SOME MORE AMPLITUHEDRA ARE CONTRACTIBLE

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Abstract. The amplituhedra arise as images of the totally nonnegative Grassmannians by projections that are induced by linear maps. They were introduced in Physics by Arkani-Hamed & Trnka (Journal of High Energy Physics, 2014) as model spaces that should provide a better understanding of the scattering amplitudes of quantum field theories. The topology of the amplituhedra has been known only in a few special cases, where they turned out to be homeomorphic to balls. The amplituhedra are special cases of Grassmann polytopes introduced by Lam (Current Developments in Mathematics 2014, Int. Press). In this paper we show that that some further amplituhedra are homeomorphic to balls, and that some more Grassmann polytopes and amplituhedra are contractible.

1. Introduction and statement of the main result

1.1. **Introduction.** Let n and k be integers such that $n \geq k \geq 1$. If $\mathrm{Mat}_{k,n}$ denotes the space of all real $k \times n$ matrices of rank k, then the real Grassmannian $\mathrm{G}_k(\mathbb{R}^n)$ — the space of all k-dimensional linear subspaces of $\mathbb{R}^n$ — can be defined as the orbit space $\mathrm{G}_k(\mathbb{R}^n) = \mathrm{GL}_k \setminus \mathrm{Mat}_{k,n}$. The totally nonnegative part of the Grassmannian is defined quite analogously.

Definition 1.1 (Postnikov [20, Sec. 3])**.** Let $n \geq k \geq 1$ be integers, let $\mathrm{Mat}^{\geq 0}_{k,n}$ be the space of all real $k \times n$ matrices of rank k all whose maximal minors are nonnegative, and let GL^+_k denote the group of all real $k \times k$ matrices with positive determinant, which acts freely on $\mathrm{Mat}^{\geq 0}_{k,n}$ by matrix multiplication from the left. The *totally nonnegative Grassmannian* $\mathrm{G}^{\geq 0}_k(\mathbb{R}^n)$ is the orbit space $\mathrm{G}^{\geq 0}_k(\mathbb{R}^n) = \mathrm{GL}^+_k \setminus \mathrm{Mat}^{\geq 0}_{k,n}$.

The totally nonnegative Grassmannian was introduced and studied by Postnikov in 2006 [20, Sec. 3], building on works by Lusztig [17] and by Fomin & Zelevinsky [9]. Subsequently, the geometric and combinatorial properties of the totally nonnegative Grassmannian were studied intensively. Rietsch & Williams showed that the totally nonnegative Grassmannian is contractible [22, Thm. 1.1]; an earlier argument by Lusztig [18, Sec. 4.4] can also be adapted to prove the same. Galashin, Karp & Lam [10, Thm. 1.1] proved that $\mathrm{G}^{\geq 0}_k(\mathbb{R}^n)$ is indeed homeomorphic to a closed $k(n-k)$-dimensional ball.

In 2014, the physicists Arkani-Hamed & Trnka [3, Sec. 9] introduced the amplituhedra as certain images of the totally nonnegative Grassmannians. They conjectured that their geometry describes scattering amplitudes in some quantum field theories. For a gentle introduction to amplituhedra in physics and mathematics consult [5]. Shortly after, Lam introduced Grassmann polytopes [16], which generalize amplituhedra.

Postnikov [20, Def. 3.2, Thm. 3.5] defined a CW structure on the totally nonnegative Grassmannian $\mathrm{G}^{\geq 0}_k(\mathbb{R}^n)$ such that each cell, also called a *positroid cell*, is indexed by the associated matroid – a positroid – of rank k on n elements, see also [21]. Furthermore, Rietsch & Williams [22] showed that the closures of positroid cells are contractible and that their boundaries are homotopy equivalent to spheres.

Definition 1.2. Let $k \geq 1$, $m \geq 0$ and $n \geq k+m$ be integers, and let Z be a real $(k+m) \times n$ matrix such that the assignment

$$\widetilde{Z}(\mathrm{span}(V)) = \mathrm{span}(VZ^\top) \tag{1}$$

induces a map

$$\widetilde{Z} : \mathrm{G}^{\geq 0}_k(\mathbb{R}^n) \longrightarrow \mathrm{G}_k(\mathbb{R}^{k+m}).$$

Here $V \in \mathrm{Mat}^{\geq 0}_{k,n}$, span denotes the row span of a matrix, and $Z^\top$ is the transpose of the matrix Z. The image $\widetilde{Z}(\bar{e})$ of a closed positroid cell $\bar{e}$ in the CW decomposition of the nonnegative Grassmannian $\mathrm{G}^{\geq 0}_k(\mathbb{R}^n)$ is called a *Grassmann polytope*, denoted by $\mathrm{P}_Z(e)$. If e is the maximal cell, which for this CW

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decomposition means $\bar{e} = \mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, and all $(k+m)\times(k+m)$ minors of the matrix Z are positive, then the Grassmann polytope $\mathrm{P}_Z(e)$ is called an *amplituhedron* and is denoted by $\mathcal{A}_{n,k,m}(Z)$.

