POSITROID CATALAN NUMBERS

PAVEL GALASHIN AND THOMAS LAM

Abstract. Given a permutation f, we study the *positroid Catalan number* C_f defined to be the torus-equivariant Euler characteristic of the associated open positroid variety. We introduce a class of *repetition-free permutations* and show that the corresponding positroid Catalan numbers count Dyck paths avoiding a convex subset of the rectangle. We show that any convex subset appears in this way. Conjecturally, the associated q, t-polynomials coincide with the *generalized q, t-Catalan numbers* that recently appeared in relation to the shuffle conjecture, flag Hilbert schemes, and Khovanov–Rozansky homology of Coxeter links.

Contents

1. Introduction

Open positroid varieties are remarkable subvarieties of the Grassmannian introduced by Knutson–Lam–Speyer in [KLS13], building on the work of Postnikov [Pos06]. They appear in numerous contexts: total positivity, Schubert calculus, Poisson geometry, scattering amplitudes, cluster algebras, and so on [Lus98, BGY06, AHBC$^+$16, GL19]. In a recent paper [GL20], we further connected positroid varieties to knot invariants, showing a relation between the cohomology of an open positroid variety Π_f° and *Khovanov–Rozansky homology* [KR08a, KR08b] of an associated *positroid link* $\hat{\beta}_f$.

1.1. **Positroid Catalan numbers.** Let $\mathrm{Cyc}(n)$ denote the set of n-cycles in the symmetric group S_n. To each $\bar{f} \in \mathrm{Cyc}(n)$ we associate a *bounded affine permutation* $f : \mathbb{Z} \to \mathbb{Z}$. The map f is uniquely determined by the conditions $f(i+n) = f(i) + n$ and $i < f(i) < i + n$ for all $i \in \mathbb{Z}$, together with $f(i) \equiv \bar{f}(i) \pmod n$ for all $1 \leq i \leq n$. Taking f modulo n recovers

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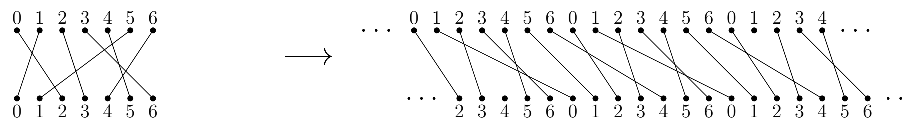

permutation $\bar{f} \in \mathrm{Cyc}(n)$ bounded affine permutation $f \in \Theta_{k,n}$

FIGURE 1. Drawing an n-cycle (left) in affine notation (right).

$\bar{f}$, and thus f and $\bar{f}$ determine each other. See Figure 1 for an example and Section 2.1 for further details.

For a bounded affine permutation f, let $\Pi_f^\circ \subset \mathrm{Gr}(k,n)$ denote the corresponding *open positroid variety* of the Grassmannian. Let $T \subset \mathrm{PGL}(n)$ denote the natural torus of diagonal matrices acting on $\mathrm{Gr}(k,n)$.

Definition 1.1. For an n-cycle $\bar{f} \in \mathrm{Cyc}(n)$, define the *positroid Catalan number*

$$C_f := \chi_T(\Pi_f^\circ)$$

to be the torus-equivariant Euler characteristic of Π_f°.

These numbers are positive integers which can be computed via an explicit combinatorial recurrence; see Section 3.3. Additionally, they have the following interpretations:

(a) C_f is equal to the number of *maximal f-Deograms* introduced in [GL20], which are in bijection with a class of distinguished subexpressions in the sense of Deodhar [Deo85]; see Section 7.4.
(b) C_f is equal to the $q = 1$ evaluation of the polynomial $\tilde{R}_f(q) := R_f(q)/(q-1)^{n-1}$, where $R_f(q)$ is the *Kazhdan–Lusztig R-polynomial* [KL79, KL80]; see Section 3. By [GL20, Theorem 1.11], $\tilde{R}_f(q)$ may be obtained as a coefficient of the *HOMFLY polynomial* [FYH+85, PT87] of $\hat{\beta}_f$.
(c) C_f is equal to the $q = t = 1$ evaluation of the *mixed Hodge polynomial* $\mathcal{P}(\Pi_f^\circ/T; q, t)$. By [GL20], $\mathcal{P}(\Pi_f^\circ/T; q, t)$ is equal to a coefficient of the Khovanov–Rozansky triply-graded link invariant of $\hat{\beta}_f$.

We showed in [GL20], using results on torus knots that date back to Jones [Jon87], that for $\gcd(k,n) = 1$ and f given by $f(i) = i + k$, the positroid Catalan number C_f recovers the famous (rational) Catalan number $C_{k,n-k} := \frac{1}{n}\binom{n}{k}$ which counts Dyck paths above the diagonal inside a $k \times (n-k)$ rectangle. This explains the nomenclature "positroid Catalan number" and points towards a deeper investigation of positroid Catalan numbers from a combinatorial perspective. In this work, we make the first step in this direction.

1.2. **Repetition-free permutations.** To each n-cycle $\bar{f} \in \mathrm{Cyc}(n)$ we associate an *inversion multiset* $\Gamma(f)$, and we introduce a natural class of *repetition-free* permutations for which the multiset $\Gamma(f)$ has no repeated elements. Let $[n] := \{1, 2, \dots, n\}$ and for $\bar{f} \in \mathrm{Cyc}(n)$, set

$$k(\bar{f}) := \#\{i \in [n] \mid \bar{f}(i) < i\}. \tag{1.1}$$

For $1 \le k \le n-1$, we denote

$$\Theta_{k,n} := \{f \mid \bar{f} \in \mathrm{Cyc}(n) \text{ is such that } k(\bar{f}) = k\}.$$

The set $\mathrm{Cyc}(n)$ is in bijection with $\bigsqcup_{k=1}^{n-1} \Theta_{k,n}$.

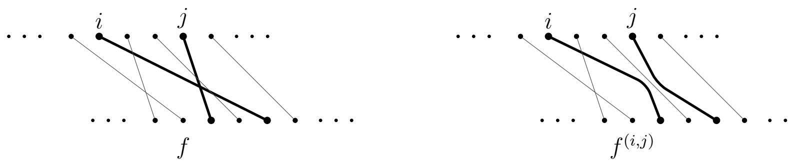

FIGURE 2. Resolving a crossing (i,j) of f.

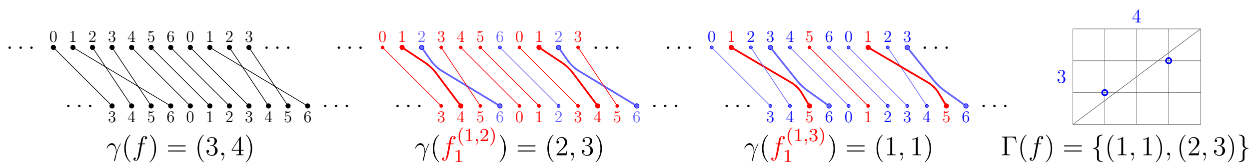

FIGURE 3. Computing $\Gamma(f)$ for $f \in \Theta_{3,7}$.

For $f \in \Theta_{k,n}$, let

$$k(f) = k(\bar{f}) := k, \quad n(f) = n(\bar{f}) := n, \quad \text{and} \quad \gamma(f) = \gamma(\bar{f}) := (k, n-k).$$

An *inversion* of $f \in \Theta_{k,n}$ is a pair (i,j) of integers such that $i < j$, $f(i) > f(j)$, and $i \in [n]$. The *length* $\ell(f)$ is the number of inversions of f. For an inversion (i,j) of f, let $f^{(i,j)} : \mathbb{Z} \to \mathbb{Z}$ be obtained by swapping the values $f(i)$ and $f(j)$ (and repeating this for $f(i+rn)$ and $f(j+rn)$ for all $r \in \mathbb{Z}$). We say that $f^{(i,j)}$ is obtained from f by *resolving the crossing* (i,j); see Figure 2. We let $\bar{f}^{(i,j)} \in S_n$ denote the permutation obtained by reducing $f^{(i,j)}$ modulo n.

The permutation $\bar{f}^{(i,j)} \in S_n$ is a product of two cycles, say, $\bar{f}^{(i,j)} = (a_1 a_2 \cdots a_{n_1})(b_1 b_2 \cdots b_{n_2})$, where $a_1 \equiv i$ and $b_1 \equiv j$ modulo n. By taking order-preserving bijections $\{a_1, a_2, \dots, a_{n_1}\} \to [n_1]$ and $\{b_1, b_2, \dots, b_{n_2}\} \to [n_2]$, we may naturally view each of the two cycles as permutations $\bar{f}_1^{(i,j)} \in \mathrm{Cyc}(n_1)$ and $\bar{f}_2^{(i,j)} \in \mathrm{Cyc}(n_2)$. We thus have $f_1^{(i,j)} \in \Theta_{k_1,n_1}$ and $f_2^{(i,j)} \in \Theta_{k_2,n_2}$, where $k_1 := k(\bar{f}_1^{(i,j)})$ and $k_2 := k(\bar{f}_2^{(i,j)})$.

Definition 1.2. For $f \in \Theta_{k,n}$, the *inversion multiset* $\Gamma(f)$ contains a point $\gamma(f_1^{(i,j)})$ for each inversion (i,j) of f. We say that f is *repetition-free* if $\Gamma(f)$ is actually a set, that is, if it contains exactly $\ell(f)$ distinct points.

See Figure 3 for an example. When we draw the set $\Gamma(f)$ inside a $k \times (n-k)$ rectangle, we swap the horizontal and vertical coordinates; cf. Notation 4.1.

We have $\gamma(f_1^{(i,j)}) + \gamma(f_2^{(i,j)}) = (k, n-k)$, but note that we only include $\gamma(f_1^{(i,j)})$ in $\Gamma(f)$ for each inversion (i,j). Nevertheless, $\Gamma(f)$ is always *centrally symmetric*, that is, invariant under the map $\gamma \mapsto (k, n-k) - \gamma$; see Corollary 4.4.

1.3. **Main result.** For a set $\Gamma \subset [k-1] \times [n-k-1]$, we let $\tilde{\Gamma} := \Gamma \sqcup \{(0,0), (k, n-k)\}$. We say that Γ is *convex* if $\tilde{\Gamma}$ contains all lattice points of its convex hull. For $f \in \Theta_{k,n}$ such that $\Gamma(f)$ is convex, let $\mathrm{Dyck}(\Gamma(f))$ denote the set of lattice paths from $(0,0)$ to $(k, n-k)$ with up/right unit steps which stay above the main diagonal and avoid $\Gamma(f)$. For each $f \in \Theta_{k,n}$, $\Gamma(f)$ contains all lattice points $(k_1, n_1 - k_1) \in [k-1] \times [n-k-1]$ that satisfy $k_1/n_1 = k/n$ (Lemma 4.6(iii)). Thus the paths in $\mathrm{Dyck}(\Gamma(f))$ always avoid the main diagonal.

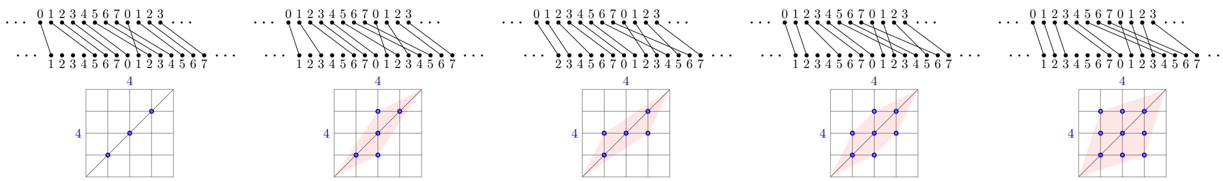

FIGURE 4. Top row: a collection of repetition-free permutations $f \in \Theta_{4,8}$ drawn in affine notation as in Section 1.1. Bottom row: their inversion sets $\Gamma(f)$; see Definition 1.2 and Example 1.4.

Theorem 1.3.

(i) *If $f \in \Theta_{k,n}$ is repetition-free then $\Gamma(f)$ is centrally symmetric and convex, and*

$$C_f = \#\operatorname{Dyck}(\Gamma(f)). \tag{1.2}$$

(ii) *For any centrally symmetric convex subset $\Gamma \subset [k-1] \times [n-k-1]$, there exists a repetition-free $f \in \Theta_{k,n}$ satisfying $\Gamma(f) = \Gamma$.*

Example 1.4. The bottom row of Figure 4 contains all possible centrally symmetric convex subsets of $[k-1] \times [n-k-1]$ for $k = 4$ and $n = 8$. For each such subset Γ, the top row contains a repetition-free bounded affine permutation $f \in \Theta_{k,n}$ satisfying $\Gamma(f) = \Gamma$.

1.4. **Other interpretations and further directions.** Even though our results are purely combinatorial, they provide a starting point for several unexpected connections to the recent results of [OR17, GHSR20, BHM$^+$21] on the rational shuffle conjecture, Coxeter links, and flag Hilbert schemes. In particular, the appearance of convex sets in [BHM$^+$21, Section 7] indicates that open positroid varieties may provide the right geometric framework for the symmetric functions considered in [BHM$^+$21]. We discuss these connections and list several conjectures in Section 7.

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2. Bounded affine permutations

2.1. **Affine permutations.** An (n-periodic) *affine permutation* is a bijection $f : \mathbb{Z} \to \mathbb{Z}$ satisfying the periodicity condition $f(i+n) = f(i) + n$. We let $\tilde{S}_n$ denote the group (under composition) of n-periodic affine permutations. Inversions, and the length function $\ell(f)$ (see Section 1.2) are defined for any $f \in \tilde{S}_n$.

For $k \in \mathbb{Z}$, let $\tilde{S}_n^{(k)} \subset \tilde{S}_n$ be the subset of affine permutations satisfying the condition

$$\sum_{i=1}^{n} (f(i) - i) = kn.$$

Then $\tilde{S}_n = \bigsqcup_{k \in \mathbb{Z}} \tilde{S}_n^{(k)}$. The subgroup $\tilde{S}_n^{(0)}$ is the Coxeter group of affine type A. The group $\tilde{S}_n$ is usually called the *extended affine Weyl group.*

A *bounded affine permutation* is an affine permutation $f \in \tilde{S}_n$ that satisfies the additional condition $i \le f(i) \le i + n$ for all $i \in \mathbb{Z}$. Denote by $\mathcal{B}_{k,n}$ the (finite) set of bounded affine permutations in $\tilde{S}_n^{(k)}$, called the set of (k,n)-*bounded affine permutations.* We see that if $\bar{f} \in \operatorname{Cyc}(n)$ and $k = k(\bar{f})$ then the associated bounded affine permutation f (cf. Section 1.1) belongs to $\mathcal{B}_{k,n}$. In other words, we have $\Theta_{k,n} \subset \mathcal{B}_{k,n}$.

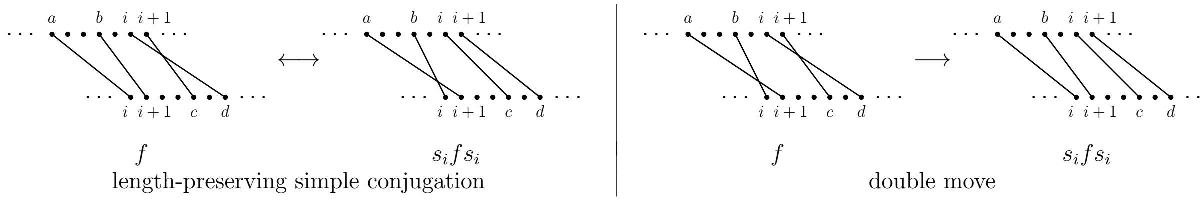

FIGURE 5. Moves for computing $\tilde{R}_f(q)$ and C_f.

For $k \in \mathbb{Z}$, let $f_{k,n} \in \tilde{S}_n^{(k)} \subset \tilde{S}_n$ be given by $i \mapsto i+k$ for all $i \in \mathbb{Z}$. Then $\{f_{k,n} \mid k \in \mathbb{Z}\}$ is exactly the set of length 0 elements in $\tilde{S}_n$, and for $0 \le k \le n$ we have $f_{k,n} \in \mathcal{B}_{k,n}$.

