THE TOTALLY NONNEGATIVE PART OF G/P IS A BALL

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Abstract. We show that the totally nonnegative part of a partial flag variety (in the sense of Lusztig) is homeomorphic to a closed ball.

1. Introduction

Let G be a simply connected, semisimple algebraic group defined and split over $\mathbb{R}$. Lusztig [Lus94] has defined a "remarkable polyhedral subspace" $\mathcal{P}^J_{\geq 0}$, called the *totally nonnegative part* of a (real) partial flag variety $\mathcal{P}^J$ of G. In this paper we establish the following result.

Theorem 1. *The totally nonnegative part $\mathcal{P}^J_{\geq 0}$ is homeomorphic to a closed ball of dimension equal to that of $\mathcal{P}^J$.*

Lusztig proved that $\mathcal{P}^J_{\geq 0}$ is contractible [Lus98b, §4.4]. He also defined a partition of $\mathcal{P}^J_{\geq 0}$ [Lus94, §8.15], which was shown to be a cell decomposition by Rietsch [Rie99, §4.2]. Williams [Wil07, §7] conjectured that this cell decomposition forms a regular CW complex. Rietsch and Williams [RW08] showed that $\mathcal{P}^J_{\geq 0}$ forms a CW complex, and have proved Williams's conjecture up to homotopy [RW10]. These results complement work by Hersh [Her14] in the unipotent group case.

Our proof of Theorem 1 employs the vector field τ on $\mathcal{P}^J$ which is the sum of all Chevalley generators of $\mathfrak{g}$, which Lusztig used to show that $\mathcal{P}^J_{\geq 0}$ is contractible. The flow defined by τ is a *contractive flow* on $\mathcal{P}^J_{\geq 0}$ in the sense of our paper [GKL18], so in particular it contracts all of $\mathcal{P}^J_{\geq 0}$ to a unique fixed point $p_0 \in \mathcal{P}^J_{\geq 0}$. The machinery of [GKL18] (see Lemma 1) generates a homeomorphism from $\mathcal{P}^J_{\geq 0}$ to a closed ball $B \subset \mathcal{P}^J_{\geq 0}$ centered at p_0, by mapping each trajectory in $\mathcal{P}^J_{\geq 0}$ to its intersection with B. This generalizes [GKL18, Theorem 1.1] from type A Grassmannians to all partial flag varieties. We remark that Theorem 1 is new in all other cases, including for the complete flag variety and multi-step flag varieties in $\mathbb{R}^n$.

2. Preliminaries

In this section, we recall some background from Lusztig [Lus94, Lus98a] and [GKL18].

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2.1. **Pinnings.** Let $\mathfrak{g}$ denote the Lie algebra of G over $\mathbb{R}$. We fix Chevalley generators $(e_i, f_i)_{i\in I}$ of $\mathfrak{g}$, so that the elements $h_i := [e_i, f_i]$ $(i \in I)$ span the Lie algebra of a split real maximal torus T of G. For $i \in I$ and $t \in \mathbb{R}$, we define the elements of G

$$x_i(t) := \exp(te_i), \quad y_i(t) := \exp(tf_i).$$

We also let $\alpha_i^\vee : \mathbb{R}^* \to T$ be the homomorphism of algebraic groups whose tangent map takes $1 \in \mathbb{R}$ to h_i. The $x_i(t)$'s (respectively, $y_i(t)$'s) generate the unipotent radical U^+ of a Borel subgroup B^+ (respectively, U^- and B^-) of G, with $B^+ \cap B^- = T$. The data $(T, B^+, B^-, x_i, y_i; i \in I)$ is called a *pinning* for G.

2.2. **Total positivity.** Let $i_1, \ldots, i_\ell$ be a sequence of elements of I such that the product of simple reflections $s_{i_1} \cdots s_{i_\ell}$ is a reduced decomposition of the longest element of the Weyl group. We define the totally positive parts

$$U_{>0}^+ := \{x_{i_1}(t_1) \cdots x_{i_\ell}(t_\ell) : t_1, \ldots, t_\ell > 0\}, \quad U_{>0}^- := \{y_{i_1}(t_1) \cdots y_{i_\ell}(t_\ell) : t_1, \ldots, t_\ell > 0\}.$$

Let $G_{>0} := U_{>0}^+ T_{>0} U_{>0}^- = U_{>0}^- T_{>0} U_{>0}^+$, where $T_{>0}$ is generated by $\alpha_i^\vee(t)$ for $i \in I$ and $t > 0$.