The previous definition in particular means that if $v_1,\dots,v_k \in \mathbb{R}^n$ are linearly independent row vectors, then

$$\widetilde{Z}(\mathrm{span}\{v_1,\dots,v_k\}) = \mathrm{span}\{v_1Z^\top,\dots,v_kZ^\top\}.$$

The map $\widetilde{Z}$ is, said to be well defined, if $\mathrm{span}(VZ^\top)$ is a k-dimensional subspace of $\mathbb{R}^{k+m}$ for every $V \in \mathrm{Mat}_{k,n}^{\geq 0}$. The fact that the map $\widetilde{Z}$ is well defined when Z is a matrix with positive maximal minors was established by Arkani-Hamed & Trnka in [3] and by Karp in [14, Thm. 4.2]. Lam [16, Prop. 15.2], however, considers a larger class of matrices Z for which the map $\widetilde{Z}$ is still well defined.

The structure of the amplituhedron is known only in a few cases. In the case $m = 0$ all amplituhedra $\mathcal{A}_{n,k,0}(Z)$ are the point $\mathrm{G}_k(\mathbb{R}^k)$, whereas when $m = 1$ Karp & Williams [15, Cor. 6.18] have shown that the amplituhedron is homeomorphic to a ball. For $k = 1$ the amplituhedron is a cyclic polytope of dimension m on n vertices [24], and for $n = k+m$ the map Z is a linear isomorphism, and consequently the amplituhedron is homeomorphic to the totally nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, which is a ball by [10, Thm. 1.1]. Finally, Galashin, Karp & Lam [10, Thm. 1.2] proved that the cyclically symmetric amplituhedra, amplituhedra arising from particularly chosen matrices Z, are homeomorphic to balls whenever m is even. The topology of other Grassmann polytopes is unknown.

1.2. **Main results.** Our first result gives a family of contractible Grassmann polytopes.

Theorem 1.3. *Let $k \geq 1$ and $m \geq 0$ be integers, and let Z be a real $(k+m)\times(k+m+1)$ matrix such that the map $\widetilde{Z} : \mathrm{G}_k^{\geq 0}(\mathbb{R}^{k+m+1}) \longrightarrow \mathrm{G}_k(\mathbb{R}^{k+m})$ is well defined. Then the Grassmann polytope $\mathrm{P}_Z(e)$ is contractible for every positroid cell e in the CW decomposition of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^{k+m+1})$.*

The proof of Theorem 1.3 relies on classical results of Smale [23, Main Thm.] and Whitehead [25, Thm. 1].

The following is a consequence of Smale's result [23, Main Thm.].

Theorem 1.4 (Smale)**.** *Let X and Y be path connected, locally compact, separable metric spaces, and in addition let X be locally contractible. Let $f\colon X \longrightarrow Y$ be a continuous surjective proper map, that is, any inverse image of a compact set is compact. If for every $y \in Y$ the inverse image $f^{-1}(\{y\})$ is contractible, then the induced homomorphism*

$$f_\#\colon \pi_i(X) \longrightarrow \pi_i(Y)$$

is an isomorphism for all $i \geq 0$.

Recall that a continuous map $f\colon X \longrightarrow Y$ between topological spaces X and Y is a *weak homotopy equivalence* if the induced map on the path connected components $f_\#\colon \pi_0(X) \longrightarrow \pi_0(Y)$ is bijective, and for every point $x_0 \in X$ and for every integer $n \geq 1$ the induced map $f_\#\colon \pi_n(X,x_0) \longrightarrow \pi_n(Y,f(x_0))$ is an isomorphism.

Theorem 1.5 ([25, Thm. 1])**.** *Let X and Y be topological spaces that are homotopy equivalent to CW complexes. Then a continuous map $f\colon X \longrightarrow Y$ is a weak homotopy equivalence if and only if it is a homotopy equivalence.*

Since Theorem 1.5 requires that spaces have the homotopy type of a CW complex, the following theorem is a necessary ingredient in the proof of Theorem 1.3.