For $i \in \mathbb{Z}$, let $s_i \in \tilde{S}_n$ be the simple transposition given by $i \mapsto i+1$, $i+1 \mapsto i$, and $j \mapsto j$ for all $j \not\equiv i, i+1$ modulo n. For $f \in \mathcal{B}_{k,n}$ and $i \in \mathbb{Z}$, we have $\ell(s_i f) = \ell(f) \pm 1$ and $\ell(f s_i) = \ell(f) \pm 1$. We write $f s_i < f$ if $\ell(f s_i) < \ell(f)$, and similarly for $f s_i > f$, $s_i f < f$, and $s_i f > f$.

Given $f \in \tilde{S}_n$, define the *cyclic shift* $\sigma f \in \tilde{S}_n$ by

$$(\sigma f)(i) := f(i-1) + 1 \quad \text{for all } i \in \mathbb{Z}. \tag{2.1}$$

In other words, we have $\sigma f = f_{1,n} f f_{1,n}^{-1}$. Note that σ preserves each of the subsets $\tilde{S}_n^{(k)}$, $\mathcal{B}_{k,n}$, and $\Theta_{k,n}$.

2.2. Conjugation and double move reduction.

Definition 2.1. We say that $f \in \mathcal{B}_{k,n}$ has a *double crossing* at some $i \in \mathbb{Z}$ if $s_i f s_i < s_i f < f$, $s_i f s_i < f s_i < f$, and $s_i f s_i, s_i f, f s_i \in \mathcal{B}_{k,n}$.

Equivalently, for $a := f^{-1}(i+1)$, $b := f^{-1}(i)$, $c := f(i+1)$, $d := f(i)$, f has a double crossing at i if and only if $a < b < i < i+1 < c < d$. See Figure 5(right). In this case, we say that $s_i f s_i$ is obtained from f by a *double move*.

Definition 2.2. Let $f \in \mathcal{B}_{k,n}$, $i \in \mathbb{Z}$, and $f' := s_i f s_i$. If $\ell(f) = \ell(s_i f s_i)$ and $s_i f s_i \in \mathcal{B}_{k,n}$ then we say that f and f' are related by a *length-preserving simple conjugation*. We say that $f, g \in \mathcal{B}_{k,n}$ are *c-equivalent* and write $f \overset{c}{\sim} g$ if f and g can be related by a sequence of length-preserving simple conjugations. See Figure 5(left).

The following result describes the structure of $\Theta_{k,n}$ under double moves and c-equivalence.

Proposition 2.3.
(i) *The minimal length elements of $\Theta_{k,n}$ are of length $d := \gcd(k,n) - 1$ and all such elements are related by cyclic shift* (2.1) *and c-equivalence.*
(ii) *Any $f \in \Theta_{k,n}$ can be reduced to a minimal length element of $\Theta_{k,n}$ by double moves and c-equivalence.*

2.3. Proof of Proposition 2.3.

We deduce these statements from the results of He and Nie [HN14] and He and Yang [HY12].

Following [HN14], we introduce the following notation. For $f, f' \in \tilde{S}_n$, we write $f \to f'$ if there is a sequence $f = f_0, f_1, f_2, \dots, f_r = f'$ such that $f_j = s_{i_j} f_{j-1} s_{i_j}$ for $j = 1, 2, \dots, r$, satisfying $\ell(f_j) \le \ell(f_{j-1})$. We write $f \overset{c}{\approx} f'$ if $f \to f'$ and $f' \to f$. Thus $f \to f'$ if f' can be obtained from f by a sequence of c-equivalences and double moves, and $f \overset{c}{\approx} f'$ if f and f' are c-equivalent, without the restriction on staying inside $\mathcal{B}_{k,n}$.

Lemma 2.4. *Let $f \in \Theta_{k,n}$ and $f' \in \tilde{S}_n$ be such that $f \to f'$. Then $f' \in \Theta_{k,n}$.*

Proof. Suppose $f \in \Theta_{k,n}$ and $f' = s_i f s_i$ satisfies $\ell(f') \leq \ell(f)$. Since $f \in \Theta_{k,n}$, we have $f(j) \in [j+1, j+n-1]$ for all $j \in \mathbb{Z}$. It follows from this that f' is also a bounded affine permutation, and thus $f' \in \Theta_{k,n}$. □

Theorem 2.5 ([HN14, Theorem 2.9])**.** *Let $\mathcal{O}'$ be an $\tilde{S}_n^{(0)}$-conjugacy class in $\tilde{S}_n$ and let $\mathcal{O}'_{\min} \subset \mathcal{O}'$ denote the set of elements of minimal length. Then for any $f \in \mathcal{O}'$, there exists $f' \in \mathcal{O}'_{\min}$ such that $f \to f'$.*

Proposition 2.6. *Suppose that $\mathcal{O}'$ is an $\tilde{S}_n^{(0)}$-conjugacy class in $\tilde{S}_n$ with a nonempty intersection with $\Theta_{k,n}$. Then for $f, f' \in \mathcal{O}'_{\min}$, we have $f \overset{c}{\approx} f'$.*

Proof. For $f \in \Theta_{k,n}$, the image $\bar{f} \in S_n$ is an n-cycle. The n-cycles are *elliptic* elements in S_n, so by [HN14, Corollary 4.7], we have that $\mathcal{O}'$ is *nice* in the sense of [HN14, Section 4.1]. It follows from the definition of *nice* that $f \overset{c}{\approx} f'$ for $f, f' \in \mathcal{O}'_{\min}$. □

Proposition 2.7. *The elements of $\Theta_{k,n}$ all belong to a single $\tilde{S}_n$-conjugacy class $\mathcal{O}$ in $\tilde{S}_n$.*

Proof. Our goal is to apply [HY12, Proposition 2.1]. In the notation of [HY12], we have $\delta = \mathrm{id}$, $\tilde{W}' = \tilde{W}$, $(P^\vee/Q^\vee)_\delta \cong \mathbb{Z}/n\mathbb{Z}$, and $\mathcal{O}_0$ is the S_n-conjugacy class consisting of n-cycles in S_n. Choosing $\nu := k \in \mathbb{Z}/n\mathbb{Z}$, we see that $\kappa_\delta^{-1}(\nu)$ contains $\Theta_{k,n}$. Similarly, $\eta^{-1}(\mathcal{O}_0)$ contains all bounded affine permutations whose reduction modulo n is an n-cycle. Thus $\Theta_{k,n}$ is a subset of $\eta^{-1}(\mathcal{O}_0) \cap \kappa_\delta^{-1}(\nu)$, which, according to [HY12, Proposition 2.1], is a single $\tilde{S}_n$-conjugacy class in $\tilde{S}_n$. □

We are ready to finish the proof of Proposition 2.3. By Proposition 2.7, there is an $\tilde{S}_n$-conjugacy class $\mathcal{O} \subset \tilde{S}_n$ containing $\Theta_{k,n}$. Since $\tilde{S}_n = \tilde{S}_n^{(0)} \rtimes \langle f_{1,n} \rangle$, there exist finitely-many distinct $\tilde{S}_n^{(0)}$-conjugacy classes $\mathcal{O}'_0, \dots, \mathcal{O}'_{r-1}$ such that $\mathcal{O} = \bigsqcup_i \mathcal{O}'_i$ and $\mathcal{O}'_i = \sigma^i \mathcal{O}'_0$. Note that the cyclic shift σ is length-preserving. Thus the minimal length elements in $\mathcal{O}'_0, \dots, \mathcal{O}'_{r-1}$ have the same length, and this length is equal to the minimal length of any element in $\Theta_{k,n}$.

It is easy to see that $\mathrm{c}(f_{k,n}) = \gcd(k,n)$, where for $f \in \tilde{S}_n$, we denote by $\mathrm{c}(f) := \mathrm{c}(\bar{f})$ the number of cycles of the permutation $\bar{f}$. Now, for $f \in \tilde{S}_n$, we have $\mathrm{c}(s_i f) = \mathrm{c}(f s_i) \in \{\mathrm{c}(f)+1, \mathrm{c}(f)-1\}$. It follows that for $f \in \Theta_{k,n}$, we have $\ell(f) \geq \gcd(k,n) - 1$. On the other hand, it is easy to see that $f_{k,n} s_1 s_2 \cdots s_{\gcd(k,n)-1} \in \Theta_{k,n}$. Thus the minimal length of $f \in \Theta_{k,n}$ is $d := \gcd(k,n) - 1$. Since $\mathrm{c}(\cdot)$ is invariant under conjugation, we find that any $f \in \Theta_{k,n}$ with $\ell(f) = d$ has minimal length in its $\tilde{S}_n^{(0)}$-conjugacy class.

Let $f, f' \in \Theta_{k,n}$ be two elements of length d. By Proposition 2.7, f and f' are $\tilde{S}_n$-conjugate. Thus f is $\tilde{S}_n^{(0)}$-conjugate to a cyclic shift $g = \sigma^i f'$ of f'. Let $\mathcal{O}'$ be the $\tilde{S}_n^{(0)}$-conjugacy class containing f and g. Since $\ell(f) = \ell(g) = d$ is minimal, by Proposition 2.6, we get $f \overset{c}{\approx} g$. By Lemma 2.4, having $f \overset{c}{\approx} g$ for $f \in \Theta_{k,n}$ implies that $f \overset{c}{\sim} g$. This proves Proposition 2.3(i).

As we showed above, the minimum length elements of $\Theta_{k,n}$ are also minimum length elements in their $\tilde{S}_n^{(0)}$-conjugacy class. Thus Proposition 2.3(ii) follows from Theorem 2.5 combined with Lemma 2.4. □

3. Positroid Catalan numbers

3.1. ***R*-polynomials.** For each (k,n)-bounded affine permutation f, let Π_f° denote the *open positroid variety* [KLS13]. The *R-polynomial* $R_f(q) := \#\Pi_f^\circ(\mathbb{F}_q)$ counts the number of points in Π_f° over a finite field $\mathbb{F}_q$ with q elements (where q is a prime power). These R-polynomials are special cases of the R-polynomials of Kazhdan and Lusztig [KL79, KL80].

The following recurrence appears in [MS16, Section 4].

Proposition 3.1. *The polynomials $R_f(q)$, $f \in \mathcal{B}_{k,n}$, may be computed from the following recurrence.*

(a) *If $n = 1$ then $R_f(q) = 1$.*
(b) *If $\bar{f}$ has some fixed points then $R_f(q) = R_{f'}(q)$, where f' is obtained from f by removing all fixed points of $\bar{f}$.*
(c) *If $f(i) = i + 1$ or $f(i + 1) = i + n$ (where $n \geq 2$) then $fs_i, s_if \in \mathcal{B}_{k,n}$ and $R_f(q) = (q - 1)R_{s_if}(q) = (q - 1)R_{fs_i}(q)$.*
(d) *If $f \overset{c}{\sim} g$ then $R_f(q) = R_g(q)$.*
(e) *If f has a double crossing at $i \in \mathbb{Z}$ then*

$$R_{s_ifs_i}(q) = (q - 1)R_{s_if}(q) + qR_f(q). \tag{3.1}$$

Proof. The results of [MS16] are formulated in the language of cluster algebras. For the convenience of the reader, we give an alternative proof of (a)–(e) not relying on cluster algebras, assuming familiarity with [KLS13]. We start by noting that the definition $R_f(q) := \#\Pi_f^\circ(\mathbb{F}_q)$ implies that $R_f(q) = R_{\sigma f}(q)$ since the open positroid varieties indexed by f and by σf are isomorphic.

The initial condition (a) is trivial. Property (b) follows from the definition of the open positroid variety Π_f°. If $f(i) = i$ (resp. $f(i) = i+n$) then Π_f° maps isomorphically to another open positroid variety $\Pi_{f'}^\circ$ under the natural projection map $\mathrm{Gr}(k, n) \to \mathrm{Gr}(k, n-1)$ (resp., $\mathrm{Gr}(k, n) \to \mathrm{Gr}(k-1, n-1)$) between Grassmannians that removes (resp., contracts) the i-th column. See e.g. [Lam16, Lemmas 7.8 and 7.9].

The Kazhdan–Lusztig R-polynomials $R_{v,w}(q)$ are indexed by pairs (v, w) of permutations. When $v \not\leq w$ (where $\leq$ denotes the Bruhat order on S_n), we have $R_{v,w}(q) = 0$, and for $v = w$, we have $R_{v,w}(q) = 1$. For $v \leq w \in S_n$, $R_{v,w}(q)$ can then be computed by a recurrence relation [KL79, Section 2]:

$$R_{v,w}(q) = \begin{cases} R_{sv,sw}(q), & \text{if } sv < v \text{ and } sw < w, \\ (q-1)R_{sv,w}(q) + qR_{sv,sw}(q), & \text{if } sv > v \text{ and } sw < w. \end{cases} \tag{3.2}$$

Here, $s = s_i$ for some $1 \leq i \leq n-1$ is a simple transposition satisfying $sw < w$.

For each $f \in \mathcal{B}_{k,n}$, there is a pair (v, w) such that $f = w\tau_{k,n}v^{-1}$ and $R_f(q) = R_{v,w}(q)$, where $\tau_{k,n} \in \tilde{S}_n$ denotes a certain *translation element*; see [KLS13, Proposition 3.15]. From this, (3.2) implies (d)–(e) whenever we have a length-preserving simple conjugation or a double crossing at $1 \leq i \leq n-1$. Applying the cyclic shift, we see that properties (d)–(e) hold also for $i = 0$, which completes their proof.

Property (c) for $1 \leq i \leq n-1$ also follows from (3.2). If $f(i) = i+1$ or $f(i+1) = i+n$ then $\ell(s_if) = \ell(fs_i) = \ell(f) + 1$, and this corresponds to the case $R_{v,w}(q) = (q-1)R_{sv,w}(q)$; here, $R_{sv,sw}(q) = 0$ since $sv \not\leq sw$. In the remaining case $i = 0$, (c) follows from applying the cyclic shift.

Finally, a constructive algorithm to compute $R_f(q)$ from (a)–(e) is given in the proof of [MS16, Theorem 3.3]. □

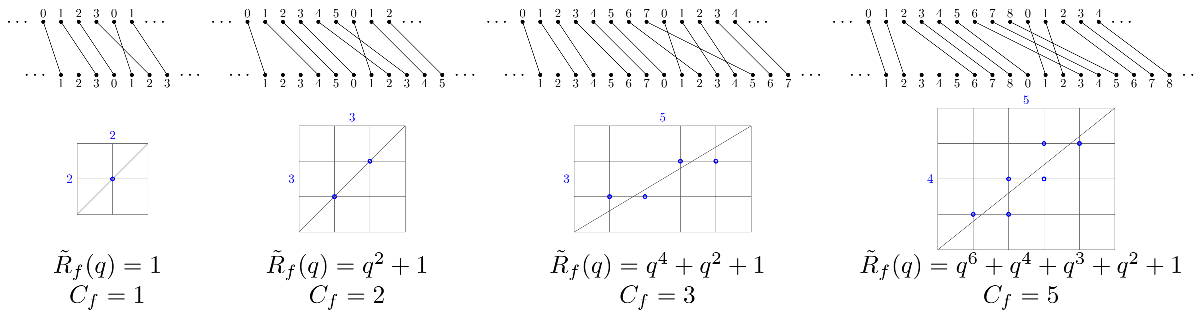

FIGURE 6. Some examples of $\tilde{R}_f(q)$ and C_f.

3.2. **Positroid Catalan numbers.** Recall that for a permutation $\bar{f} \in S_n$, we let $\mathrm{c}(\bar{f}) = \mathrm{c}(f)$ denote its number of cycles. For $f \in \mathcal{B}_{k,n}$, we let

$$\tilde{R}_f(q) := R_f(q)/(q-1)^{n-\mathrm{c}(f)}. \tag{3.3}$$

It is easy to see (for example using (3.5) below; see also [GL20, Proposition 4.5]) that $\tilde{R}_f(q)$ is always a polynomial in q.