The *complete flag variety* $\mathcal{B}$ of G is the space of all Borel subgroups B of G. Its totally positive part is

$$\mathcal{B}_{>0} := \{uB^+u^{-1} : u \in U_{>0}^-\} = \{uB^-u^{-1} : u \in U_{>0}^+\},$$

and its totally nonnegative part $\mathcal{B}_{\geq 0}$ is the closure of $\mathcal{B}_{>0}$ in $\mathcal{B}$.

We now fix a subset $J \subset I$ and define P_J^+ to be the subgroup of G generated by B^+ and $\{y_j(t) : j \in J, t \in \mathbb{R}\}$. Let $\mathcal{P}^J := \{gP_J^+g^{-1} : g \in G\}$ be a *partial flag variety* of G. (In the case $J = \emptyset$, we have $\mathcal{P}^J = \mathcal{B}$.) For each $B \in \mathcal{B}$, there is a unique $P \in \mathcal{P}^J$ such that $B \subset P$; denote by $\pi^J : \mathcal{B} \to \mathcal{P}^J$ the natural projection map that sends $B \in \mathcal{B}$ to the $P \in \mathcal{P}^J$ that contains it. We define the totally positive and totally nonnegative parts

$$\mathcal{P}_{>0}^J := \pi^J(\mathcal{B}_{>0}), \quad \mathcal{P}_{\geq 0}^J := \pi^J(\mathcal{B}_{\geq 0}). \tag{2.1}$$

Example 1. Let $G := \mathrm{SL}_3(\mathbb{R})$. We may take a pinning with $I = \{1, 2\}$ and

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ f_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ x_1(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ y_1(t) = \begin{bmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \alpha_1^\vee(t) = \begin{bmatrix} t & 0 & 0 \\ 0 & t^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},\ f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ x_2(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix},\ y_2(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & t & 1 \end{bmatrix},\ \alpha_2^\vee(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t^{-1} \end{bmatrix}.$$

Then B^+, B^-, and T are the subgroups of upper-triangular, lower-triangular, and diagonal matrices, respectively. Taking $J := \emptyset$, we can identify $\mathcal{P}^J = \mathcal{B}$ with the space of complete flags in $\mathbb{R}^3$. Explicitly, if $g = [g_1 \,|\, g_2 \,|\, g_3] \in G$, then $gB^+g^{-1} \in \mathcal{P}^J$ corresponds to the flag $\{0\} \subset \langle g_1 \rangle \subset \langle g_1, g_2 \rangle \subset \mathbb{R}^3$. One can check that the totally nonnegative part $\mathcal{P}_{\geq 0}^J$ consists of flags $\{0\} \subset V_1 \subset V_2 \subset \mathbb{R}^3$ where V_1 is spanned by a vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ with $v_1, v_2, v_3 \geq 0$, and V_2 is orthogonal to a vector $(w_1, -w_2, w_3) \in \mathbb{R}^3$ with $w_1, w_2, w_3 \geq 0$. Hence we may identify $\mathcal{P}_{\geq 0}^J$ with the subset of $\mathbb{R}^6$ of points $(v_1, v_2, v_3, w_1, w_2, w_3)$ satisfying the (in)equalities

$$v_1 + v_2 + v_3 = 1, \quad w_1 + w_2 + w_3 = 1, \quad v_1w_1 - v_2w_2 + v_3w_3 = 0, \quad v_1, v_2, v_3, w_1, w_2, w_3 \geq 0.$$

Theorem 1 implies that this region is homeomorphic to a 3-dimensional closed ball. Its cell decomposition (determined by which of the six coordinates are zero) is shown in Figure 1.