Theorem 1.6. *Let $k \geq 1$, $m \geq 0$ and $n \geq k+m$ be integers, and let Z be a real $(k+m)\times n$ matrix such that the map $\widetilde{Z}$ is well defined. Then for every positroid cell e in $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, the Grassmann polytope $\mathrm{P}_Z(e)$ is homotopy equivalent to a countable CW complex. Moreover, if $n = k+m+1$, the Grassmann polytope $\mathrm{P}_Z(e)$ is homotopy equivalent to a finite CW complex.*

In order to apply Theorem 1.4 to the map $\widetilde{Z}$, we need to understand its fibers. Thus we prove the following theorem.

Theorem 1.7. *Let $k \geq 1$ and $m \geq 0$ be integers, and let Z be a real $(k+m)\times(k+m+1)$ matrix such that the map $\widetilde{Z}$ is well defined. Then for every positroid cell e and for every point $y \in \mathrm{P}_Z(e)$, the inverse image $(\widetilde{Z}|_{\bar{e}})^{-1}(\{y\}) = \widetilde{Z}^{-1}(\{y\}) \cap \bar{e}$ under the restriction map $\widetilde{Z}|_{\bar{e}} : \bar{e} \longrightarrow \mathrm{P}_Z(e)$ is contractible.*

The proof of Theorem 1.7 is postponed to the next section, whereas the proof of Theorem 1.6 is given in Section 4. Here we show that Theorem 1.7 in combination with Theorem 1.4 and Theorem 1.5 implies our main result.

Proof of Theorem 1.3. Let e be a positroid cell in the CW decomposition of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^{k+m+1})$. We apply Theorem 1.4 to the map $\widetilde{Z} : \bar{e} \longrightarrow \mathrm{P}_Z(e)$. The spaces $\bar{e}$ and $\mathrm{P}_Z(e)$, as well as the map $\widetilde{Z}$, satisfy assumptions of Theorem 1.4. Furthermore, Theorem 1.7 implies that for every $y \in \bar{e}$, the fiber $\widetilde{Z}^{-1}(\{y\})$ is contractible. Thus, from Theorem 1.4 we have that the map $\widetilde{Z}$ is a weak homotopy equivalence.

The closed positroid cell $\bar{e}$ is a CW complex. Furthermore, the Grassmann polytope $\mathrm{P}_Z(e)$ is homotopy equivalent to a CW complex, by Theorem 1.6. Thus, from Theorem 1.5 we conclude that the map $\widetilde{Z}$ is a homotopy equivalence. Hence, the Grassmann polytope $\mathrm{P}_Z(e)$ is homotopy equivalent to the closed positroid cell $\bar{e}$, which is contractible, see [22]. □

In Theorem 1.6, we show that Grassmann polytopes are homotopy equivalent to CW complexes, using classical topological results. However, an even stronger result holds.

Theorem 1.8. *Every Grassmann polytope is a semialgebraic set. In particular, it admits a triangulation.*

Note that Theorem 1.8 claims that every Grassmann polytope $\mathrm{P}_Z(e)$ can be triangulated in a classical sense, thus there exists a simplicial complex T and a homeomorphism $T \longrightarrow \mathrm{P}_Z(e)$. This is, however, not a triangulation in terms of [16].

In particular, the above theorem gives an implicit answer to [16, Problem 15.9], which asks to describe a Grassmann polytope by inequalities. A related question in the case $m = 2$ was investigated in [2].

The proof of Theorem 1.8 is given in Section 5. We note that a very similar argument to ours was also given by Arkani-Hamed, Bai & Lam in [1, Appendix J].

Theorem 1.3 in particular implies that all amplituhedra $\mathcal{A}_{k+m+1,k,m}(Z)$ are contractible. Our next result shows that if in addition m is even, they are homeomorphic to balls.

Theorem 1.9. *Let $k \geq 1$ be an integer, let $m \geq 0$ be an even integer, and let $Z \in \mathrm{Mat}_{k+m,k+m+1}$ be a matrix with all $(k+m) \times (k+m)$ minors positive. Then the amplituhedron $\mathcal{A}_{k+m+1,k,m}(Z)$ induced by the matrix Z is homeomorphic to a km-dimensional ball.*

The proof of Theorem 1.9 is presented in Section 3. We remark that the combinatorics of the amplituhedron in the case $n = k + m + 1$ with m even has been recently studied in detail in [11].