The definition of a positroid Catalan number C_f (Definition 1.1) can be extended to all $f \in \mathcal{B}_{k,n}$ by setting

$$C_f := \tilde{R}_f(1). \tag{3.4}$$

The relation to Definition 1.1 is given in Section 7.1. See Figure 6 for examples.

3.3. **Recurrence for positroid Catalan numbers.** If f has a double crossing at $i \in \mathbb{Z}$ then (3.1) implies

$$\tilde{R}_{s_i f s_i}(q) = \begin{cases} \tilde{R}_{s_i f}(q) + q\tilde{R}_f(q), & \text{if } i, i+1 \text{ belong to the same cycle of } \bar{f}; \\ (q-1)^2 \tilde{R}_{s_i f}(q) + q\tilde{R}_f(q), & \text{if } i, i+1 \text{ belong to different cycles of } \bar{f}. \end{cases} \tag{3.5}$$

Here, $i, i+1$ are considered modulo n.

The next result follows from Proposition 3.1 combined with (3.4)–(3.5).

Proposition 3.2. *The positroid Catalan numbers C_f, $f \in \mathcal{B}_{k,n}$, may be computed from the following recurrence.*

- (a′) *If $n = 1$ then $C_f = 1$.*
- (b′) *If $\bar{f}$ has some fixed points then $C_f = C_{f'}$, where f' is obtained from f by removing all fixed points of $\bar{f}$.*
- (c′) *If $f(i) = i+1$ or $f(i+1) = i+n$ (where $n \geq 2$) then $fs_i, s_i f \in \mathcal{B}_{k,n}$ and $C_f = C_{s_i f} = C_{f s_i}$.*
- (d′) *If $f \overset{\mathrm{c}}{\sim} g$ then $C_f = C_g$.*
- (e′) *If $f \in \mathcal{B}_{k,n}$ has a double crossing at $i \in \mathbb{Z}$ then*

$$C_{s_i f s_i} = \begin{cases} C_{s_i f} + C_f, & \textit{if } i, i+1 \textit{ belong to the same cycle of } \bar{f}; \\ C_f, & \textit{if } i, i+1 \textit{ belong to different cycles of } \bar{f}. \end{cases} \tag{3.6}$$

Proposition 3.3. *Let $f \in \mathcal{B}_{k,n}$. Then C_f is a positive integer.*

Proof. The proof of [MS16, Theorem 3.3] shows that C_f may be expressed using (a′)–(e′) in terms of C_g for bounded affine permutations g satisfying either $n(g) < n(f)$ or $n(g) = n(f)$ and $\ell(g) > \ell(f)$. In particular, the recurrence in Proposition 3.2 is subtraction-free, which shows the result. See also [GL20, Remark 9.4 and Proposition 9.5]. □

Remark 3.4. It is not always true that $\tilde{R}_f(q)$ has positive coefficients: see [GL20, Example 4.22]. This question is closely related to the *odd cohomology vanishing* phenomenon which appears for $\gcd(k, n) = 1$ and $f = f_{k,n}$ (i.e., for torus knots) but not for all $f \in \Theta_{k,n}$. It is an important open problem to describe a wider class of positroids (or more generally, knots) for which this phenomenon occurs. We expect this class to contain all $f \in \Theta_{k,n}$ which are repetition-free; see Conjecture 7.1.

Let $f \in \mathcal{B}_{k,n}$ be such that $\bar{f} = (a_1^{(1)} \cdots a_{n_1}^{(1)})(a_1^{(2)} \cdots a_{n_2}^{(2)}) \cdots (a_1^{(r)} \cdots a_{n_r}^{(r)})$ is a product of r cycles. (The case $r = 2$ was considered in Section 1.2.) For each $j \in [r]$, denote by $f|_{S_j} \in \mathcal{B}(k_j, n_j)$ the restriction of f to the set S_j of all integers congruent to one of $a_1^{(j)}, \ldots, a_{n_j}^{(j)}$ modulo n. We deduce the following *decoupling property* from Proposition 3.2.

Corollary 3.5 (Decoupling)**.** *Let $f \in \mathcal{B}_{k,n}$ and $i \in \mathbb{Z}$. If i and $i+1$ belong to different cycles of $\bar{f}$ then $s_i f s_i \in \mathcal{B}_{k,n}$ and*

$$C_f = C_{s_i f s_i}. \tag{3.7}$$

In particular, in the above notation, for $f \in \mathcal{B}_{k,n}$ we have

$$C_f = \prod_{j=1}^{r} C_{f|_{S_j}}. \tag{3.8}$$

Proof. Eq. (3.7) follows easily from Proposition 3.2. To deduce (3.8), we apply (3.7) repeatedly until each cycle of $\bar{f}$ is supported on a cyclically consecutive interval $[a, b] \subset [n]$ for some $a, b \in [n]$. After that, C_f may be computed via Proposition 3.2 independently on each interval, which results in the product formula (3.8). □

We will use a special case of (3.8) when $r = 2$.

Corollary 3.6. *Suppose that $f \in \Theta_{k,n}$ has a double crossing at $i \in [n]$. Then*

$$C_{s_i f s_i} = C_{f_1^{(i,i+1)}} C_{f_2^{(i,i+1)}} + C_f. \tag{3.9}$$

Our eventual goal will be to relate (3.9) to the recurrence for Dyck paths shown in Figure 12. One other simple result we will need is the cyclic shift invariance of C_f and $\Gamma(f)$.

Proposition 3.7. *For any $f \in \mathcal{B}_{k,n}$, we have*

$$\Gamma(f) = \Gamma(\sigma f), \quad C_f = C_{\sigma f}, \quad \textit{and} \quad \tilde{R}_f(q) = \tilde{R}_{\sigma f}(q).$$

Proof. It is obvious that both the definition of $\Gamma(f)$ and the recurrence in Propositions 3.1 and 3.2 are invariant under the action of σ. □

4. Big paths

The next few sections contain the main body of the proof of Theorem 1.3. From now on, we switch from working in the $(k, n-k)$-coordinates to working in the (k, n)-coordinates. For $f \in \Theta_{k,n}$, we let

$$\delta(f) := (k, n) \tag{4.1}$$

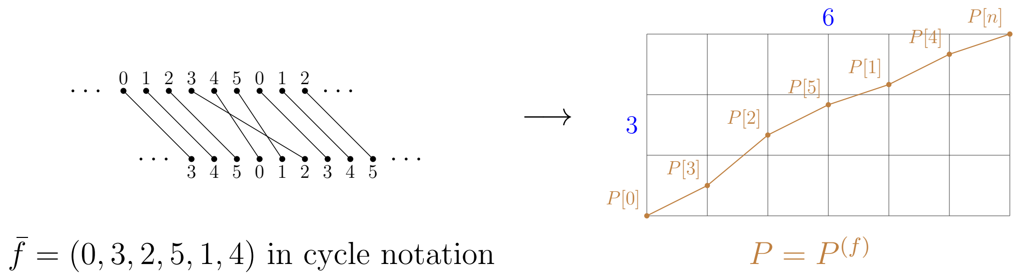

FIGURE 7. Computing the small path $P^{(f)}$. Its points are labeled according to Notation 5.4.

and define the multiset $\Gamma'(f)$ to be the image of $\Gamma(f)$ under the map $(k_1, n_1 - k_1) \mapsto (k_1, n_1)$. We let $\tilde\Gamma'(f) := \Gamma'(f) \sqcup \{(0,0),(k,n)\}$.

Let $f \in \Theta_{k,n}$. Our goal is to give a geometric interpretation of the multiset $\Gamma'(f)$.

Notation 4.1. When referring to points in the plane, we swap their coordinates. For a point $\alpha = (a,b) \in \mathbb{Z}^2$, we denote by $n(\alpha) := b$ (resp., $k(\alpha) := a$) its horizontal (resp., vertical) coordinate.

Definition 4.2. The *big path* $P_\infty^{(f)}$ of f is the path in the plane through the points $p_{f,r} := (f^r(0)/n, r)$ for all $r \in \mathbb{Z}$. The *small path* $P^{(f)}$ is the subpath of $P_\infty^{(f)}$ through the points $p_{f,0}, p_{f,1}, \dots, p_{f,n}$.

See Figure 7. We usually drop the superscript and denote $P_\infty := P_\infty^{(f)}$. We refer to the points $p_{f,r}$ for $r \in \mathbb{Z}$ as the *integer points* of P_∞.

Set $\delta := (k,n)$ and choose some $\alpha \in \mathbb{Z}^2$. We will be interested in the *intersection points* of P_∞ with $Q_\infty := P_\infty + \alpha$. First, observe that if $\alpha \in \mathbb{Z}\delta$ then $P_\infty = Q_\infty$. If $\alpha \notin \mathbb{Z}\delta$ then it is easy to see that no integer point of P_∞ belongs to Q_∞, and that the set $P_\infty \cap Q_\infty$ is invariant under adding multiples of δ. We denote by $|P_\infty \cap Q_\infty|$ the size of this set when considered "modulo δ," that is, as a subset of the cylinder[1] $\mathbb{Z}^2/\mathbb{Z}\delta$. For $l := |P_\infty \cap Q_\infty|$, we say that P_∞ *and* Q_∞ *intersect* l *times.* The number l is always finite and even.

Proposition 4.3. *Let $f \in \Theta_{k,n}$. Then f is repetition-free if and only if for all $\alpha \in \mathbb{Z}^2 \setminus \mathbb{Z}\delta$, P_∞ and $Q_\infty := P_\infty + \alpha$ intersect at most two times. In this case, we have*

$$\Gamma'(f) = \{\alpha \in [k-1] \times [n-1] \mid P_\infty \text{ intersects } Q_\infty\}.$$

Proof. Let $\alpha = (a,b) \neq (0,0)$. If $a \le 0, b \ge 0$ or $a \ge 0, b \le 0$ then clearly P_∞ does not intersect Q_∞. Thus if P_∞ intersects Q_∞ then there exists a unique $t \in \mathbb{Z}$ such that $\alpha + t\delta \in [k-1] \times [n-1]$. From now on, we assume that $\alpha \in [k-1] \times [n-1]$.

We will prove the more general statement that for all $f \in \Theta_{k,n}$, the multiplicity of α in the multiset $\Gamma'(f)$ is given by $\frac{1}{2}|P_\infty \cap Q_\infty|$. Indeed, suppose that P_∞ crosses Q_∞ *from below* at some non-integer point x. (That is, P_∞ is below Q_∞ when approaching x from the left and above Q_∞ when approaching x from the right.) Then x belongs to the segment of P_∞ connecting $p_{f,r}$ to $p_{f,r+1}$ and to the segment of Q_∞ connecting $\alpha + p_{f,r-b}$ to $\alpha + p_{f,r-b+1}$,

[1]Some of our constructions are most naturally described in terms of the cylinder $\mathbb{Z}^2/\mathbb{Z}\delta$. However, we choose to work with the full plane $\mathbb{Z}^2$ since we need to talk about convexity. For example, we will see that the set $\tilde\Gamma'(f)$ is convex as a subset of the plane but not as a subset of the cylinder.

where $\alpha = (a, b)$. Let $i := f^r(0)$ and $j := an + f^{r-b}(0)$. Then we have $i < j$ and $f(i) > f(j)$, and thus $(i', j') := (i - tn, j - tn)$ form an inversion of f, where $t \in \mathbb{Z}$ is such that $i - tn \in [n]$. Moreover, it is easy to see that $\delta(f_1^{(i',j')}) = \alpha$, where $\delta(\cdot)$ was defined in (4.1).

Conversely, given an inversion (i, j) of f with $\delta(f_1^{(i,j)}) = \alpha$, we may find a (unique modulo n) index $r \in \mathbb{Z}$ such that $f^r(0) \equiv i$ modulo n, and we can also find a (unique modulo δ) shift $\alpha \in \mathbb{Z}^2$ such that Q_∞ passes through the point $p_{f,r} + (\frac{j-i}{n}, 0)$. This shows that the inversions (i, j) of f satisfying $\delta(f_1^{(i,j)}) = \alpha$ are in bijection with the $\frac{1}{2}|P_\infty \cap Q_\infty|$ points where P_∞ crosses Q_∞ from below. □

We say that a multiset Γ' is *centrally symmetric* if for each $\alpha \in [k-1] \times [n-1]$, the multiplicities of α and of $\delta - \alpha$ in Γ' coincide.

Corollary 4.4. *For all $f \in \Theta_{k,n}$, the inversion multiset $\Gamma'(f)$ is centrally symmetric.*

Proof. We showed above that the multiplicity of α in $\Gamma'(f)$ is given by $\frac{1}{2}|P_\infty \cap (P_\infty + \alpha)|$. Since $P_\infty = P_\infty + \delta$, we find $|P_\infty \cap (P_\infty + \delta - \alpha)| = |P_\infty \cap (P_\infty + \alpha)|$, and the result follows. □

We discuss how $\Gamma'(f)$ changes under length-preserving simple conjugations and double moves. The following result is immediate.

Lemma 4.5. *Let $f \in \Theta_{k,n}$ be repetition-free.*

(i) *If $g \overset{c}{\sim} f$ then g is repetition-free and $\Gamma'(f) = \Gamma'(g)$.*
(ii) *If f has a double crossing at $i \in [n]$ then $f' := s_i f s_i$ is repetition-free and*

$$\Gamma'(f') = \Gamma'(f) \setminus \{\delta(f_1^{(i,i+1)}), \delta(f_2^{(i,i+1)})\}.$$

For a point $\alpha = (a, b) \in [k-1] \times [n-1]$, let $\operatorname{slope}(\alpha) := \frac{a}{b}$. Part (iii) of the next lemma confirms that $\Gamma'(f)$ always contains all points on the main diagonal of $[k-1] \times [n-1]$.

Lemma 4.6. *Let $f \in \Theta_{k,n}$, $\alpha \in [k-1] \times [n-1]$, and $Q_\infty := P_\infty + \alpha$.*

(i) *If $\operatorname{slope}(\alpha) \le \operatorname{slope}(\delta)$ then Q_∞ contains integer points below P_∞.*
(ii) *If $\operatorname{slope}(\alpha) \ge \operatorname{slope}(\delta)$ then Q_∞ contains integer points above P_∞.*
(iii) *If $\operatorname{slope}(\alpha) = \operatorname{slope}(\delta)$ then α belongs to $\Gamma'(f)$.*

Proof. For $r \in \mathbb{Z}$, let $q_{f,r}$ be the integer point of Q_∞ with horizontal coordinate $n(q_{f,r}) = r$. Let $\delta^\perp := (1, -k/n)$ and denote by $\langle \cdot, \cdot \rangle$ the standard dot product on $\mathbb{R}^2$. We have $\langle \delta^\perp, \delta \rangle = 0$, and the sign of $\langle \delta^\perp, \alpha \rangle$ coincides with the sign of $\operatorname{slope}(\alpha) - \operatorname{slope}(\delta)$. Let

$$\langle \delta^\perp, P_\infty \rangle := \sum_{r=0}^{n-1} \langle \delta^\perp, p_{f,r} \rangle \quad \text{and} \quad \langle \delta^\perp, Q_\infty \rangle := \sum_{r=0}^{n-1} \langle \delta^\perp, q_{f,r} \rangle. \tag{4.2}$$

Since $p_{f,r+n} = p_{f,r} + \delta$ for all $r \in \mathbb{Z}$, we have $\langle \delta^\perp, P_\infty \rangle = \sum_{r=j}^{j+n-1} \langle \delta^\perp, p_{f,r} \rangle$ for all $j \in \mathbb{Z}$, and similarly for $\langle \delta^\perp, Q_\infty \rangle$. In particular, we have

$$\langle \delta^\perp, Q_\infty \rangle = \sum_{r=0}^{n-1} \langle \delta^\perp, p_{f,r} + \alpha \rangle = \langle \delta^\perp, P_\infty \rangle + n \langle \delta^\perp, \alpha \rangle.$$

Observe that for each $r \in \mathbb{Z}$, $q_{f,r}$ is above P_∞ if and only if it is above $p_{f,r}$, which happens if and only if $\langle \delta^\perp, q_{f,r} - p_{f,r} \rangle > 0$, since the vertical coordinate of $\delta^\perp$ is positive. Thus (i)–(ii) follow, and (iii) follows by combining (i)–(ii) with (the proof of) Proposition 4.3, since if $\operatorname{slope}(\alpha) = \operatorname{slope}(\delta)$ then Q_∞ contains integer points both below and above P_∞, and therefore intersects P_∞. □

5. Convexity of the inversion multiset

Similarly to Section 1.3, we say that $\Gamma'(f)$ is *convex* if $\tilde{\Gamma}'(f)$ contains all lattice points of its convex hull. (These sets were defined in the beginning of Section 4.) The goal of this section is to prove the following result.