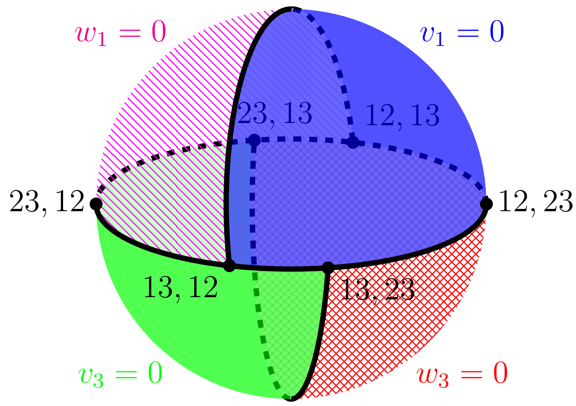

FIGURE 1. The totally nonnegative part of the complete flag variety in $\mathbb{R}^3$. A vertex labeled ab, cd denotes the cell $v_a = v_b = w_c = w_d = 0$ (see Example 1).

2.3. **Canonical basis.** We now assume that G is simply laced. Let Y (respectively, X) be the free abelian group of homomorphisms of algebraic groups $\mathbb{R}^* \to T$ (respectively, $T \to \mathbb{R}^*$), with standard pairing $\langle\cdot,\cdot\rangle : Y \times X \to \mathbb{Z}$. Let X^+ be the set of $\lambda \in X$ such that $\langle\alpha_i^\vee, \lambda\rangle \geq 0$ for all $i \in I$, and for such λ let $\operatorname{supp}(\lambda) := \{i \in I : \langle\alpha_i^\vee, \lambda\rangle > 0\}$ be its *support*.

For each $\lambda \in X^+$, there is a unique irreducible G-module Λ_λ with highest weight λ. Lusztig [Lus94, §3.1] defined a *canonical basis* ${}_\lambda\mathbf{B}$ of Λ_λ; see [Lus90] for more details. For L in the projective space $\mathbb{P}(\Lambda_\lambda)$, we write $L > 0$ (respectively, $L \geq 0$) if the line L is spanned by a vector $x \in \Lambda_\lambda$ whose coordinates in the canonical basis ${}_\lambda\mathbf{B}$ are all positive (respectively, nonnegative). We denote

$$\mathbb{P}(\Lambda_\lambda)_{>0} := \{L \in \mathbb{P}(\Lambda_\lambda) : L > 0\}, \quad \mathbb{P}(\Lambda_\lambda)_{\geq 0} := \{L \in \mathbb{P}(\Lambda_\lambda) : L \geq 0\}.$$

2.4. **Contractive flows.** We recall a topological lemma from [GKL18] that we used to show that various spaces appearing in total positivity (including the totally nonnegative part of a type A Grassmannian) are homeomorphic to closed balls.

Definition 1 ([GKL18, Definition 2.1])**.** A map $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is called a *contractive flow* if the following conditions are satisfied:

(1) the map f is continuous;
(2) for all $p \in \mathbb{R}^N$ and $t_1, t_2 \in \mathbb{R}$, we have $f(0,p) = p$ and $f(t_1 + t_2, p) = f(t_1, f(t_2, p))$; and
(3) for all $p \neq 0$ and $t > 0$, we have $\|f(t,p)\| < \|p\|$.

Here $\|p\|$ denotes the Euclidean norm[1] of $p \in \mathbb{R}^N$. For a subset $K \subset \mathbb{R}^N$ and $t \in \mathbb{R}$, let $f(t,K)$ denote $\{f(t,p) : p \in K\}$.

Lemma 1 ([GKL18, Lemma 2.3])**.** *Let $Q \subset \mathbb{R}^N$ be a smooth embedded submanifold of dimension $d \leq N$, and $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ a contractive flow. Suppose that Q is bounded and satisfies the condition*

$$f(t, \overline{Q}) \subset Q \quad \textit{for } t > 0. \tag{2.2}$$

[1]In [GKL18], Lemma 1 was proved more generally for an arbitrary norm $\|\cdot\|$ on $\mathbb{R}^N$.