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2. Proof of Theorem 1.7

Let $k \geq 1, m \geq 0$ and $n \geq k + m$ be integers and let Z be a real $(k+m) \times (k+m+1)$ matrix such that the map $\widetilde{Z}$ is well defined. Since the action of the group GL_k^+ on $\mathrm{Mat}_{k,n}^{\geq 0}$ is free, there is a fibration

$$\mathrm{GL}_k^+ \longrightarrow \mathrm{Mat}_{k,n}^{\geq 0} \longrightarrow \mathrm{G}_k^{\geq 0}(\mathbb{R}^n). \tag{2}$$

The matrix Z, as in Definition 1.2, induces a map

$$\begin{aligned} \widehat{Z} : \mathrm{Mat}_{k,n}^{\geq 0} &\longrightarrow \mathrm{Mat}_{k,k+m}, \\ V &\longmapsto VZ^\top, \end{aligned}$$

which is again well defined, see for example [16, Prop. 15.2].

Let e be a positroid cell in the CW decomposition of $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, and let $I_e \subseteq \binom{[n]}{k}$ be the family of nonbases (dependent sets) of cardinality k of the matroid that defines the cell e. The maximal minors of a $k \times n$ matrix are indexed by the set $\binom{[n]}{k}$. Denote by $\mathrm{Mat}_{k,n}^{\geq 0}(e)$ the set of all matrices $V \in \mathrm{Mat}_{k,n}^{\geq 0}$ whose minors indexed by elements of I_e are equal to zero. Then every point in $\bar{e} \subseteq \mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$ is represented by a matrix in $\mathrm{Mat}_{k,n}^{\geq 0}(e)$, and the row span of every such matrix lies in $\bar{e}$. In other words, $\bar{e} = \mathrm{GL}_k^+ \setminus \mathrm{Mat}_{k,n}^{\geq 0}(e)$. Thus the restriction of the fibration (2) is a fibration

$$\mathrm{GL}_k^+ \longrightarrow \mathrm{Mat}_{k,n}^{\geq 0}(e) \longrightarrow \bar{e}. \tag{3}$$

Note that if e is the maximal positroid cell, the set $\mathrm{Mat}_{k,n}^{\geq 0}(e)$ is the whole set $\mathrm{Mat}_{k,n}^{\geq 0}$.

Denote by $\widehat{\mathrm{P}_Z(e)}$ the image of the set $\mathrm{Mat}_{k,n}^{\geq 0}(e)$ under the map $\widehat{Z}$. With a usual abuse of notation, we consider maps $\widehat{Z} : \mathrm{Mat}_{k,n}^{\geq 0}(e) \longrightarrow \widehat{\mathrm{P}_Z(e)}$ and $\widetilde{Z} : \bar{e} \longrightarrow \mathrm{P}_Z(e)$. Then there exists a commutative diagram of spaces and continuous maps

$$\begin{array}{ccc} \mathrm{Mat}^{\geq 0}_{k,n}(e) & \xrightarrow{\widehat{Z}} & \widehat{\mathrm{P}_Z(e)} \\ \downarrow & & \downarrow \\ \bar{e} & \xrightarrow{\widetilde{Z}} & \mathrm{P}_Z(e), \end{array}$$

where vertical maps send any matrix to its row span.

The proof of Theorem 1.7 splits into the following two lemmas.

Lemma 2.1. *Let $k \geq 1$ and $m \geq 0$ be integers, $n = k + m + 1$, and let Z be a real $(k+m) \times n$ matrix such that the map $\widetilde{Z}$ is well defined. Then for every positroid cell e in the CW decomposition of $\mathrm{G}^{\geq 0}_k(\mathbb{R}^n)$ and for every $W \in \widehat{\mathrm{P}_Z(e)}$, the inverse image $\widehat{Z}^{-1}(\{W\}) \subseteq \mathrm{Mat}^{\geq 0}_{k,n}(e)$ is nonempty and convex.*

Proof. The matrix Z induces a linear map

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^{k+m} \\ v & \longmapsto & vZ^\top, \end{array} \tag{4}$$

where $v \in \mathbb{R}^n$ is a row vector. Since $n = k + m + 1$, the kernel of the map (4) is 1-dimensional. Fix a generator $a \in \mathbb{R}^n$ of that kernel.