Theorem 5.1. *Let $f \in \Theta_{k,n}$ be repetition-free. Then the set $\Gamma'(f)$ is convex.*

We start by stating some consequences of the results obtained in Section 2. Let

$$\Gamma_{k,n}^{\min} = \{\alpha \in [k-1] \times [n-1] \mid \mathrm{slope}(\alpha) = \mathrm{slope}(\delta)\}.$$

By Lemma 4.6(iii), we have $\Gamma_{k,n}^{\min} \subseteq \Gamma'(f)$ for all $f \in \Theta_{k,n}$. The next two statements follow directly from Proposition 2.3.

Corollary 5.2. *Let $f \in \Theta_{k,n}$ be repetition-free. Then at least one of the following holds:*

- $\Gamma'(f) = \Gamma_{k,n}^{\min}$.
- *There exists $g \in \Theta_{k,n}$ such that $f \overset{c}{\sim} g$ and g has a double crossing at some $i \in \mathbb{Z}$.*

Corollary 5.3. *Suppose that $f, g \in \Theta_{k,n}$ are repetition-free and $\Gamma'(f) = \Gamma'(g) = \Gamma_{k,n}^{\min}$. Then $C_f = C_g$.*

Throughout the rest of this section, the following data is fixed:

- a repetition-free $f \in \Theta_{k,n}$ that has a double crossing at 0;
- a big path $P_\infty := P_\infty^{(f)}$ and a small path $P := P^{(f)}$ for f;
- $f_1 := f_1^{(0,1)} \in \Theta_{k_1,n_1}$ and $f_2 := f_2^{(0,1)} \in \Theta_{k_2,n_2}$ obtained by resolving the crossing $(0,1)$ as in Section 1.2;
- $\delta := (k,n)$, $\delta_1 := (k_1,n_1)$, and $\delta_2 := (k_2,n_2)$.

Notation 5.4. For each $0 \le r < n$, let $0 \le j_r < n$ be the unique index equal to $f^r(0)$ modulo n. Then we label $p_{f,r}$ by $P[j_r]$ as in Figure 7. We extend this to all $r \in \mathbb{Z}$ using the convention that $j_{r+n} := j_r + n$, and we label $p_{f,r}$ by $P[j_r]$ for $r \in \mathbb{Z}$. Thus $P[j+n] = P[j] + \delta$ for all $j \in \mathbb{Z}$. If $P[i]$ appears to the left of $P[j]$ for some $i, j \in \mathbb{Z}$, we denote by $P[i \to j]$ the subpath of P_∞ connecting $P[i]$ to $P[j]$. Thus $P = P[0 \to n]$ and we will be particularly interested in the subpaths $P[0 \to 1]$ and $P[1 \to n]$ of P (under the above assumption that f has a double crossing at 0).

We establish several elementary properties of $\Gamma'(f)$. For a subset $\Gamma' \subset \mathbb{Z}^2$, we let $\Gamma' + \mathbb{Z}\delta := \{\alpha + t\delta \mid \alpha \in \Gamma', t \in \mathbb{Z}\}$. Recall also that we set $\delta^\perp := (1, -k/n)$ and that the sign of $\langle \delta^\perp, \alpha \rangle$ is positive if and only if α is above the line spanned by δ. Finally, we adopt the convention that when we shift a (big or small) path, its labeling of points from Notation 5.4 is preserved; for example, $(P + \alpha)[1 \to n] := P[1 \to n] + \alpha$, etc.

Lemma 5.5.

(i) *Let $\alpha, \beta \in \mathbb{Z}^2$ be such that $\langle \delta^\perp, \alpha \rangle < 0$, $\langle \delta^\perp, \beta \rangle < 0$, and $\alpha, \beta \notin (\tilde{\Gamma}'(f) + \mathbb{Z}\delta)$. Then $\alpha + \beta \notin (\tilde{\Gamma}'(f) + \mathbb{Z}\delta)$.*

(ii) *We have* $\mathrm{slope}(\delta_1) < \mathrm{slope}(\delta) < \mathrm{slope}(\delta_2)$.

(iii) *Let $\alpha \in \Gamma'(f)$ be such that $\langle \delta^\perp, \alpha \rangle < 0$. Then for all $\beta \in \mathbb{Z}^2$ satisfying $\langle \delta^\perp, \beta \rangle \le 0$, $n(\beta) \le n(\alpha)$, and $k(\beta) \ge k(\alpha)$, we have $\beta \in \Gamma'(f)$.*

Proof. (i): We showed in the proof of Lemma 4.6 that if $\langle \delta^\perp, \alpha \rangle < 0$ then $P_\infty + \alpha$ contains integer points below P_∞. If in addition $\alpha \notin (\tilde{\Gamma}'(f) + \mathbb{Z}\delta)$ then $P_\infty + \alpha$ and P_∞ do not

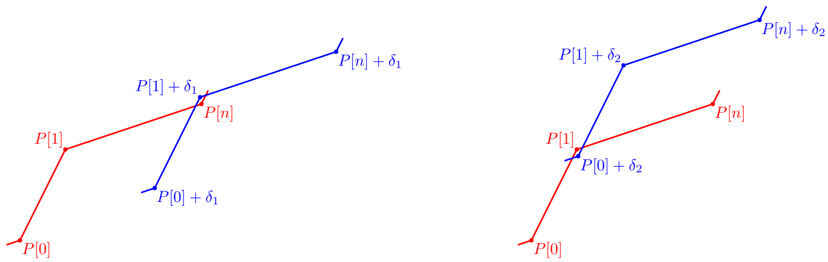

FIGURE 8. We write $P[n] \approx P[1]+\delta_1$ and $P[1] \approx P[0]+\delta_2$; see Notation 5.7.

intersect. Thus $P_\infty+\alpha$ and $P_\infty+\beta$ are both below P_∞. But then $P_\infty+\alpha+\beta$ is below $P_\infty+\alpha$, and therefore it is below P_∞, so $\alpha+\beta \notin (\tilde{\Gamma}'(f)+\mathbb{Z}\delta)$.

(ii): The only integer points of $P_\infty+\delta_1$ above P_∞ are $P[1]+\mathbb{Z}\delta$. Thus $P_\infty+2\delta_1$ is below P_∞ (cf. Remark 5.6 below). By Lemma 4.6(ii), we must have $\operatorname{slope}(\delta_1)=\operatorname{slope}(2\delta_1)<\operatorname{slope}(\delta)$. Similarly, $\operatorname{slope}(\delta)<\operatorname{slope}(\delta_2)$.

(iii): If $\beta=\alpha$ then clearly $\beta\in\Gamma'(f)$. Assume that $\beta\neq\alpha$ and let $e:=\alpha-\beta$. We have $e\neq 0$, $n(e)\geq 0$, and $k(e)\leq 0$, so $e\notin\tilde{\Gamma}'(f)+\mathbb{Z}\delta$ and $\langle\delta^\perp,e\rangle<0$. If $\beta\notin(\tilde{\Gamma}'(f)+\mathbb{Z}\delta)$ then by (i), we must have $\alpha\notin(\tilde{\Gamma}'(f)+\mathbb{Z}\delta)$, a contradiction. Thus $\beta\in(\tilde{\Gamma}'(f)+\mathbb{Z}\delta)$, and the conditions on the coordinates of β ensure that in fact $\beta\in\Gamma'(f)$. ☐

Remark 5.6. The proof of (i)–(ii) above shows the stronger statement that if $\langle\delta^\perp,\alpha\rangle<0$, $\langle\delta^\perp,\beta\rangle<0$, and $\alpha,\beta\notin((\tilde{\Gamma}'(f)\setminus\{\delta_1\})+\mathbb{Z}\delta)$ then $\alpha+\beta\notin(\tilde{\Gamma}'(f)+\mathbb{Z}\delta)$.

Notation 5.7. Observe that the points $P[1]$ and $P[0]+\delta_2$ differ by $(1/n,0)$. Moreover, the two paths P_∞ and $P_\infty+\delta_2$ form a double crossing at these two points, thus they form a small region as in Figure 8. Therefore no shift of P_∞ can contain an integer point in this region. In our analysis, we usually treat this region as a "single point" and write $P[1]\approx P[0]+\delta_2$ and $P[n]\approx P[1]+\delta_1$. By an abuse of terminology, we will say that P_∞ is *below* $P_\infty+\delta_2$ and *above* $P_\infty+\delta_1$.

Lemma 5.8. *The bounded affine permutations f_1 and f_2 are repetition-free.*

Proof. Let us compare the big path $P_\infty^{(f_1)}$ with $(P[1\to n])_\infty := \bigcup_{t\in\mathbb{Z}}(P[1\to n]+t\delta_1)$, where we identify the points $P[n]+(t-1)\delta_1\approx P[1]+t\delta_1$ for all $t\in\mathbb{Z}$. It is easy to see that these two paths are *equivalent* in the sense that for each $\alpha=(a,b)\in\mathbb{Z}^2$, we have

$$|P_\infty^{(f_1)}\cap(P_\infty^{(f_1)}+\alpha)| = |(P[1\to n])_\infty\cap((P[1\to n])_\infty+\alpha)|,$$

where the intersection points are counted modulo δ_1. Thus we need to analyze the intersections of $(P[1\to n])_\infty$ with its shifts.

Let $\alpha\in[k_1-1]\times[n_1-1]$ and $Q:=P+\alpha$. Let $s:=|(P[1\to n])_\infty\cap(Q[1\to n])_\infty|$. Since f is repetition-free, $P[1\to n]$ intersects $Q[1\to n]$ at most twice. Moreover, $P[1\to n]$ can intersect $Q[1\to n]+t\delta_1$ only for $t\in\{-1,0\}$. Thus $s\leq 4$.

Suppose that $s>2$. Since s is even, we have $s\geq 4$, and thus $s=4$. We see that $P[1\to n]$ intersects each of $Q[1\to n]$ and $Q':=Q[1\to n]-\delta_1$ twice. Suppose first that $Q[1]$ is above P_∞. Then $Q[1-n\to 1]$ stays below $Q'[1\to n]$ which intersects $P[1\to n]$, and therefore $Q[1-n\to 1]$ intersects $P[1\to n]$. We have found three intersection points of $P[1\to n]$ with Q_∞, a contradiction. Suppose now that $Q[1]$ is below P_∞. Then $Q'[n\to 2n]$ stays above

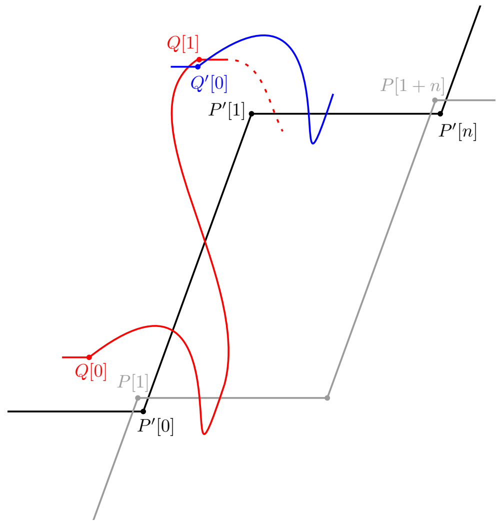

FIGURE 9. Proof of Proposition 5.9.

$Q[1 \to n]$ which intersects $P[1 \to n]$, and thus $Q'[n \to 2n]$ intersects $P[1 \to n]$. We have found three intersection points of $P[1 \to n]$ with Q'_∞, a contradiction. We have shown that f_1 is repetition-free.

Similarly, we check that $P_\infty^{(f_2)}$ is equivalent to $(P[0 \to 1])_\infty := \bigcup_{t\in\mathbb{Z}}(P[0 \to 1] + t\delta_2)$ and use it to deduce that f_2 is repetition-free. □

Note that $P[1 \to n]$ has *low slope* since it connects $P[1]$ to $P[n] \approx P[1]+\delta_1$ while $P[0 \to 1]$ has *high slope* since it connects $P[0]$ to $P[1] \approx P[0] + \delta_2$; cf. Lemma 5.5(ii).The next result states that a shifted segment of high slope cannot cross a segment of low slope from above.

Proposition 5.9. *Let $\alpha \in \mathbb{Z}^2$ and $Q := P + \alpha$. Then $Q[0 \to 1]$ cannot cross $P[1 \to n]$ from above.*

Proof. Suppose otherwise that $Q[0 \to 1]$ crosses $P[1 \to n]$ from above. We consider the cases according to the positions of $Q[0]$ and $Q[1]$ relative to P_∞. First, assume that $Q[0]$ is below P_∞ and $Q[1]$ is above P_∞. Then $Q[0 \to 1]$ intersects P_∞ at least 3 times, a contradiction.

From now on we assume that $Q[0]$ is above P_∞. (The case of $Q[1]$ being below P_∞ is completely analogous.) Let $P' := P + \delta_2$ and $Q' := Q + \delta_2$. Since $Q[0]$ is above P_∞, $Q[1] \approx Q'[0]$ are both above P'_∞. Moreover, Q crosses $P[1 \to n]$ from above, thus Q' crosses $P'[1 \to n]$ from above.

Definition 5.10. For an integer point q of Q, we say that q *is vertically above* P if there exists an integer point p of P with $n(q) = n(p)$, and q is above p.

Since $Q'[0 \to 1]$ intersects $P'[1 \to n]$, it follows that $Q'[0]$ is vertically above P'. Consider the path $Q[1 \to 1+n]$. It crosses P_∞ from above at a single point which belongs to $Q[n \to 1+n] \cap P[1+n \to 2n]$. Moreover, it stays below Q' which crosses $P'[1 \to n]$ from above. Thus $Q[1 \to 1+n]$ crosses P' from above. Since $Q[1 \to 1+n]$ cannot cross $P[1 \to 1+n]$ from above, it must cross P' from below. The remaining part of $Q[1 \to 1+n]$ still has to cross P_∞ from above, however, it cannot cross P'_∞ since it has already crossed P' twice. Since P_∞ is below P'_∞, we get a contradiction. See Figure 9. □

Lemma 5.11. *Let $\alpha, \beta \in [k-1] \times [n-1]$ be such that* $\mathrm{slope}(\alpha) > \mathrm{slope}(\beta)$. *Suppose that there are two subpaths $P[a \to b]$ and $P[c \to d]$ of P_∞ such that $P[a \to b]$ crosses $P[a \to b]+\alpha$ while $P[c \to d]$ crosses $P[c \to d] + \beta$. Then there exist $s, t \in \mathbb{Z}$ such that $P[a \to b] + s\alpha$ crosses $P[c \to d] + t\beta$ from below.*

Proof. Consider the two infinite unions $R_\alpha := P[a \to b] + \mathbb{Z}\alpha$ and $R_\beta := P[c \to d] + \mathbb{Z}\beta$. Observe that R_α (resp., R_β) is a path-connected subset of $\mathbb{R}^2$. Thus it contains an infinite piecewise linear curve S_α (resp., S_β) such that for each $r \in \mathbb{Z}$, S_α (resp., S_β) contains a unique point $x_{\alpha,r}$ (resp., $x_{\beta,r}$) satisfying $n(x_{\alpha,r}) = n(x_{\beta,r}) = r$. Here, we are additionally assuming that the vertical coordinates of $x_{\alpha,r}$ and $x_{\beta,r}$ are increasing functions of r.

