Homework #3

Theory part:

$$Y^* = Y_k + h \cdot F(Y_k)$$

$$Y_{k+1} = Y_k + \frac{h}{2} F(Y_k) + \frac{h}{2} F(Y^*)$$

Truncation error:

$$\tau_{k+1}(h) = \frac{Y_{k+1} - Y_k}{h} - \frac{1}{2} F(Y_k) - \frac{1}{2} F(Y_k + hF(Y_k))$$

expand $Y_{k+1} = Y_k + hY' + \frac{h^2}{2} Y'' + \frac{h^3}{6} Y''' + O(h^4)$

Then:

$$\tau_{k+1}(h) = Y' + \frac{h}{2} Y'' + \frac{h^2}{6} Y''' + O(h^3) - \frac{1}{2} F(Y_k) - \frac{1}{2} F(Y_k + hF(Y_k))$$

$$Y' = \frac{dY}{dt} = F(Y(t))$$

$$Y'' = \frac{d}{dt} F(Y(t))$$

$$= \underbrace{\frac{\partial F}{\partial t}}_{=0} + \frac{\partial F}{\partial Y} \frac{\partial Y}{\partial t} = \frac{\partial F}{\partial Y} F = F \frac{dF}{dY}$$

because F is autonomous

Expand last term:

$F(y_k + h\,F(y_k))$

$$= F(y_k) + h\,F(y_k)\,\frac{dF(y_k)}{dy} + \frac{h^2}{2}\,F(y_k)^2\,\frac{d^2F}{dy^2} + O(h^3)$$

putting it all together:

$$\tau_{k+1} = \underbrace{F(y,t)}_{} + \cancel{\frac{h}{2}\frac{\partial F}{\partial t}} + \frac{h}{2}\frac{dF}{dy}F + \frac{h^2}{6}y''' - \frac{1}{2}F$$
$$- \frac{1}{2}F - \frac{h}{2}F\frac{dF}{dy} - \frac{h^2}{4}F^2\frac{d^2F}{dy^2} + O(h^3)$$

$$\tau_{k+1} = \frac{h^2}{6}y''' - \frac{h^2}{4}F^2\frac{d^2F}{dy^2} + O(h^3)$$

so, we need to work out y''':

$$y''' = \frac{d}{dt}\left(\frac{dF}{dy}F\right) = \frac{dF}{dy}\frac{dF}{dt} + F\frac{d^2F}{dy\,dt}$$

$$= \frac{dF}{dy}\frac{dF}{dy}\cdot F + F\frac{d^2F}{d^2y}\cdot\frac{dy}{dt} = \left(\frac{dF}{dy}\right)^2 F + F^2\frac{d^2F}{dy^2}$$

$\Rightarrow$

$$\tau_{k+1} = h^2\left[\frac{1}{6}F\left(\frac{dF}{dy}\right)^2 + -\frac{1}{12}F^2\frac{d^2F}{dy^2}\right]$$