Choose an arbitrary point $W \in \widehat{\mathrm{P}_Z(e)}$, and let U and V be any two points in $\widehat{Z}^{-1}(\{W\})$. Our goal is to show that for every $\lambda \in [0,1]$ the convex combination $(1-\lambda)U + \lambda V$ also belongs to $\widehat{Z}^{-1}(\{W\})$.

Since $UZ^\top = VZ^\top = W$, the rows of the matrix $V - U$ belong to $\ker(Z)$. Consequently, there exists a row vector $x \in \mathbb{R}^k$ such that $V - U = x^\top a$, where a is also considered as a row vector. Thus we have to show that for every $\lambda \in [0,1]$ the convex combination

$$(1-\lambda)U + \lambda V = U + \lambda x^\top a \tag{5}$$

belongs to the space $\mathrm{Mat}^{\geq 0}_{k,n}(e)$, this means that every $k \times k$ minor of the matrix (5) is nonnegative, and in addition that all the minors of the matrix (5) indexed by the nonbases $I_e \subseteq \binom{[n]}{k}$ of the matroid corresponding to e are equal to zero.

A $k \times k$ submatrix of the matrix (5) is of the form

$$\begin{pmatrix} u_{1i_1} + \lambda x_1 a_{i_1} & \dots & u_{1i_k} + \lambda x_1 a_{i_k} \\ \vdots & & \vdots \\ u_{ki_1} + \lambda x_k a_{i_1} & \dots & u_{ki_k} + \lambda x_k a_{i_k} \end{pmatrix}, \tag{6}$$

where

$$U = \begin{pmatrix} u_{11} & \dots & u_{1n} \\ \vdots & & \vdots \\ u_{k1} & \dots & u_{kn} \end{pmatrix}, \; x = (x_1 \dots x_k), \; a = (a_1 \dots a_n),$$

and $1 \leq i_1 < \cdots < i_k \leq n$. The matrix (6) can be transformed using row operations into a matrix that contains the variable λ only in one row. Therefore, every $k \times k$ minor of the matrix (5) is a polynomial of degree at most 1 in the variable λ. Since it takes nonnegative values for $\lambda = 0$ and $\lambda = 1$, it is also nonnegative for all $\lambda \in [0,1]$. Thus for every $\lambda \in [0,1]$, the point $(1-\lambda)U + \lambda V$ belongs to $\mathrm{Mat}^{\geq 0}_{k,n}$. Similarly, if $\{i_1, \dots, i_k\}$ is a nonbasis of the matroid corresponding to e, then the determinant of the matrix (6) is zero for $\lambda = 0$ and $\lambda = 1$, so it is a constant zero-polynomial, meaning that the matrix (5) belongs to $\mathrm{Mat}^{\geq 0}_{k,n}(e)$ for every $\lambda \in [0,1]$. Consequently the set $\widehat{Z}^{-1}(\{W\})$ is convex. □

Lemma 2.2. *Let $k \geq 1, m \geq 0$ and $n \geq k + m$ be integers. For every positroid cell e and for every $W \in \widehat{\mathrm{P}_Z(e)}$, the inverse images*

$$\widehat{Z}^{-1}(\{W\}) \subseteq \mathrm{Mat}^{\geq 0}_{k,n}(e) \subseteq \mathrm{Mat}^{\geq 0}_{k,n} \qquad \text{and} \qquad \widetilde{Z}^{-1}(\{\mathrm{span}(W)\}) \subseteq \bar{e} \subseteq \mathrm{G}^{\geq 0}_k(\mathbb{R}^n)$$

are homeomorphic.

Proof. Let $\varphi\colon \widehat{Z}^{-1}(\{W\}) \longrightarrow \widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$ be defined by $\varphi(U) = \mathrm{span}(U)$, where $U \in \widehat{Z}^{-1}(\{W\})$, and span denotes the row span. We prove that φ is a homeomorphism.

Clearly, φ is continuous, so it suffices to find a continuous map $\psi : \widetilde{Z}^{-1}(\{\mathrm{span}(W)\}) \longrightarrow \widehat{Z}^{-1}(\{W\})$ such that $\varphi \circ \psi$ is the identity map on $\widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$ and $\psi \circ \varphi$ is the identity map on $\widehat{Z}^{-1}(\{W\})$.