When $r \ll 0$, $x_{\alpha,r}$ is below $x_{\beta,r}$, and when $r \gg 0$, $x_{\alpha,r}$ is above $x_{\beta,r}$. Let $r \in \mathbb{Z}$ be the smallest integer such that $x_{\alpha,r}$ is *not* below $x_{\beta,r}$. Thus $x_{\alpha,r-1}$ is below $x_{\beta,r-1}$ and either $x_{\alpha,r} = x_{\beta,r}$ or $x_{\alpha,r}$ is above $x_{\beta,r}$. In each case, it is straightforward to check that a shift $P[a \to b] + s\alpha$ (passing through either $x_{\alpha,r}$ or $x_{\alpha,r-1}$ or both) crosses a shift $P[c \to d] + t\beta$ (passing through either $x_{\beta,r}$ or $x_{\beta,r-1}$ or both) from below. □

Remark 5.12. The same argument applies when either $(a, b, \alpha) = (1, n, \delta_1)$ or $(c, d, \beta) = (0, 1, \delta_2)$. (Since $\mathrm{slope}(\delta_1) < \mathrm{slope}(\delta_2)$ by Lemma 5.5(ii), we cannot have both.) Suppose for instance that $(c, d, \beta) = (0, 1, \delta_2)$. Even though $P[0 \to 1]$ does not intersect $P[0 \to 1] + \delta_2$, since we identify $P[1] \approx P[0] + \delta_2$, the union $R_\beta = P[0 \to 1] + \mathbb{Z}\delta_2$ still contains an infinite connected (modulo our identification) piecewise linear curve.

Given two paths Q, P, we say that Q *is above* P if whenever two integer points $q \in Q$ and $p \in P$ satisfy $n(q) = n(p)$, we have that q is above p. (This condition is vacuously true if the projections of Q and P onto the horizontal axis do not overlap.)

Lemma 5.13. *Let $\alpha \in [k-1] \times [n-1]$ and $Q := P + \alpha$. Assume that $Q[1]$ is above P_∞. Then $Q[1 \to n]$ and $P[1 \to n]$ cannot intersect twice.*

Proof. Assume otherwise that they intersect twice. Our temporary goal is to show that

$$(P[0 \to 1])_\infty \text{ is below } (Q[0 \to 1])_\infty. \tag{5.1}$$

We observe that Q satisfies the following properties:

(a) $Q[1]$ is above P_∞;
(b) $Q[1 \to n]$ intersects $P[1 \to n]$ twice;
(c) $n(Q[1]) \geq n(P[1])$.

Let $Q' := Q - \delta_2$ so that $Q'[1] \approx Q[0]$. In order to show (5.1), it suffices to prove that if Q satisfies (a)–(c) then either

(i) Q' satisfies (a)–(c), or
(ii) $Q'[0 \to 1]$ is above $P[0 \to 1]$ with $n(Q'[0]) < n(P[0])$.

If (i) holds for Q' then we proceed by induction, applying the same argument to $Q' - t\delta_2$ for $t = 1, 2, \ldots$, until we find that (ii) holds for some $Q - s\delta_2$ with $s > 0$. But then all integer points of $P[0 \to 1]$ are below $\bigcup_{t=0}^{s}(Q - t\delta_2)$, which proves (5.1).

Assume that Q satisfies (a)–(c). Since $Q[1 \to n]$ and $P[1 \to n]$ intersect twice, $Q[0 \to 1]$ is above P_∞ and $P[0 \to 1]$ is below Q_∞. Then $Q'[0 \to 1]$ is above P'_∞, where $P' := P - \delta_2$. We have the following situation:

- apart from the double crossing, $Q'[1 - n \to 1]$ is below Q_∞;
- $Q'[1 - n]$ and $Q'[1]$ are above P_∞;

- $Q'[1-n \to 0]$ intersects $P'[1-n \to 0]$ twice;
- $P'[1] \approx P[0]$ is below Q'_∞.

These statements imply that $Q'[1-n \to 1]$ crosses $P[-n \to 0]$ twice, first from above and then from below. Moreover, the second crossing (from below) must belong to $P[1-n \to 0]$ since it has to come after both crossings of $Q[1-n \to 0]$ with $P'[1-n \to 0]$. In particular, no part of $Q'[1-n \to 1]$ is below $P[0 \to 1]$, and thus $Q'[0 \to 1]$ is above $P[0 \to 1]$.

Suppose that $n(Q'[1]) < n(P[1])$. Then $n(Q'[0]) < n(P[0])$. We have just shown that $Q'[0 \to 1]$ is above $P[0 \to 1]$, so we arrive at case (ii).

Suppose now that $n(Q'[1]) \geq n(P[1])$. Then Q' satisfies (a) and (c). Moreover, we also have $n(Q'[1-n]) \geq n(P[1-n])$ and $n(Q[0]) \geq n(P[0])$. In view of the above statements, this implies that $Q'[1-n \to 0]$ intersects $P[1-n \to 0]$ twice, i.e., Q' also satisfies (b). We arrive at case (i). We are done with the proof of (5.1), and now we will use it to finish off the proof of the lemma.

Observe that if $\operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_2)$ then we get a contradiction by Lemma 4.6(i) and (5.1). Thus $\operatorname{slope}(\alpha) > \operatorname{slope}(\delta_2)$. By Lemma 5.11 and Remark 5.12, for some $s, t, \in \mathbb{Z}$, we have that $P[1 \to n] + s\alpha$ crosses $P[0 \to 1] + t\delta_2$ from below, contradicting Proposition 5.9. □

Lemma 5.14. *Let $\alpha \in \Gamma'(f)$.*

(i) *If $\operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_1)$ then $\alpha \in \tilde{\Gamma}'(f_1) + \mathbb{Z}\delta_1$.*
(ii) *If $\operatorname{slope}(\alpha) \geq \operatorname{slope}(\delta_2)$ then $\alpha \in \tilde{\Gamma}'(f_2) + \mathbb{Z}\delta_2$.*

Proof. We prove (i). The proof of (ii) is completely analogous.

First, if $\operatorname{slope}(\alpha) = \operatorname{slope}(\delta_1)$ then $\alpha \in \tilde{\Gamma}'(f_1) + \mathbb{Z}\delta_1$ by Lemma 4.6(iii). Assume that $\operatorname{slope}(\alpha) < \operatorname{slope}(\delta_1)$. By Lemma 5.11 and Remark 5.12, there are $s, t \in \mathbb{Z}$ such that $P[1 \to n] + s\delta_1$ crosses $P + t\alpha$ from below. This crossing cannot belong to $P[0 \to 1] + t\alpha$ by Proposition 5.9. Thus it belongs to $P[1 \to n] + t\alpha$, so $t\alpha \in \tilde{\Gamma}'(f_1) + \mathbb{Z}\delta_1$. By Lemma 5.5(i), we get $\alpha \in \tilde{\Gamma}'(f_1) + \mathbb{Z}\delta_1$. □

The following straightforward result describes a natural transformation that swaps the notions of "above" and "below." We refer to it as the $180°$-*rotation*.

Proposition 5.15. *For $f \in \Theta_{k,n}$, let $g : \mathbb{Z} \to \mathbb{Z}$ be given by*

$$g(j) := n - f^{-1}(-j) \quad \text{for all } j \in \mathbb{Z}.$$

Then $g \in \Theta_{k,n}$ and the paths $P^{(f)}$ and $P^{(g)}$ are related as

$$P^{(g)} = \delta - P^{(f)}. \tag{5.2}$$

For each point $x \in \mathbb{R}^2$, x is above (resp., below) $P^{(f)}_\infty$ if and only if $\delta - x$ is below (resp., above) $P^{(g)}_\infty$. □

Given $\alpha, \beta \in \mathbb{R}^2$, we say that α is *weakly southwest* of β and write $\alpha \preceq \beta$ if $n(\alpha) \leq n(\beta)$ and $k(\alpha) \leq k(\beta)$. We write $\alpha \prec \beta$ if $\alpha \preceq \beta$ and $\alpha \neq \beta$.

Lemma 5.16. *Let $\alpha \in \Gamma'(f)$ and $\beta := \delta - \alpha$.*

(i) *If $\operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_1)$ and $\alpha \prec \delta_1$ then $\beta \notin \tilde{\Gamma}'(f_2) + \mathbb{Z}\delta_2$.*
(ii) *If $\operatorname{slope}(\beta) \geq \operatorname{slope}(\delta_2)$ and $\beta \prec \delta_2$ then $\alpha \notin \tilde{\Gamma}'(f_1) + \mathbb{Z}\delta_1$.*

Proof. In view of Proposition 5.15, we only prove (i). By Corollary 4.4, we have $\beta \in \Gamma'(f)$. Let $Q := P + \alpha$. By Lemma 5.14, $\alpha \in \tilde{\Gamma}'(f_1) + \mathbb{Z}\delta_1$ and $\beta \in \tilde{\Gamma}'(f_2) + \mathbb{Z}\delta_2$, or equivalently,

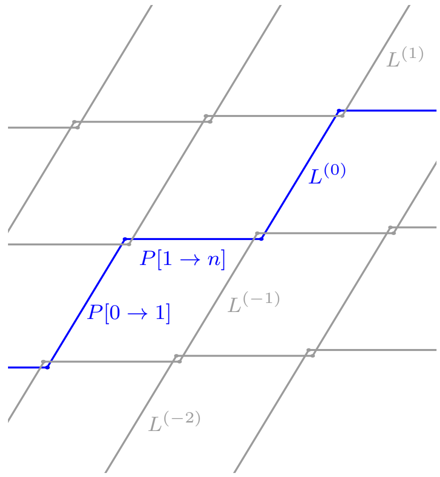

FIGURE 10. The proof of Lemma 5.16.

$Q[1 \to n]$ intersects $(P[1 \to n])_\infty$ twice and $Q[0 \to 1]$ intersects $L := (P[n \to 1+n])_\infty$ twice. Moreover, since $(0,0) \prec \alpha \prec \delta_1$, we see that

$$P[0] \prec Q[0] \prec Q[1] \prec P[n]. \tag{5.3}$$

For $t \in \mathbb{Z}$, we let $P_\infty^{(t)} := P_\infty + t\delta_2 = P_\infty - t\delta_1$ and $L^{(t)} := L \cap P_\infty^{(t)}$; see Figure 10. Thus there exists a unique integer $t \in \mathbb{Z}$ such that $Q[1]$ is above $P_\infty^{(t)}$ and below $P_\infty^{(t+1)}$. We know that $Q[0 \to 1]$ intersects L, and by (5.3), it can only intersect $L^{(<0)} := \bigcup_{s<0} L^{(s)}$. Similarly, we observe that $Q[1 \to n]$ must intersect $P[1 \to n] \cup R[1 \to n]$ exactly twice, where $R := P + \delta_1$ is a subpath of $P_\infty^{(-1)}$. We consider four cases.

Case 1: $t \geq 0$ and $Q[0]$ is above P_∞. Since $L^{(<0)}$ is below P_∞ and $Q[0 \to 1]$ intersects it twice, we see that $Q[0 \to 1]$ intersects P_∞ twice. Then $Q[1 \to n]$ cannot intersect P_∞, so it has to stay above P_∞. Thus $Q[1 \to n]$ must intersect $R[1 \to n]$ twice, which is impossible since $R[1 \to n]$ is below P_∞.

Case 2: $t \geq 0$ and $Q[0]$ is below P_∞. Thus $Q[0 \to 1]$ intersects P_∞ once, and therefore so does $Q[1 \to n]$. Thus $Q[1 \to n]$ must also intersect $R[1 \to n]$. Since $Q[1]$ is above P_∞, $Q[n] \approx Q[1] + \delta_1$ is above $P_\infty^{(-1)} = P_\infty + \delta_1$. Since $Q[1]$ is also above $P_\infty^{(-1)}$, we see that $Q[1 \to n]$ intersects $P_\infty^{(-1)}$ twice. Since $Q[n]$ is above $P_\infty^{(-1)}$, so is $Q[0]$. In order for $Q[0 \to 1]$ to intersect $L^{(<0)}$, it must intersect $P_\infty^{(-1)}$, since each point of $L^{(<0)}$ is either on or below $P_\infty^{(-1)}$. Thus Q intersects $P_\infty^{(-1)}$ at least three times, a contradiction.

Case 3: $t < 0$ and $Q[1]$ is above L. Then $Q[0]$ is above L. Recall that $Q[1]$ is above $P_\infty^{(t)}$ and below $P_\infty^{(t+1)}$, thus $Q[0]$ and $Q[n]$ are above $P_\infty^{(t-1)}$ and below $P_\infty^{(t)}$. We see that each of $Q[0 \to 1]$ and $Q[1 \to n]$ intersects $P_\infty^{(t)}$ exactly once. Let q be the first intersection point of $Q[0 \to 1]$ with $P_\infty^{(t)} \cup P_\infty^{(t-1)}$. We claim that q belongs to $P_\infty^{(t-1)}$. Indeed, suppose otherwise that $q \in P_\infty^{(t)}$. Since $Q[0 \to 1]$ has to intersect L but it can no longer intersect $P_\infty^{(t)}$, the first intersection point ℓ of $Q[0 \to 1]$ with L has to belong to $L^{(s)}$ for some $s > t$. In order for this to happen, $Q[0 \to 1]$ must intersect $P_\infty^{(s)}$ twice (with the second crossing at ℓ), and therefore the remaining part of $Q[0 \to 1]$ will stay below $P_\infty^{(s)}$, and thus below L. We see that $Q[1]$ is below L, contradicting our assumption. Thus $q \in P_\infty^{(t-1)}$. Specifically, we have $q \in (P[1 \to 1+n] + (t-1)\delta_2)$, which is the lower boundary of the region bounded by $P_\infty^{(t-1)} \cup P_\infty^{(t)}$ containing $Q[0]$.

Consider $Q' := Q + \delta_2$. The first intersection point $q' = q + \delta_2$ of Q' with $P_\infty^{(t)} \cup P_\infty^{(t+1)}$ belongs to $P_\infty^{(t)}$. Since $Q[1 \to n]$ stays below Q' and intersects $P_\infty^{(t)}$, we see that $Q[1 \to n]$ stays below $P_\infty^{(t+1)}$, and the unique intersection point of $Q[1 \to n]$ with $P_\infty^{(t)}$ belongs to $P[1 \to 1+n] + t\delta_2$. Since $t < 0$, $P[1 \to n]$ stays above $P_\infty^{(t)}$, and thus $Q[1 \to n]$ must intersect $R[1 \to n]$. If $t < -1$ then $R[1 \to n]$ is above $P_\infty^{(t)}$ and we get a contradiction. If $t = -1$ then we have already shown that the only intersection point of $Q[1 \to n]$ with $P_\infty^{(-1)}$ belongs to $P[1 \to 1+n] + t\delta_2$ which is disjoint from $R[1 \to n]$.

Case 4: $t < 0$ and $Q[1]$ is below L. Then $Q[0]$ is below L. It is still true that each of $Q[0 \to 1]$ and $Q[1 \to n]$ intersects $P_\infty^{(t)}$ exactly once. Thus the second point of $Q[0 \to 1] \cap L$ belongs to $L^{(s)}$ for some $s > t$. Thus $Q[0 \to 1]$ intersects $P_\infty^{(s)}$ twice, so $Q[1 \to n]$ stays below $P_\infty^{(s)}$. Since $Q[0 \to 1]$ can only intersect $L^{(<0)}$, we find that $s < 0$. Thus $Q[1 \to n]$ cannot intersect $P[1 \to n] \cup R[1 \to n]$, a contradiction. □

Using Lemma 5.16, the result of Lemma 5.14 can be strengthened as follows.

Corollary 5.17. *Let* $\alpha \in \Gamma'(f)$.

(i) *If* $\operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_1)$ *then* $\alpha \in \Gamma'(f_1) \sqcup \{\delta_1\}$.
(ii) *If* $\operatorname{slope}(\alpha) \geq \operatorname{slope}(\delta_2)$ *then* $\alpha \in \Gamma'(f_2) \sqcup \{\delta_2\}$.

