Solution to Practice Problems for 1st Midterm

Exercise 3.1:

1.	$\forall v_2 \neg P v_2 \to \neg P v_1$	QAx
2.	$\forall v_2 \neg P v_2 \to \forall v_1 \neg P v_1$	1;QR
3.	$\neg \forall v_1 \neg P v_1 \to \neg \forall v_2 \neg P v_2$	2; SL
	$[\exists v_1 P v_1 \to \exists v_2 P v_2]$	

Exercise 3.2:

1.	$\forall v_1 P v_1$	Premise
2.	$\forall v_1 P v_1 \to P v_1$	QAx
3.	$\forall v_1 \neg P v_1 \to \neg P v_1$	QAx
4.	$\exists v_1 P v_1$	1,2,3;SL

Exercise 3.3: We show that $\vdash (Pc \to \forall x(x = c \to Px))$ and that $\vdash (\forall x)((x = c \to Px) \to Pc)$. This is enough, since $(Pc \leftrightarrow \forall x(x = c \to Px))$ follows by SL from these two sentences.

By the Deduction Theorem, it is enough to show that $\{Pc\} \vdash \forall x(x = c \to Px)$ and that $\{\forall x(x = c \to Px)\} \vdash Pc$.

1.	Pc	Premise
2.	$(x = c \to (\neg Px \to \neg Pc))$	IdAx(b)
3.	$(x = c \to Px)$	1,2; SL
4.	$\forall x(x = c \to Px)$	3; Gen

1.	$\forall x(x = c \to Px)$	Premise
2.	$(\forall x(x = c \to Px) \to (c = c \to Pc))$	QAx
3.	$(c = c \to Pc)$	1,2; MP
4.	$c = c$	QAx(a)
5.	Pc	3,4; MP

Exercise 3.4:

1.	$\forall v_1 \forall v_2 (P v_1 v_2 \vee P v_2 v_1)$	Premise
2.	$\forall v_1 \forall v_2 (P v_1 v_2 \vee P v_2 v_1) \to \forall v_2 (P v_1 v_2 \vee P v_2 v_1)$	QAx
3.	$\forall v_2 (P v_1 v_2 \vee P v_2 v_1)$	1,2; MP
4.	$\forall v_2 (P v_1 v_2 \vee P v_2 v_1) \to (P v_1 v_1 \vee P v_1 v_1)$	QAx
5.	$P v_1 v_1 \vee P v_1 v_1$	3,4; MP
6.	$P v_1 v_1$	5; SL
7.	$\forall v_1 P v_1 v_1$	6; Gen

Exercise 2.7. Define $\mathrm{tv}_{\mathfrak{A}}^{s}(\Gamma)$ to be $\mathbf{T}$ if $\mathrm{tv}_{\mathfrak{A}}^{s}(\varphi) = \mathbf{T}$ for all $\varphi \in \Gamma$ and to be $\mathbf{F}$ otherwise.

$$\begin{aligned}
\Gamma \models \varphi \quad &\Leftrightarrow \text{ for all } \mathfrak{A} \text{ and all } s, \text{ if } \mathrm{tv}_{\mathfrak{A}}^{s}(\Gamma) = \mathbf{T}, \text{ then } \mathrm{tv}_{\mathfrak{A}}^{s}(\varphi) = \mathbf{T} \\
&\Leftrightarrow \text{ for all } \mathfrak{A} \text{ and all } s, \text{ if } \mathrm{tv}_{\mathfrak{A}}^{s}(\Gamma) = \mathbf{T}, \text{ then } \mathrm{tv}_{\mathfrak{A}}^{s}(\neg\varphi) = \mathbf{F} \\
&\Leftrightarrow \text{ for all } \mathfrak{A} \text{ and all } s, \ \mathrm{tv}_{\mathfrak{A}}^{s}(\Gamma \cup \{\neg\varphi\}) = \mathbf{F} \\
&\Leftrightarrow \Gamma \cup \{\neg\varphi\} \text{ is not satisfiable.}
\end{aligned}$$

Extra Problem 1. Here is the statement of the problem. Let $\mathfrak{A}$ be a model and let s be a variable assignment. Let x be a variable and let t be a term. Let s' be like s except that $s'(x) = \mathrm{den}_{\mathfrak{A}}^{s}(t)$. Prove by induction on length that, for all terms t^*,

$$\mathrm{den}_{\mathfrak{A}}^{s'}(t^*) = \mathrm{den}_{\mathfrak{A}}^{s}(t^*(x;t)).$$

Proof. *Case 1.* t^* is a constant or a variable different from x. Then $t^*(x;t)$ is the same as t^*. Since x does not occur in t^*, $\mathrm{den}_{\mathfrak{A}}^{s'}(t^*) = \mathrm{den}_{\mathfrak{A}}^{s}(t^*) = \mathrm{den}_{\mathfrak{A}}^{s}(t^*(x;t))$.

Case 2. t^* is x. Then $t^*(x;t)$ is t. Hence $\mathrm{den}_{\mathfrak{A}}^{s'}(t^*) = \mathrm{den}_{\mathfrak{A}}^{s'}(x) = s'(x) = \mathrm{den}_{\mathfrak{A}}^{s}(t) = \mathrm{den}_{\mathfrak{A}}^{s}(t^*(x;t))$.

Case 3. t^* is $ft_1^* \dots t_n^*$ where f is an n-place function symbol and $t_1^*, \dots, t_n^*$ are terms. Since $t_1^*, \dots, t_n^*$ are all shorter than t^*,

$$\mathrm{den}_{\mathfrak{A}}^{s'}(t_i^*) = \mathrm{den}_{\mathfrak{A}}^{s}(t_i^*(x;t))$$

for $1 \le i \le n$. Hence

$$\begin{aligned}
\mathrm{den}_{\mathfrak{A}}^{s'}(t^*) \quad &= \quad f_{\mathfrak{A}}(\mathrm{den}_{\mathfrak{A}}^{s'}(t_1^*), \dots, \mathrm{den}_{\mathfrak{A}}^{s'}(t_n^*)) \\
&= \quad f_{\mathfrak{A}}(\mathrm{den}_{\mathfrak{A}}^{s}(t_1^*(x;t)), \dots, \mathrm{den}_{\mathfrak{A}}^{s}(t_n^*(x;t))) \\
&= \quad \mathrm{den}_{\mathfrak{A}}^{s}(t^*(x;t)).
\end{aligned}$$