Let $L \in \widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$. Then there exists a matrix $K \in \mathrm{Mat}_{k,n}^{\geq 0}(e)$ whose rows span the subspace L. Since

$$\mathrm{span}(KZ^\top) = \mathrm{span}(W),$$

there exists a unique $C \in \mathrm{GL}_k$ such that $KZ^\top = CW$. Now define ψ as $\psi(L) = C^{-1}K$. Clearly, $C^{-1}K \in \mathrm{Mat}_{k,n}^{\geq 0}(e)$. Even though we have defined the map ψ using an arbitrarily chosen matrix K such that $\mathrm{span}(K) = L$, it can be checked directly that the definition of ψ does not depend on a choice of K.

In order to prove that the map ψ is continuous, we need to show that the choice of a matrix K can be made continuously on $\widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$. The choice of a matrix K is equivalent to the choice of a positively oriented basis for the subspace $L \subseteq \mathbb{R}^n$. Therefore, we need a continuous section of the fiber bundle (3) restricted to the set $\widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$. Since the base space $\bar{e}$ is contractible, the fiber bundle (3) is trivial. In particular, its restriction on $\widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$ is also trivial, so it admits a continuous section. Therefore, the bases for elements of $\widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$ can be chosen continuously. On the other hand, the matrix C is a solution of the linear system $KZ^\top = CW$, which depends continuously on K, thus it also depends continuously on L.

Lastly,

$$\varphi(\psi(L)) = \varphi(C^{-1}K) = \mathrm{span}(C^{-1}K) = \mathrm{span}(K) = L,$$

holds for every $L \in \widetilde{Z}^{-1}(\{\mathrm{span}(W)\})$, and

$$\psi(\varphi(U)) = \psi(\mathrm{span}(U)) = UC^{-1},$$

for every $U \in \widehat{Z}^{-1}(\{W\})$, where C is the unique $k \times k$ matrix such that $W = \widehat{Z}(U) = UZ^\top = CW$, hence C is the identity matrix. $\square$

Finally, Lemma 2.1 and Lemma 2.2 complete the proof of Theorem 1.7.

3. Proof of Theorem 1.9

Let $k \geq 1$, $m \geq 0$ and $n \geq k+m$ be integers, and suppose in addition that m is even. Let $S \in \mathrm{GL}_n$ be given by

$$S(x_1, \dots, x_n) = (x_2, \dots, x_n, (-1)^{k-1}x_1).$$

Denote by $Z_0 \in \mathrm{Mat}_{k+m,n}$ the matrix whose rows are the eigenvectors of the matrix $S + S^\top$ that correspond to the largest $k+m$ eigenvalues. It was shown in [10, Lemma 3.1] that all $(k+m)\times(k+m)$ minors of the matrix Z_0 are positive, thus it defines an amplituhedron $\mathcal{A}_{n,k,m}(Z_0)$, called *cyclically symmetric amplituhedron*. Galashin, Karp & Lam [10, Thm. 1.2] showed that $\mathcal{A}_{n,k,m}(Z_0)$ is homeomorphic to a closed km-dimensional ball whenever the parameter m is even.

We conclude the proof of Theorem 1.9 by showing that the amplituhedra $\mathcal{A}_{n,k,m}(Z)$ and $\mathcal{A}_{n,k,m}(Z_0)$ are homeomorphic.

From [14, Cor. 1.12(ii)] we know that entries of every nonzero vector of $\ker(Z_0)$ and of $\ker(Z)$ are nonzero, and they alternate in sign. Since $n = k+m+1$, the kernels of matrices Z and Z_0 are 1-dimensional. Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ be a generator of the kernel of Z and let $b = (b_1, \dots, b_n) \in \mathbb{R}^n$ be a generator of the kernel of Z_0[1]. Choose them in such a way that a_1 and b_1 have the same sign. Consequently, for every $1 \leq i \leq n$, the entries a_i and b_i have the same sign. Let D be an $n \times n$ diagonal matrix $D = \mathrm{diag}(\frac{a_1}{b_1}, \dots, \frac{a_n}{b_n})$. The matrix ZD has the same kernel as the matrix Z_0, and since the diagonal entries of the matrix D are positive, all maximal minors of the matrix ZD are positive. The fact that the matrices ZD and Z_0 have the same kernel implies that they have the same row spans, as well. In particular, there exists a matrix $C \in \mathrm{GL}_{k+m}^+$ such that $Z_0 = CZD$.