Proof. Again, by Proposition 5.15, it suffices to prove (i). By Lemma 5.14(i), we have $\alpha \in \tilde{\Gamma}'(f_1) + \mathbb{Z}\delta_1$, and recall that $\alpha \in [k-1] \times [n-1]$ since $\alpha \in \Gamma'(f)$. Let $m \in \mathbb{Z}$ be the unique integer satisfying $\alpha \in \tilde{\Gamma}'(f_1) + m\delta_1$. Clearly, $m \geq 0$. Our goal is to show that $m = 0$. Assume for the sake of contradiction that $m > 0$. Thus $\delta_1 \prec \alpha$. Let $\beta := \delta - \alpha$, then $\beta \prec \delta_2$ and $\operatorname{slope}(\beta) \geq \operatorname{slope}(\delta_2)$. We get a contradiction by Lemma 5.16(ii). □

Lemma 5.18. *Let* $\alpha \in \Gamma'(f) \setminus \{\delta_1\}$ *be such that* $\operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_1)$. *Then* $\delta_1 - \alpha \in \Gamma'(f)$.

Proof. Let $x := \delta_1 - \alpha$, and assume $x \notin \Gamma'(f)$. By Corollary 5.17(i), we have $\alpha \in \Gamma'(f_1)$, so there exists $s \in \mathbb{Z}$ such that for $Q := P + \alpha - s\delta_1$, we have that $Q[1 \to n]$ crosses $P[1 \to n]$. Since $x \notin \Gamma'(f)$, we cannot have $s = 1$. If $s \notin \{0, 1\}$ then $Q[1 \to n]$ and $P[1 \to n]$ cannot intersect at all because $\alpha \in \Gamma'(f_1)$ implies $0 \prec \alpha \prec \delta_1$.

Thus $s = 0$ and $Q = P + \alpha$. So $Q[1 \to n]$ intersects $P[1 \to n]$, and also $Q[1 \to n]$ does not intersect $P[1 \to n] + \delta_1$ (because $x \notin \Gamma'(f)$). Thus $Q[1 \to n]$ intersects $P[1 \to n]$ twice. Assume first that $Q[1]$ is below P. Let $Q' := Q + \delta_2$. Thus $Q'[0] \approx Q[1]$ and Q' is above Q. Therefore, Q' must intersect $P[1 \to n]$. Since $P = Q' + x - \delta$, we see that $x \in \Gamma'(f)$, a contradiction. If $Q[1]$ is above P then we get a contradiction by Lemma 5.13. □

Let Δ_1 be the convex hull of $\{0, \delta_1, \delta\}$ and Δ_2 be the convex hull of $\{0, \delta_2, \delta\}$. Denote $(\Delta_1 \cup \Delta_2)_{\mathbb{Z}} := (\Delta_1 \cup \Delta_2) \cap \mathbb{Z}^2$.

Lemma 5.19. *We have* $(\Delta_1 \cup \Delta_2)_{\mathbb{Z}} \subset \tilde{\Gamma}'(f)$.

Proof. Let $x \in \mathbb{Z}^2 \cap \Delta_1$, and suppose that $x \notin \tilde{\Gamma}'(f)$. First, x cannot be northwest of δ_1 by Lemma 5.5(iii). Thus either $x \prec \delta_1$ or $\delta_1 \prec x$.

Assume first that $x \prec \delta_1$. Then for $\alpha := \delta_1 - x$, we have $\operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_1)$ and $\langle \delta^\perp, \alpha \rangle < 0$. If $\alpha \notin \Gamma'(f)$ then by Lemma 5.5(i), we get $\delta_1 \notin \Gamma'(f)$, a contradiction. Thus $\alpha \in \Gamma'(f)$, in which case we are done by Lemma 5.18.

Assume now that $\delta_1 \prec x$. Applying the dual argument (cf. Proposition 5.15) to $y := \delta - x$, we find $y \in \Gamma'(f)$, and thus $x \in \Gamma'(f)$ by Corollary 4.4. Thus $\tilde{\Gamma}'(f)$ contains all lattice points of Δ_1, and by Corollary 4.4 again, it contains all lattice points of Δ_2 as well. □

Proposition 5.20. *We have* $\Gamma'(f_1) = G_1$ *and* $\Gamma'(f_2) = G_2$*, where*

$$G'_1 := \{\alpha \mid \alpha \in \Gamma'(f) \setminus \{\delta_1\} \text{ is such that } \operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_1)\},$$
$$G'_2 := \{\beta \mid \beta \in \Gamma'(f) \setminus \{\delta_2\} \text{ is such that } \operatorname{slope}(\beta) \geq \operatorname{slope}(\delta_2)\},$$
$$G_1 := G'_1 \cup (\delta_1 - G'_1), \quad \text{and} \quad G_2 := G'_2 \cup (\delta_2 - G'_2).$$

See Figure 12 for an example.

Proof. By Proposition 5.15, it suffices to prove $\Gamma'(f_1) = G_1$. By Corollary 4.4, $\Gamma'(f_1)$ is symmetric with respect to the map $\alpha \mapsto \delta_1 - \alpha$, so by Corollary 5.17, $G_1 \subset \Gamma'(f_1)$. Conversely, suppose that we have found $\alpha \in \Gamma'(f_1) \setminus G_1$. Since both sets are symmetric with respect to the map $\alpha \mapsto \delta_1 - \alpha$, we may assume that $\operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_1)$. By the definition of G_1, we have $\alpha \notin \Gamma'(f)$, so $\operatorname{slope}(\alpha) < \operatorname{slope}(\delta_1)$. By Lemma 4.6(i), $Q := P + \alpha$ is below P_∞.

Since $\alpha \in \Gamma'(f_1)$, $Q[1 \to n]$ intersects $(P[1 \to n])_\infty$ twice, and both intersections must belong to $P[1 \to n] \cup P'[1 \to n]$, where $P' := P + \delta_1$. Since $\alpha \notin \Gamma'(f)$, we see that $Q[1 \to n]$ cannot intersect $P[1 \to n]$, so it intersects $P'[1 \to n]$ twice. Observe that $P'[1] \approx P[n]$ is above Q_∞. We get a contradiction by Lemma 5.13 (applied to $\alpha' := \delta_1 - \alpha$, Q, and P'). □

Corollary 5.21. *We have*

$$\tilde{\Gamma}'(f) = (\Delta_1 \cup \Delta_2)_{\mathbb{Z}} \cup \Gamma'(f_1) \cup \Gamma'(f_2) \cup (\Gamma'(f_1) + \delta_2) \cup (\Gamma'(f_2) + \delta_1). \tag{5.4}$$

Proof. By Remark 5.6, $\Gamma'(f)$ contains no points which are southeast of δ_1 or northwest of δ_2. By Lemma 5.19, $\tilde{\Gamma}'(f)$ contains $(\Delta_1 \cup \Delta_2)_{\mathbb{Z}}$. For any $\alpha \in [k-1] \times [n-1]$ satisfying $0 \prec \alpha \prec \delta_1$ and $\operatorname{slope}(\alpha) \leq \operatorname{slope}(\delta_1)$, we have $\alpha \in \Gamma'(f)$ if and only if $\alpha \in \Gamma'(f_1)$ by Proposition 5.20. The case $0 \prec \alpha \prec \delta_2$ and $\operatorname{slope}(\alpha) \geq \operatorname{slope}(\delta_2)$ is handled similarly. The remaining two cases follow from the observation that both sides of (5.4) are centrally symmetric. □

Lemma 5.22. *For all* $\alpha \in \Gamma'(f_1)$ *and* $\beta \in \Gamma'(f_2)$*, we have* $\operatorname{slope}(\alpha) < \operatorname{slope}(\beta)$.

Proof. Assume otherwise that $\operatorname{slope}(\alpha) \geq \operatorname{slope}(\beta)$ for some $\alpha \in \Gamma'(f_1)$ and $\beta \in \Gamma'(f_2)$. We will consider the cases $\operatorname{slope}(\alpha) = \operatorname{slope}(\beta)$ and $\operatorname{slope}(\alpha) > \operatorname{slope}(\beta)$ separately.

Suppose that $\operatorname{slope}(\alpha) = \operatorname{slope}(\beta)$. Denote $\alpha = (a, b)$ and let $x := \frac{1}{\gcd(a,b)}\alpha$. Thus each of α and β is a positive integer multiple of x. By Lemma 5.5(i), we have $x \in \Gamma'(f_1) \cap \Gamma'(f_2)$. Suppose first that $\operatorname{slope}(x) \geq \operatorname{slope}(\delta_1)$. Let $y := \delta_1 - x$, thus $\operatorname{slope}(y) \leq \operatorname{slope}(\delta_1)$ and $y \prec \delta_1$. By Lemma 5.16, $\delta - y \notin \tilde{\Gamma}'(f_2) + \mathbb{Z}\delta_2$. On the other hand, $\delta - y = \delta_2 + x$ which clearly belongs to $\tilde{\Gamma}'(f_2) + \mathbb{Z}\delta_2$ since $x \in \Gamma'(f_2)$, a contradiction. Applying a dual argument (cf. Proposition 5.15) yields a contradiction when $\operatorname{slope}(x) \leq \operatorname{slope}(\delta_2)$. Since $\operatorname{slope}(\delta_1) < \operatorname{slope}(\delta_2)$, we are done with the case $\operatorname{slope}(\alpha) = \operatorname{slope}(\beta)$.

Suppose now that $\operatorname{slope}(\alpha) > \operatorname{slope}(\beta)$. By Lemma 5.11, we find that for some $s, t, \in \mathbb{Z}$, we have that $P[1 \to n] + s\alpha$ crosses $P[0 \to 1] + t\beta$ from below, contradicting Proposition 5.9. □

Corollary 5.23. *The points* δ_1 *and* δ_2 *are vertices of the convex hull of* $\tilde{\Gamma}'(f)$.

Proof. Indeed, let $\alpha \in \Gamma'(f_1)$ have the maximal slope and $\beta \in \Gamma'(f_2)$ have the minimal slope. Then the convex hull of $\tilde{\Gamma}'(f)$ is bounded from below by the rays $\delta_1 - \mathbb{R}_{\geq 0}\alpha$ and $\delta_1 + \mathbb{R}_{\geq 0}\beta$ and from above by the rays $\delta_2 - \mathbb{R}_{\geq 0}\beta$ and $\delta_2 + \mathbb{R}_{\geq 0}\alpha$. □

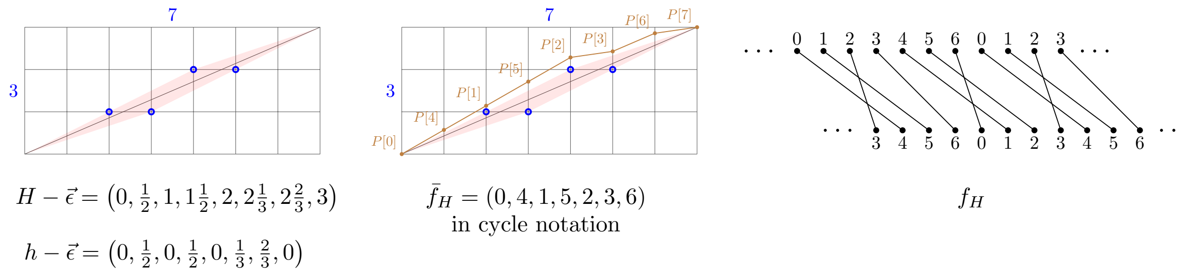

FIGURE 11. Constructing a concave profile H and a repetition-free permutation f_H for a given convex set Γ'. See Proposition 6.2 and Definition 6.3.

Proof of Theorem 5.1. We proceed by induction on k and n. Suppose that the statement is known for all smaller k and n, and consider some lattice point $x \notin \tilde{\Gamma}'(f)$ which belongs to the convex hull of $\tilde{\Gamma}'(f)$. By Lemma 5.19, we have $x \notin (\Delta_1 \cup \Delta_2)_{\mathbb{Z}}$. By Corollary 5.23, δ_1 and δ_2 are vertices of the convex hull of $\tilde{\Gamma}'(f)$. By the induction hypothesis, we know that the sets $\Gamma'(f_1)$ and $\Gamma'(f_2)$ are convex. We obtain a contradiction with Corollary 5.21, so we must have $x \in \tilde{\Gamma}'(f)$. □

6. Concave profiles and the counting formula

By Corollary 4.4 and Theorem 5.1, if $f \in \Theta_{k,n}$ is repetition-free then $\Gamma(f)$ is convex and centrally symmetric. In this section, we show that each convex centrally symmetric set arises in this way, as stated in Theorem 1.3(ii). We will use this construction to prove the counting formula (1.2) in Section 6.2, completing the proof of Theorem 1.3.

6.1. Concave profiles.

Definition 6.1. A sequence $H := (0 = H_0, H_1, \dots, H_n = k)$ of real numbers is called a *concave profile* if

- $0 < H_{i+1} - H_i < 1$ for all $0 \le i < n$,
- $H_{i+1} - H_i \ge H_{j+1} - H_j$ for all $0 \le i \le j < n$, and
- $h_i \ne h_j$ for $0 \le i \ne j < n$, where we set

$$h_r := H_r - \lfloor H_r \rfloor \quad \text{for} \quad 0 \le r \le n. \tag{6.1}$$

Given a concave profile H, we let

$$\Gamma'(H) := \{(a,b) \in [k-1] \times [n-1] \mid k - H_{n-b} \le a \le H_b\}.$$

As before, we let $\tilde{\Gamma}'(H) := \Gamma'(H) \sqcup \{(0,0),(k,n)\}$. We also let $P^{(H)}$ be the path connecting the points (r, H_r) for $r = 0, 1, \dots, n$. Thus $\tilde{\Gamma}'(H)$ consists of all lattice points weakly below $P^{(H)}$ and weakly above the 180°-rotation $(k,n) - P^{(H)}$ of $P^{(H)}$.

Denote

$$\Gamma^{\max}_{k,n} := \{(a, a+b) \mid (a,b) \in [k-1] \times [n-k-1]\}.$$

Proposition 6.2. *Let $\Gamma' \subset \Gamma^{\max}_{k,n}$ be convex and centrally symmetric. Then there exists a concave profile H satisfying $\Gamma' = \Gamma'(H)$.*

Proof. Choose a nonnegative strictly concave sequence $\vec{\epsilon} = (\epsilon_0, \epsilon_1, \dots, \epsilon_n)$ whose values are sufficiently small, and let H be such that the difference $H - \vec{\epsilon}$ records the maximal vertical coordinates of the intersection of the convex hull of $\tilde{\Gamma}'$ with the vertical line $n(x) = i$ for

$i = 0, 1, \dots, n$. Then clearly H is a concave profile and we have $\Gamma' = \Gamma'(H)$. See Figure 11 for an example. □

The following construction uses H to find a bounded affine permutation $f_H \in \Theta_{k,n}$ satisfying the desired properties.

Definition 6.3. Given a concave profile H, let $f = f_H \in \Theta_{k,n}$ be the unique bounded affine permutation such that for all $0 \le i, j < n$, we have $\bar{f}^i(0) < \bar{f}^j(0)$ if and only $h_i < h_j$, where h_i, h_j are defined in (6.1). In other words, writing $\bar{f} = (0, j_1, j_2, \dots, j_{n-1})$ in cycle notation, the indices $(j_1, j_2, \dots, j_{n-1})$ have the same relative order as $(h_1, h_2, \dots, h_{n-1})$. See Figure 11 for an example.

Proposition 6.4. *Let H be a concave profile and $f := f_H$. We have:*

(i) $\left\lfloor \frac{f^r(0)}{n} \right\rfloor = \lfloor H_r \rfloor$ *for all* $0 \le r \le n$;
(ii) $f \in \Theta_{k,n}$ *is repetition-free;*
(iii) $\Gamma'(f) = \Gamma'(H)$.