Then the closure $\overline{Q}$ is homeomorphic to a closed ball of dimension d, and $\overline{Q} \setminus Q$ is homeomorphic to a sphere of dimension $d-1$.

3. Proof of Theorem 1

We first prove Theorem 1 in the case that G is simply laced, in Section 3.1. We will then deduce the result for all G using the *folding* technique, in Section 3.2. Our arguments employ the vector field $\tau := \sum_{i\in I}(e_i + f_i) \in \mathfrak{g}$, and

$$a(t) := \exp(t\tau) \text{ defined for } t \in \mathbb{R}.$$

By [Lus94, Proposition 5.9(c)], we have $a(t) \in G_{>0}$ for $t > 0$.

Remark 1. In our earlier proof [GKL18, §3] that the totally nonnegative part of a type A Grassmannian is homeomorphic to a closed ball, we worked with a closely related vector field, which is a cyclically symmetric (or loop group) analogue of τ. We did so because exploiting cyclic symmetry appears to be necessary to show that the compactification of the space of electrical networks is homeomorphic to a ball, as we proved in [GKL18, §6].

3.1. **Simply-laced case.** Choose $\lambda \in X^+$ such that $\operatorname{supp}(\lambda) = I \setminus J$. Then by [Lus98a, §1.6], for each $P \in \mathcal{P}^J$, there is a unique line L_P^λ in Λ_λ that is stable under the action of P on Λ_λ. Moreover, the map $P \mapsto L_P^\lambda$ is a smooth embedding of $\mathcal{P}^J$ into $\mathbb{P}(\Lambda_\lambda)$, which we denote by $\psi_J : \mathcal{P}^J \to \mathbb{P}(\Lambda_\lambda)$. Lusztig [Lus98a, Theorem 3.4] showed that we may select λ so that the following hold:

(i) we have $P \in \mathcal{P}^J_{>0}$ if and only if $L_P^\lambda > 0$; and
(ii) we have $P \in \mathcal{P}^J_{\geq 0}$ if and only if $L_P^\lambda \geq 0$.

By [Lus94, Lemma 5.2(a)], any $g \in G_{>0}$ has a unique stable line L_g in $\mathbb{P}(\Lambda_\lambda)_{\geq 0}$, and this line satisfies $L_g > 0$. Taking $g = a(t)$, we obtain an $a(t)$-stable line $L_{a(t)}$ for each $t > 0$. We see that $L_{a(t)}$ does not depend on t, and we denote this common line by L_0.

Since $a(t) \in G_{>0}$ for $t > 0$, by [Lus94, Theorem 5.6] we have that τ is regular and semisimple over $\mathbb{R}$. Let $v_0, v_1, \ldots, v_N \in \Lambda_\lambda$ be an eigenbasis of τ with eigenvalues $\mu_0, \mu_1, \ldots, \mu_N \in \mathbb{R}$, such that v_0 spans L_0. We have a smooth open chart

$$\phi : \mathbb{R}^N \hookrightarrow \mathbb{P}(\Lambda_\lambda),$$

which sends $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$ to the line spanned by $v_0 + p_1 v_1 + \cdots + p_N v_N$. By [Lus94, Lemma 5.2(a)] again, we have $\mu_k < \mu_0$ for $1 \leq k \leq N$, and the image of ϕ contains $\mathbb{P}(\Lambda_\lambda)_{\geq 0}$. Consider the sets $Q \subset \overline{Q} \subset \mathbb{R}^N$ defined by

$$Q := \phi^{-1}(\psi_J(\mathcal{P}^J_{>0})) \simeq \mathcal{P}^J_{>0}, \quad \overline{Q} := \phi^{-1}(\psi_J(\mathcal{P}^J_{\geq 0})) \simeq \mathcal{P}^J_{\geq 0},$$

so that $\overline{Q}$ is compact and is the closure of Q. Note that ϕ maps the origin 0 of $\mathbb{R}^N$ to L_0.