Multiplication by D on the right gives a homeomorphism $\widehat{D} : \mathrm{Mat}_{k,n}^{\geq 0} \longrightarrow \mathrm{Mat}_{k,n}^{\geq 0}$, which induces a homeomorphism $\widetilde{D} : \mathrm{G}_k^{\geq 0}(\mathbb{R}^n) \longrightarrow \mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$. Furthermore, multiplication by $C^\top$ on the right gives a homeomorphism $\widehat{C} : \mathrm{Mat}_{k,k+m} \longrightarrow \mathrm{Mat}_{k,k+m}$, thus the induced map $\widetilde{C} : \mathrm{G}_k(\mathbb{R}^{k+m}) \longrightarrow \mathrm{G}_k(\mathbb{R}^{k+m})$ is also a homeomorphism. Hence, we obtain the commutative diagram of spaces and maps

$$\begin{tikzcd}
\mathrm{Mat}_{k,n}^{\geq 0} \arrow[r, "\widehat{D}"] \arrow[d] & \mathrm{Mat}_{k,n}^{\geq 0} \arrow[r, "\widehat{Z}"] \arrow[d] & \mathrm{Mat}_{k,k+m} \arrow[r, "\widehat{C}"] \arrow[d] & \mathrm{Mat}_{k,k+m} \arrow[d] \\
\mathrm{G}_k^{\geq 0}(\mathbb{R}^n) \arrow[r, "\widetilde{D}"] & \mathrm{G}_k^{\geq 0}(\mathbb{R}^n) \arrow[r, "\widetilde{Z}"] & \mathrm{G}_k(\mathbb{R}^{k+m}) \arrow[r, "\widetilde{C}"] & \mathrm{G}_k(\mathbb{R}^{k+m}).
\end{tikzcd}$$

[1]It follows from the cyclic symmetry of Z_0 that $b_i = (-1)^{i-1}$ for $1 \leq i \leq n$. See [10] for details.

The image of the composition $\widetilde{C} \circ \widetilde{Z} \circ \widetilde{D}$ of the maps in the lower row of the diagram is the cyclically symmetric amplituhedron $\mathcal{A}_{n,k,m}(Z_0)$ and the image of the map $\widetilde{Z}$ is the amplituhedron $\mathcal{A}_{n,k,m}(Z)$. Since the maps $\widetilde{C}$ and $\widetilde{D}$ are homeomorphisms, these two amplituhedra are homeomorphic. Finally, the fact that the cyclically symmetric amplituhedron $\mathcal{A}_{n,k,m}(Z_0)$ is homeomorphic to a km-dimensional ball [10, Thm. 1.2], when m is even, concludes the argument that every amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a km-dimensional ball whenever $n = k + m + 1$ and m is even.

4. Proof of Theorem 1.6

Let e be a positroid cell in the nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$, and let Z be a matrix that defines the Grassmann polytope $\mathrm{P}_Z(e)$. By [19, Thm. 1], the Grassmann polytope $\mathrm{P}_Z(e)$ has the homotopy type of a countable CW complex if and only if it has the homotopy type of an absolute neighborhood retract (ANR). Furthermore, by [4, p. 240] the space $\mathrm{P}_Z(e)$ is an ANR if it is compact and locally contractible, see also [12, p. 389]. Since the closed positroid cell $\bar{e}$ is compact, the Grassmann polytope $\mathrm{P}_Z(e)$ is also compact. Thus, it remains to show that $\mathrm{P}_Z(e)$ is locally contractible.

Applying the Gram-Schmidt orthogonalization on the fibration (3), we obtain a fibration

$$\mathrm{SO}(k) \longrightarrow E_1 \longrightarrow \bar{e}, \tag{7}$$

where the total space E_1 is a subspace of the orthonormal Stiefel manifold. Similarly, we obtain a fibration

$$\mathrm{SO}(k) \longrightarrow E_2 \longrightarrow \mathrm{P}_Z(e). \tag{8}$$

We also consider a commutative diagram of spaces and continuous maps

$$\begin{tikzcd} E_1 \arrow[r, "\widehat{Z}"] \arrow[d] & E_2 \arrow[d] \\ \bar{e} \arrow[r, "\widetilde{Z}"] & \mathrm{P}_Z(e), \end{tikzcd}$$

where the horizontal maps are induced by the matrix Z, and the vertical maps send any frame to its span.

By [8, p. 81], every Euclidean neighborhood retract (ENR) is locally contractible. On the other hand, if E_2 is an $\mathrm{SO}(k)$-ENR, then the orbit space $\mathrm{P}_Z(e)$ is an ENR, [7, Prop. II.8.9]. Finally, since E_2 is a compact space with a free $\mathrm{SO}(k)$-action, it is an $\mathrm{SO}(k)$-ENR, [13, Thm. 2.1], which completes the argument that the Grassmann polytope $\mathrm{P}_Z(e)$ has a homotopy type of a countable CW complex.