Proof. (i): We prove the result by induction on r. The base case $r = 0$ is clear. Suppose that the result holds for $0 \le r < n$. We have $h_{r+1} \ne h_r$. If $h_{r+1} > h_r$ then $\bar{f}^{r+1}(0) > \bar{f}^r(0)$, and thus $\left\lfloor \frac{f^{r+1}(0)}{n} \right\rfloor = \left\lfloor \frac{f^r(0)}{n} \right\rfloor$. It is also clear that $h_{r+1} > h_r$ implies $\lfloor H_{r+1} \rfloor = \lfloor H_r \rfloor$. Similarly, if $h_{r+1} < h_r$ then $\bar{f}^{r+1}(0) < \bar{f}^r(0)$, which implies $\left\lfloor \frac{f^{r+1}(0)}{n} \right\rfloor = \left\lfloor \frac{f^r(0)}{n} \right\rfloor + 1$ and $\lfloor H_{r+1} \rfloor = \lfloor H_r \rfloor + 1$.

(ii): Let $P_\infty^{(H)} := \bigcup_{t \in \mathbb{Z}} (P^{(H)} + t\delta)$ be the corresponding infinite path. Observe that for each $\alpha = (a, b) \in \mathbb{Z}^2$, we have $|P_\infty^{(H)} \cap (P_\infty^{(H)} + \alpha)| = |P_\infty^{(f)} \cap (P_\infty^{(f)} + \alpha)|$. Indeed, if $p \in P^{(H)}$ and $q \in P^{(H)} + \alpha$ have the same horizontal coordinate $r \in \mathbb{Z}$ then p is above q if and only if $H_r > a + H_{r-b}$. This condition is equivalent to having either $\lfloor H_r \rfloor > a + \lfloor H_{r-b} \rfloor$ or $\lfloor H_r \rfloor = a + \lfloor H_{r-b} \rfloor$ and $h_r > h_{r-b}$. By (i), this is equivalent to having $\frac{f^r(0)}{n} > a + \frac{f^{r-b}(0)}{n}$, which means that for the integer points $p' \in P^{(f)}$ and $q' \in P^{(f)} + \alpha$ satisfying $n(p') = n(q') = r$, the point p' is above q'. Since the path $P^{(H)}$ is the plot of a concave function, it intersects $P^{(H)} + \alpha$ at most once for each $\alpha \in \mathbb{Z}^2$. Thus $P_\infty^{(H)}$ intersects $P_\infty^{(H)} + \alpha$ at most twice, and therefore the same holds for $P_\infty^{(f)}$. The result follows by Proposition 4.3.

(iii): For $\alpha \in [k-1] \times [n-1]$, $P_\infty^{(H)}$ intersects $P_\infty^{(H)} + \alpha$ if and only if α is below $P^{(H)}$ and (k, n) is below $P^{(H)} + \alpha$. This is equivalent to $\alpha \in \Gamma'(H)$. Since $|P_\infty^{(H)} \cap (P_\infty^{(H)} + \alpha)| = |P_\infty^{(f)} \cap (P_\infty^{(f)} + \alpha)|$, this is equivalent to $\alpha \in \Gamma'(f)$. □

6.2. **Counting formula for concave profiles.** We prove (1.2) in two steps. We start by treating the case where $f = f_H$ arises from a concave profile. The case of arbitrary repetition-free $f \in \Theta_{k,n}$ is considered in Section 6.3 below.

Proposition 6.5. *Let H be a concave profile and let $f := f_H$. Then*

$$C_f = \# \operatorname{Dyck}(\Gamma(f)).$$

Proof. Let us say that a *slanted Dyck path* is a lattice path connecting $(0, 0)$ to (k, n) which stays above the main diagonal and consists of right steps $(0, 1)$ and up-right steps $(1, 1)$. Thus $\# \operatorname{Dyck}(\Gamma(f))$ counts the number of slanted Dyck paths which stay above $P^{(H)}$ (and do not share any points with $P^{(H)}$ except for the endpoints $(0, 0)$ and (k, n)).

In order to keep track of the size of the rectangle in which H lives, let us refer to H as a (k, n)-*concave profile*. We proceed by induction on n using Proposition 3.2. The base case

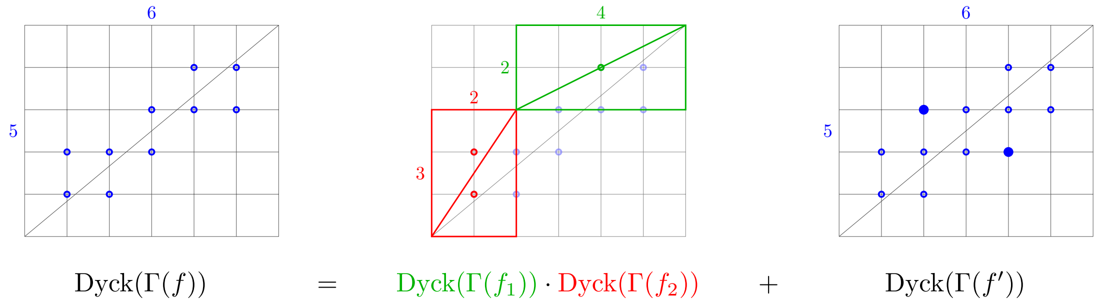

FIGURE 12. The Dyck path recurrence in the proofs of Proposition 6.5 and Theorem 1.3(i). Here $\Gamma(f') = \Gamma(f) \sqcup \{(a,b),(n-k-a,k-b)\}$ for some a, b. Each Dyck path above $\Gamma(f)$ either passes through (a,b) or stays above $\Gamma(f')$.

$n = 1$ is clear. Suppose now that $n > 1$ and that the claim has been shown for all $n' < n$ and also for all (k,n)-concave profiles H' satisfying $\Gamma'(H') \supsetneq \Gamma'(H)$. Let $0 < r < n$ be the index such that $0 < h_r < 1$ is maximal among $h_0, h_1, \dots, h_n$. Thus $\bar{f}^r(0) = n-1$, and we let $\epsilon := 1 - h_r$.

Assume first that $r = 1$. Let $g \in \Theta_{k-1,n-1}$ be given by $\bar{g}^i(0) := \bar{f}^{i+1}(0)$ for $0 < i < n$, and let $H' := (0 = H'_0, H'_1, \dots, H'_{n-1} = k-1)$ be given by $H'_i := H_{i+1} + \epsilon - 1$ for $0 \le i < n-1$. It is easy to check that $g = f_{H'}$ and that removing the first step (which must be up-right) of a slanted Dyck path above $P^{(H)}$ yields a slanted Dyck path above $P^{(H')}$ and vice versa, thus $\#\,\mathrm{Dyck}(\Gamma(f)) = \#\,\mathrm{Dyck}(\Gamma(g))$. Applying parts (b′)–(c′) of Proposition 3.2, we find $C_f = C_g$.

Assume next that $r = n-1$. Let $g \in \Theta_{k,n-1}$ be given by $\bar{g}^i(0) := \bar{f}^i(0)$ for $0 \le i < n-1$, and let $H' := (0 = H'_0, H'_1, \dots, H'_{n-1} = k)$ be given by $H'_i := H_i + \epsilon$ for $0 < i < n$. Similarly to the above, we have $\#\,\mathrm{Dyck}(\Gamma(f)) = \#\,\mathrm{Dyck}(\Gamma(g))$ and $C_f = C_g$.

Finally, assume that $1 < r < n-1$. Let $i := -1$, $j = 0$, $a := f^{-1}(i)$, $b := f^{-1}(j)$, $c := f(i)$, $d := f(j)$. By Definition 6.1, we have $H_1 - H_0 \ge H_{r+1} - H_r$, and since h_r is maximal, we get $h_1 \ge 1 + h_{r+1} - h_r > h_{r+1}$. Similarly, using $H_n - H_{n-1} \le H_r - H_{r-1}$ we get $h_{n-1} > h_{r-1}$. By Definition 6.3, this implies $d > c$ and $b > a$. We thus have $a < b < i < j < c < d$.

Let $f' := s_i f s_i$. Our goal is to relate $C_{f'}$ to C_{f_1}, C_{f_2}, and C_f as shown in Figure 12. It follows that $f' \in \Theta_{k,n}$ and that f' has a double crossing at i. Let f_1, f_2 be obtained from f by resolving the crossing $(i, i+1)$. By Corollary 3.6, we have

$$C_f = C_{f_1} C_{f_2} + C_{f'}. \tag{6.2}$$

Let $g = \sigma f' \in \Theta_{k,n}$ be the cyclic shift of f' defined in (2.1). We have $f^r(0) \equiv n-1$ and $g^r(0) \equiv 1$ modulo n, and for $1 \le s \le n$ such that $s \ne r$, we have $g^s(0) = f^s(0) + 1$. Choose $\epsilon' > \epsilon$ such that $\epsilon' < 1 - h_s$ for $s \ne r$ and let $H' := (0 = H'_0, H'_1, \dots, H'_n = k)$ be given by $H'_s := H_s + \epsilon'$ for all $0 < s < n$. One easily checks that $g = f_{H'}$. Since $\Gamma'(g) \supsetneq \Gamma'(f)$, by the induction hypothesis, we have $C_g = \#\,\mathrm{Dyck}(\Gamma(g))$. By Proposition 3.7, we have $C_{f'} = C_g$ and $\Gamma'(f') = \Gamma'(g)$, thus $C_{f'} = \#\,\mathrm{Dyck}(\Gamma(f'))$. It is straightforward to check that there exist concave profiles $H^{(1)}$ and $H^{(2)}$ such that $f_1 = f_{H^{(1)}}$ and $f_2 = f_{H^{(2)}}$, thus by the induction hypothesis, (6.2) becomes

$$C_f = \#\,\mathrm{Dyck}(\Gamma(f_1)) \cdot \#\,\mathrm{Dyck}(\Gamma(f_2)) + \#\,\mathrm{Dyck}(\Gamma(f')). \tag{6.3}$$

On the other hand, it is clear that $\Gamma'(f') = \Gamma'(f) \sqcup \{(k_2, r), (k-k_2, n-r)\}$, where $k_2 := \frac{f^r(0)+1}{n}$. The number of slanted Dyck paths above $P^{(H)}$ passing through the point (k_2, r) equals $\#\,\mathrm{Dyck}(\Gamma(f_1)) \cdot \#\,\mathrm{Dyck}(\Gamma(f_2))$. The slanted Dyck paths above $P^{(H)}$ which do not pass through (k_2, r) must stay above $P^{(H')}$. Therefore $C_f = \#\,\mathrm{Dyck}(\Gamma(f))$. □

6.3. Finishing the proof of Theorem 1.3.

Proof of Theorem 1.3(i). Let $f \in \Theta_{k,n}$ be repetition-free. By Corollary 4.4, $\Gamma(f)$ is centrally symmetric. By Theorem 5.1, $\Gamma(f)$ is convex. It remains to show the counting formula (1.2). Recall from Proposition 6.5 that the formula holds when $f = f_H$ arises from a concave profile. In particular, we may choose H to be such that $\Gamma'(H) = \Gamma^{\min}_{k,n}$. We now proceed by induction. By Corollary 5.3, the counting formula extends to all repetition-free $f \in \Theta_{k,n}$ satisfying $\Gamma'(f) = \Gamma^{\min}_{k,n}$, which is the base case. Suppose now that $\Gamma'(f) \supsetneq \Gamma^{\min}_{k,n}$ and that the result has been shown for all $n' < n$ and for all repetition-free $f' \in \Theta_{k,n}$ such that $\Gamma'(f') \subsetneq \Gamma'(f)$. (This induction proceeds in the opposite direction to the one in the proof of Proposition 6.5.) By Corollary 5.2, after applying some length-preserving simple conjugations, we may assume that f has a double crossing at some $i \in \mathbb{Z}$. Let $f' := s_i f s_i$ (thus $\ell(f) = \ell(f') + 2$) and f_1, f_2 be obtained from f by resolving the crossing $(i, i+1)$. By Corollary 3.6, we have

$$C_{f'} = C_{f_1} C_{f_2} + C_f. \tag{6.4}$$

This is different from (6.2) in that f and f' are swapped. By induction, we have $C_{f_1} = \#\,\mathrm{Dyck}(\Gamma(f_1))$, $C_{f_2} = \#\,\mathrm{Dyck}(\Gamma(f_2))$, and $C_{f'} = \#\,\mathrm{Dyck}(\Gamma(f'))$. Similarly to the proof of Proposition 6.5 (cf. Figure 12), we obtain the desired result $C_f = \#\,\mathrm{Dyck}(\Gamma(f))$. □

Proof of Theorem 1.3(ii). As explained in Section 6.1, for any convex centrally symmetric set $\Gamma' \subset \Gamma^{\max}_{k,n}$, there exists a concave profile H such that $\Gamma'(H) = \Gamma'$. The result follows from Proposition 6.4. □

7. Other interpretations and further directions

Computer experimentation reveals many other remarkable properties of repetition-free bounded affine permutations which we state below in conjectural form. We discuss them from a knot-theoretic perspective and in relation to positroid varieties, motivated by our recent results [GL20]. We also discuss the various interpretations of positroid Catalan numbers mentioned in the introduction.

7.1. Euler characteristic of open positroid varieties.

The relation between Definition 1.1 and (3.4) follows from [GL20]; here we give a brief explanation. When $f \in \Theta_{k,n}$, the torus T acts freely on Π°_f and the quotient $\mathcal{X}^\circ_f := \Pi^\circ_f / T$ is a smooth affine variety, called the *positroid configuration space* in [GL20]. The torus-equivariant Euler characteristic of Π°_f is simply the usual Euler characteristic of $\mathcal{X}^\circ_f$. The point count is given by $\#(\mathcal{X}^\circ_f)(\mathbb{F}_q) = \tilde{R}_f(q)$. By the Grothendieck–Lefschetz fixed-point formula, when a smooth variety X has polynomial point count, its Euler characteristic is equal to $\#X(\mathbb{F}_q)|_{q=1}$. This shows the agreement of Definition 1.1 and (3.4).

When $f \in \mathcal{B}_{k,n} \setminus \Theta_{k,n}$, the torus T no longer acts freely on Π°_f, and the torus-equivariant cohomology $H^*_T(\Pi^\circ_f)$ (or compactly supported cohomology $H^*_{T,c}(\Pi^\circ_f)$) is typically infinite-dimensional. In this case, Definition 1.1 does not immediately apply, but a q, t-power series is studied in [GL20]. In the present work, we use (3.4) for all $f \in \mathcal{B}_{k,n}$, but caution the reader that the situation is more subtle when $f \notin \Theta_{k,n}$.

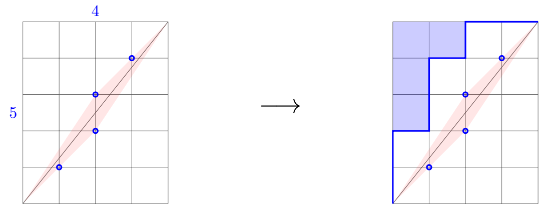

FIGURE 13. Constructing a Young diagram $\lambda(f)$ and the sequence $\mathbf{a}(f)$ from $\Gamma(f)$. Here $f \in \Theta_{k,n}$ for $k = 5$ and $n = 9$.

7.2. **Generalized q, t-Catalan numbers.** The most exciting computational evidence arises when comparing our results to the constructions in [GHSR20, BHM$^+$21].

Given a repetition-free $f \in \Theta_{k,n}$, let $\lambda(f) = (\lambda_1, \lambda_2, \dots, \lambda_k)$ be the partition consisting of all boxes inside the $k \times (n - k)$ rectangle which are above the diagonal and are strictly above all points of $\Gamma(f)$; see Figure 13. Let $\mathbf{a}(f) = (a_2, \dots, a_k)$ be given by $a_i = \lambda_{i-1} - \lambda_i$ for $2 \le i \le k$. To an arbitrary sequence $\mathbf{a} = (a_2, \dots, a_k)$ of nonnegative integers, the authors of [GHSR20] associate a *generalized q, t-Catalan number*[2] $C_{\mathbf{a}}(q, t)$, which may be explicitly described as a combinatorial sum over *Tesler matrices*. According to [GHSR20, Conjecture 1.3] attributed to A. Neguţ, if $a_2 \ge a_3 \ge \cdots \ge a_k \ge 0$ then $C_{\mathbf{a}}(q, t)$ has positive integer coefficients. If this condition is satisfied then we have $\mathbf{a} = \mathbf{a}(f)$ for some repetition-free $f \in \Theta_{k,n}$ in view of Theorem 1.3(ii). However, the convexity condition is more general: for instance, the sequence $\mathbf{a}(f) = (1, 0, 1, 0)$ in Figure 13 is not weakly decreasing.

