma*, 9:Paper No. e66, 89, 2021.

[WY07] Ben Webster and Milen Yakimov. A Deodhar-type stratification on the double flag variety. *Transform. Groups*, 12(4):769–785, 2007.

[Zel00] Andrei Zelevinsky. Connected components of real double Bruhat cells. *Internat. Math. Res. Notices*, (21):1131–1154, 2000.

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