Finally, if $n = k + m + 1$ by Theorem 1.7 and Theorem 1.4, $\mathrm{P}_Z(e)$ is simply connected, so by [19, Prop. 1 + Remark] it is homotopy equivalent to a finite CW complex.

5. Proof of Theorem 1.8

Let e be a positroid cell in the CW decomposition of the nonnegative Grassmannian $\mathrm{G}_k^{\geq 0}(\mathbb{R}^n)$. Set $d = \binom{k+m}{k}$, and consider the Veronese embedding

$$\nu : \mathbb{R}\mathrm{P}^{d-1} \longrightarrow \mathbb{R}^{d \times d}$$

that maps every point $x = (x_1 : \ldots : x_d) \in \mathbb{R}\mathrm{P}^{d-1}$ to the matrix

$$\left(\frac{x_i x_j}{x_1^2 + \cdots + x_d^2} \right)_{ij} \in \mathbb{R}^{d \times d}.$$

The embedding ν maps every linear line $x \in \mathbb{R}^d$ to the matrix of the projection $\mathbb{R}^d \to x$.

Consider also the map

$$\nu : \mathbb{R}^d \longrightarrow \mathbb{R}^{d \times d}$$

given by

$$(x_1, \ldots, x_d) \longmapsto \left(\frac{x_i x_j}{x_1^2 + \cdots + x_d^2} \right)_{ij} \in \mathbb{R}^{d \times d}.$$

Now we obtain the commutative diagram of spaces and maps

$$\begin{tikzcd} \mathrm{Mat}_{k,n}^{\geq 0} \arrow[r, "\widehat{Z}"] \arrow[d] & \mathrm{Mat}_{k,k+m} \arrow[r, "\gamma"] \arrow[d] & \mathbb{R}^d \setminus \{0\} \arrow[r, "\nu"] \arrow[d, "\pi"] & \mathbb{R}^{d \times d} \arrow[d, "\mathrm{id}"] \\ \mathrm{G}_k^{\geq 0}(\mathbb{R}^n) \arrow[r, "\widetilde{Z}"] & \mathrm{G}_k(\mathbb{R}^{k+m}) \arrow[r, "\gamma"] & \mathbb{R}\mathrm{P}^{d-1} \arrow[r, "\nu"] & \mathbb{R}^{d \times d}, \end{tikzcd}$$

where $\gamma : \mathrm{G}_k(\mathbb{R}^{k+m}) \longrightarrow \mathbb{R}\mathrm{P}^{d-1}$ is the Plücker embedding, $\gamma : \mathrm{Mat}_{k,k+m} \longrightarrow \mathbb{R}^d \setminus \{0\}$ maps every matrix to the tuple of its $k \times k$ minors, and $\pi : \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}\mathrm{P}^{d-1}$ is the quotient map.

Since the Grassmann polytope $\mathrm{P}_Z(e)$ is embedded into $\mathbb{R}\mathrm{P}^{d-1}$ via γ, and the projective space $\mathbb{R}\mathrm{P}^{d-1}$ is embedded in $\mathbb{R}^{d\times d}$ via ν, we show that $\nu(\gamma(\mathrm{P}_Z(e)))$ is semialgebraic. The commutativity of the diagram above implies that

$$\nu(\gamma(\mathrm{P}_Z(e))) = \nu(\pi(\gamma(\widehat{\mathrm{P}_Z(e)}))) = \nu(\gamma(\widehat{\mathrm{P}_Z(e)})) = \nu(\gamma(\widehat{Z}(\mathrm{Mat}_{k,n}^{\geq 0}(e)))).$$

The set $\mathrm{Mat}_{k,n}^{\geq 0}(e) \subseteq \mathbb{R}^{k\times n}$ is semialgebraic. Since the map $\widehat{Z}$ is multiplication by a matrix, the set $\widehat{\mathrm{P}_Z(e)}$ is also semialgebraic. Furthermore, every coordinate of the map γ is given by a polynomial, thus $\gamma(\widehat{\mathrm{P}_Z(e)}) \subseteq \mathbb{R}^d \setminus \{0\}$ is semialgebraic, as well. Finally, the map $\nu : \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ is a regular rational map, thus it maps semialgebraic sets to semialgebraic sets, see [6, Sec. 2.2.1].

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