Each sequence $\mathbf{a}$ also gives rise to a *Coxeter link* $\hat{\beta}(\mathbf{a})$. See [GN15, GNR21, OR17, GHSR20] and references therein for further details, such as an interpretation of $C_{\mathbf{a}}(q, t)$ in terms of flag Hilbert schemes and a conjectural relation between $C_{\mathbf{a}}(q, t)$ and Khovanov–Rozansky homology [KR08a, KR08b] of $\hat{\beta}(\mathbf{a})$.

In [GL20, Definition 1.9], we have associated a knot $\hat{\beta}_f$ to each $f \in \Theta_{k,n}$ and we showed in [GL20, Theorem 1.11] that $\tilde{R}_f(q)$ may be computed from the *HOMFLY* polynomial of $\hat{\beta}_f$. More generally, we gave a simple relation between the mixed Hodge polynomial $\mathcal{P}(\mathcal{X}_f^\circ; q, t)$ and Khovanov–Rozansky homology of $\hat{\beta}_f$ in [GL20, Equation (1.25)].

Conjecture 7.1. *Let $f \in \Theta_{k,n}$ be repetition-free.*

(i) *The knots $\hat{\beta}(\mathbf{a}(f))$ and $\hat{\beta}_f$ are isotopic.*
(ii) *Up to a monomial in q and t, we have $\mathcal{P}(\mathcal{X}_f^\circ; q, t) = C_{\mathbf{a}(f)}(q, t)$.*
(iii) *Up to a monomial in q, we have $\tilde{R}_f(q) = C_{\mathbf{a}(f)}(q, t = 1/q)$.*
(iv) *The polynomials $\mathcal{P}(\mathcal{X}_f^\circ; q, t)$, $C_{\mathbf{a}(f)}(q, t)$, and $\tilde{R}_f(q)$ have positive integer coefficients.*

Remark 7.2. Combining Theorem 1.3(i) with [GHSR20, Proposition 1.1], we see that $\tilde{R}_f(1) = C_{\mathbf{a}(f)}(1, 1)$, in agreement with Conjecture 7.1(iii).

Conjecture 7.1 becomes especially intriguing in view of [BHM$^+$21, Section 7]. Namely, to each sequence $\mathbf{a} = (a_2, \dots, a_k)$ of nonnegative integers, the authors of [BHM$^+$21] associate a symmetric function $\omega(D_{\mathbf{a}} \cdot 1)$ and show that one of the coefficients in its Schur expansion equals $C_{\mathbf{a}}(q, t)$. They conjecture that when $\mathbf{a}$ is obtained from a Young diagram above a concave curve (that is, precisely when $\mathbf{a} = \mathbf{a}(f)$ for some repetition-free $f \in \Theta_{k,n}$) then

[2] What we denote by $C_{\mathbf{a}}(q, t)$ was denoted by $F(a_2, \dots, a_k)$ in [GHSR20].

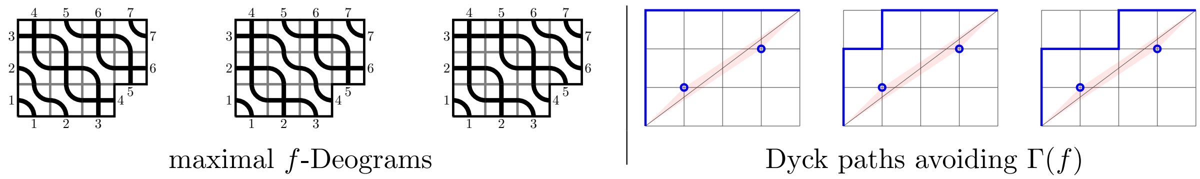

FIGURE 14. For each repetition-free $f \in \Theta_{k,n}$, the number of maximal f-Deograms (left) equals the number of Dyck paths which stay above $\Gamma(f)$ (right); see Example 7.5 and Problem 7.6.

the function $\omega(D_{\mathbf{a}} \cdot 1)$ is q,t Schur positive. The appearance of the convexity condition in both of these settings suggests that the whole symmetric function $\omega(D_{\mathbf{a}(f)} \cdot 1)$ may have an interpretation in terms of the geometry of Π_f°, which may explain the Schur positivity phenomenon.

A related promising direction would be to "categorify" the recurrence (3.1) to the level of Khovanov–Rozansky homology, in the spirit of [Hog18, Mel17, HM19]. Conversely, it would be interesting to "decategorify" the *categorified Young symmetrizers* of [Hog18] and interpret them in the positroid language. We hope to return to these questions in future work.

7.3. **c-equivalence classes.** Our experiments indicate that the combinatorics of c-equivalence classes has very rigid structure. Some statements describing this structure were shown in Section 2.2. The following conjecture implies that the objects C_f, $\tilde{R}_f(q)$, $\mathcal{P}(\Pi_f^\circ; q,t)$, and $\hat{\beta}_f$ depend only on $\Gamma'(f)$ when $f \in \Theta_{k,n}$ is repetition-free.

Conjecture 7.3. *Let $\Gamma \subset [k-1] \times [n-k-1]$ be centrally symmetric and convex. Then the set*

$$\{f \in \Theta_{k,n} \mid \Gamma'(f) = \Gamma\} \tag{7.1}$$

is a union of $\gcd(k,n)$*-many c-equivalence classes. They are cyclically permuted by the map σ from* (2.1).

Remark 7.4. Let $\epsilon_{k,n}$ be equal to $1/2$ if both k and n are even and to 0 otherwise. For $f \in \Theta_{k,n}$, denote

$$\nu(f) := \langle \delta^\perp, P_\infty^{(f)} \rangle - \epsilon_{k,n},$$

cf. (4.2). It is not hard to see that $\nu(f)$ is always an integer, so we let $0 \le \bar{\nu}(f) \le d-1$ be obtained by taking $\nu(f)$ modulo $d := \gcd(k,n)$.

Let $f \in \Theta_{k,n}$ be repetition-free. Observe that if $g := \sigma f$ is the cyclic shift of f then $\nu(g) - \nu(f) = 1$, however, if $f' \overset{c}{\sim} f$ then $\bar{\nu}(f') = \bar{\nu}(f)$. We therefore see that the set (7.1) contains at least d distinct c-equivalence classes, cyclically permuted by σ. The content of Conjecture 7.3 is that if $f, g \in \Theta_{k,n}$ are repetition-free and satisfy $(\Gamma(f), \bar{\nu}(f)) = (\Gamma(g), \bar{\nu}(g))$ then $f \overset{c}{\sim} g$.

7.4. **Deograms.** In [GL20, Section 9], we explained that for each $f \in \Theta_{k,n}$, the positroid Catalan number C_f equals the number $\#\operatorname{Deo}_f^{\max}$ of certain combinatorial objects called *maximal f-Deograms*, see [GL20, Definition 9.3]. Here, $\operatorname{Deo}_f^{\max}$ denotes the set of maximal f-Deograms, defined as follows. First, by [KLS13, Proposition 3.15], there exists a unique pair (v,w) of permutations in S_n such that $v \le w$, $w^{-1}(1) < \cdots < w^{-1}(k)$, $w^{-1}(k+1) < \cdots < w^{-1}(n)$, and $\bar{f} = wv^{-1}$. Thus w is *k-Grassmannian*, and each such permutation

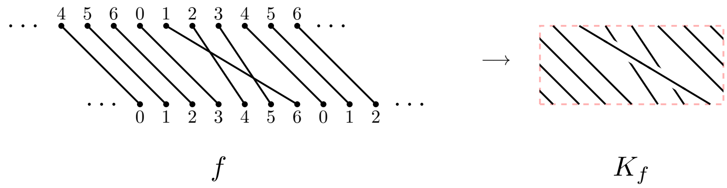

FIGURE 15. Associating a knot K_f inside $\mathbb{T}^2 \times \mathbb{R}$ to $f \in \Theta_{k,n}$. The dashed rectangle on the right represents the fundamental domain of $\mathbb{T}^2$.

corresponds to a Young diagram $\lambda(w)$ that fits inside a $k \times (n-k)$ rectangle. An f-Deogram is obtained by placing a crossing or an elbow inside each box of $\lambda(w)$ so that (i) the resulting strand permutation is v, and (ii) a certain *distinguished condition* is satisfied. An f-Deogram is *maximal* if it contains the maximal possible number of crossings, equivalently, assuming $f \in \Theta_{k,n}$, if it contains exactly $n-1$ elbows. In view of Theorem 1.3(i), when f is repetition-free, $C_f = \#\operatorname{Dyck}(\Gamma(f))$ also counts Dyck paths avoiding $\Gamma(f)$.

Example 7.5. Let $\bar{f} = (1,4,6,2,5,7,3)$ in cycle notation, and thus $f \in \Theta_{k,n}$ for $k = 3$ and $n = 7$. The unique factorization $\bar{f} = wv^{-1}$ as above is given by $v = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 4 & 3 & 5 & 6 & 7 \end{pmatrix}$ and $w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 1 & 7 & 2 & 3 \end{pmatrix}$ in two-line notation. The three maximal f-Deograms are shown in Figure 14(left). The three Dyck paths avoiding $\Gamma(f)$ are shown in Figure 14(right).

The following problem extends [GL20, Problem 9.6].

Problem 7.6. Let $f \in \Theta_{k,n}$ be repetition-free. Find a bijection between $\mathrm{Deo}_f^{\max}$ and $\operatorname{Dyck}(\Gamma(f))$.

7.5. **Fiedler invariant and knots in a thickened torus.** Let $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$ be a torus and $K : S^1 \hookrightarrow \mathbb{T}^2 \times \mathbb{R}$ be a knot inside a thickened torus. To this data, Fiedler [Fie93] associates an isotopy invariant W_K called the *small state sum*. Let us instead identify $\mathbb{T}^2$ with $\mathbb{R}^2/\langle (0,n),(1,0)\rangle$. For $f \in \Theta_{k,n}$ and $P := P^{(f)}$, let $\bar{P}$ be the image of $P \subset \mathbb{R}^2$ under the quotient map $\mathbb{R}^2 \to \mathbb{T}^2$. The points where $\bar{P}$ intersects itself correspond precisely to the inversions of f. Thus we may define a knot K_f inside $\mathbb{T}^2 \times \mathbb{R}$ whose projection to $\mathbb{T}^2$ coincides with $\bar{P}$, and for each inversion (i,j) of f, the line segment connecting $P[i]$ to $P[f(i)]$ lies above the line segment connecting $P[j]$ to $P[f(j)]$. See Figure 15.

It is straightforward to check that the formal sum W_{K_f} contains essentially the same information as the inversion multiset $\Gamma'(f)$. This leads to the following question: which parts of our story generalize to arbitrary *repetition-free knots* inside $\mathbb{T}^2 \times \mathbb{R}$? Here we say that a knot K inside $\mathbb{T}^2 \times \mathbb{R}$ is repetition-free if each nonzero coefficient of W_{K_f} is equal to ± 1. For example, it would be interesting to determine which subsets of $\mathbb{Z}^2/\mathbb{Z}\delta$ may appear with nonzero coefficients inside W_K for a repetition-free K, and whether the HOMFLY polynomial of K (or its Khovanov–Rozansky homology) have nice properties when K is repetition-free.

REFERENCES

[AHBC+16] Nima Arkani-Hamed, Jacob Bourjaily, Freddy Cachazo, Alexander Goncharov, Alexander Postnikov, and Jaroslav Trnka. *Grassmannian Geometry of Scattering Amplitudes*. Cambridge University Press, Cambridge, 2016.

[BGY06] K. A. Brown, K. R. Goodearl, and M. Yakimov. Poisson structures on affine spaces and flag varieties. I. Matrix affine Poisson space. *Adv. Math.*, 206(2):567–629, 2006.

[BHM+21] Jonah Blasiak, Mark Haiman, Jennifer Morse, Anna Pun, and George H. Seelinger. A Shuffle Theorem for Paths Under Any Line. `arXiv:2102.07931v1`, 2021.

[Deo85] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.*, 79(3):499–511, 1985.

[Fie93] Thomas Fiedler. A small state sum for knots. *Topology*, 32(2):281–294, 1993.

[FYH+85] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu. A new polynomial invariant of knots and links. *Bull. Amer. Math. Soc. (N.S.)*, 12(2):239–246, 1985.

[GHSR20] Eugene Gorsky, Graham Hawkes, Anne Schilling, and Julianne Rainbolt. Generalized q,t-Catalan numbers. *Algebr. Comb.*, 3(4):855–886, 2020.

[GL19] Pavel Galashin and Thomas Lam. Positroid varieties and cluster algebras. `arXiv:1906.03501v1`, 2019.

[GL20] Pavel Galashin and Thomas Lam. Positroids, knots, and q,t-Catalan numbers. `arXiv:2012.09745v2`, 2020.

[GN15] Eugene Gorsky and Andrei Neguţ. Refined knot invariants and Hilbert schemes. *J. Math. Pures Appl. (9)*, 104(3):403–435, 2015.

[GNR21] Eugene Gorsky, Andrei Neguţ, and Jacob Rasmussen. Flag Hilbert schemes, colored projectors and Khovanov-Rozansky homology. *Adv. Math.*, 378:107542, 115, 2021.

[HM19] Matthew Hogancamp and Anton Mellit. Torus link homology. `arXiv:1909.00418v1`, 2019.

[HN14] Xuhua He and Sian Nie. Minimal length elements of extended affine Weyl groups. *Compos. Math.*, 150(11):1903–1927, 2014.

[Hog18] Matthew Hogancamp. Categorified Young symmetrizers and stable homology of torus links. *Geom. Topol.*, 22(5):2943–3002, 2018.

[HY12] Xuhua He and Zhongwei Yang. Elements with finite Coxeter part in an affine Weyl group. *J. Algebra*, 372:204–210, 2012.

[Jon87] V. F. R. Jones. Hecke algebra representations of braid groups and link polynomials. *Ann. of Math. (2)*, 126(2):335–388, 1987.

[KL79] David Kazhdan and George Lusztig. Representations of Coxeter groups and Hecke algebras. *Invent. Math.*, 53(2):165–184, 1979.

[KL80] David Kazhdan and George Lusztig. Schubert varieties and Poincaré duality. In *Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979)*, Proc. Sympos. Pure Math., XXXVI, pages 185–203. Amer. Math. Soc., Providence, R.I., 1980.

[KLS13] Allen Knutson, Thomas Lam, and David E. Speyer. Positroid varieties: juggling and geometry. *Compos. Math.*, 149(10):1710–1752, 2013.

[KR08a] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. *Fund. Math.*, 199(1):1–91, 2008.

[KR08b] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. II. *Geom. Topol.*, 12(3):1387–1425, 2008.

[Lam16] Thomas Lam. Totally nonnegative Grassmannian and Grassmann polytopes. In *Current developments in mathematics 2014*, pages 51–152. Int. Press, Somerville, MA, 2016.

[Lus98] George Lusztig. Introduction to total positivity. In *Positivity in Lie theory: open problems*, volume 26 of *De Gruyter Exp. Math.*, pages 133–145. de Gruyter, Berlin, 1998.

[Mel17] Anton Mellit. Homology of torus knots. `arXiv:1704.07630v1`, 2017.

[MS16] Greg Muller and David E. Speyer. Cluster algebras of Grassmannians are locally acyclic. *Proc. Amer. Math. Soc.*, 144(8):3267–3281, 2016.

[OR17] Alexei Oblomkov and Lev Rozansky. HOMFLYPT homology of Coxeter links. `arXiv:1706.00124v2`, 2017.

[Pos06] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, `http://math.mit.edu/~apost/papers/tpgrass.pdf`, 2006.

[PT87] Józef H. Przytycki and Paweł Traczyk. Conway algebras and skein equivalence of links. *Proc. Amer. Math. Soc.*, 100(4):744–748, 1987.

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