Consider a map $f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ defined for $t \in \mathbb{R}$ and $p = (p_1, \ldots, p_N) \in \mathbb{R}^N$ by

$$f(t,p) := \left(e^{t(\mu_1 - \mu_0)} p_1, \ldots, e^{t(\mu_N - \mu_0)} p_N\right). \tag{3.1}$$

We have $\phi(f(t,p)) = a(t) \cdot \phi(p)$. We claim that f is a contractive flow. Indeed, parts (1) and (2) of Definition 1 hold for f. By (3.1), we see that f satisfies the property

$$\|f(t,p)\| \leq C^{-t}\|p\| \quad \text{for all } t > 0 \text{ and } p \in \mathbb{R}^N,$$

where $C > 1$ is the minimum of $e^{\mu_0 - \mu_j}$ for $1 \le j \le N$. Therefore f satisfies part (3) of Definition 1, as claimed.

Lusztig [Lus98a, Proposition 2.2] showed that $\mathcal{P}^J_{>0}$ is an open submanifold of $\mathcal{P}^J$. Any $g \in G_{>0}$ sends $\mathbb{P}(\Lambda_\lambda)_{\ge 0}$ inside $\mathbb{P}(\Lambda_\lambda)_{>0}$ [Lus94, Proposition 3.2], which implies that (2.2) holds. Therefore Theorem 1 in the simply-laced case follows by applying Lemma 1.

Example 2. Adopting the setup of Example 1, we have $\tau = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. The unique fixed point of $a(t)$ in $\mathcal{P}^J = \mathcal{B}$ is the flag generated by the eigenvectors of τ, ordered by decreasing eigenvalue. Its coordinates (see Example 1) are $v_1 = v_3 = w_1 = w_3 = \frac{1}{2+\sqrt{2}}, v_2 = w_2 = \frac{\sqrt{2}}{2+\sqrt{2}}$.

3.2. **General case, via folding.** For general G, by [Lus94, §1.6] there exists an algebraic group $\dot{G}$ of simply-laced type with pinning $(\dot{T}, \dot{B}^+, \dot{B}^-, \dot{x}_i, \dot{y}_i; i \in \dot{I})$ and an automorphism $\sigma : \dot{I} \to \dot{I}$ that extends to an automorphism of $\dot{G}$ (also denoted σ), such that $G \cong \dot{G}^\sigma$. (Here S^σ denotes the set of fixed points of σ in S.) We have

$$G_{>0} = (\dot{G}_{>0})^\sigma, \quad G_{\ge 0} = (\dot{G}_{\ge 0})^\sigma, \quad \mathcal{B}_{>0} = (\dot{\mathcal{B}}_{>0})^\sigma, \quad \mathcal{B}_{\ge 0} = (\dot{\mathcal{B}}_{\ge 0})^\sigma;$$

see [Lus94, §8.8]. It follows from (2.1) that

$$\mathcal{P}^J_{>0} = (\dot{\mathcal{P}}^{\dot{J}}_{>0})^\sigma, \quad \mathcal{P}^J_{\ge 0} = (\dot{\mathcal{P}}^{\dot{J}}_{\ge 0})^\sigma,$$

where we identify I with the set of equivalence classes of $\dot{I}$ modulo the action of σ, and $\dot{J}$ is defined to be the preimage of J under the projection map $\dot{I} \to I$.

Note that $\sum_{i \in I}(e_i + f_i) \in \mathfrak{g}$, so $a(t) \in G$ for all $t \in \mathbb{R}$. Thus $a(t)$ acts on $\mathcal{P}^J_{\ge 0} = (\dot{\mathcal{P}}^{\dot{J}}_{\ge 0})^\sigma$. The smooth embedding $\dot{\mathcal{P}}^{\dot{J}}_{>0} \hookrightarrow \mathbb{P}(\Lambda_\lambda)_{>0}$ restricts to a smooth embedding $\mathcal{P}^J_{>0} \hookrightarrow \mathbb{P}(\Lambda_\lambda)_{>0}$. Therefore we may apply Lemma 1 again to deduce that $\mathcal{P}^J_{\ge 0}$ is homeomorphic to a closed ball, completing the proof of Theorem 1 in the general case.

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