Homework Assignment #7

(due: May 27 in class)

Theory Part

Problem 1 (30%):
Show that the following linear system of equations converges for both, Gauss Jacobi and Gauss Seidel methods:

E1: 4*x1 - x2 -x4 = 0
E2: -x1 +4*x2 -x3 -x5 = 5
E3: -x2 +4*x3 -x6 = 0
E4: -x1 +4*x4 -x5 = 6
E5: -x2 -x4 +4*x5 -x6 = -2
E6: -x3 -x5 + 4*x6 = 6

Bonus Problem(20%):
Proof part ii) of Corollary 7.20.
i.e., proof that for ||T||<1, the following error bound holds for the sequence $\{x^{(k)}\}$, defined by $x^{(k)} = Tx^{(k-1)} + c$:

$||x^{(k)} - x|| <= ||T||^{(k)} / (1-||T||) * ||x^{(1)} - x^{(0)}||$

Hints
i) Consider $||x^{(m)} - x^{(k)}||$ for any m > k > 1. Show that this is less or equal then $||x^{(m)} - x^{(m-1)}|| + ||x^{(m-1)} - x^{(m-2)}|| + ... + ||x^{(k+1)} - x^{(k)}||$
ii) Now use that $||x^{(k+1)} - x^{(k)}|| = ||Tx^{(k)} - Tx^{(k-1)}||$
iii) Since ||T|| < 1, the series converges, i.e. $x^{(m)}$ --> x as m --> infinity.
iv) The geometrical series sum(i=0,infinity) (x_i) = 1/(1-x) for 0 < x < 1.

Numerics Part

Problem 3 (70%):
a) Solve problem 1 above using the Gauss Jacobi Method.

b) Solve problem 1 above using the Gauss Seidel Method.
c) Solve problem 1 above using the SOR method with omega = 1.1, 1.2, 1.3, 1.4, 1.5

Use a tolerance of 0.001. Set the maximum number of iterations to 100 (you will need fewer !). Use the following stopping criteria:
$\| \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \|_{\text{infinity}} / \| \mathbf{x}^{(k)} \|_{\text{infinity}} <$ tolerance.

d) Compare the efficiency of the algorithms in (a), (b), and (c) (i.e., compare the number of iterations)

What you should turn in:
The codes you used in (a), (b), and (c).
Your final results for x for (a), (b), and (c).
Your answer to (d).

Homework 7 :

Problem 1)

To show that Gauss-Jacobi, and Gauss-Seidel converge, we need to check if matrix is strictly diagonally dominant :

row 1 :		$4 > 2$
2		$4 > 3$
3		$4 > 2$
4		$4 > 2$
5		$4 > 3$
6		$4 > 2$

o.k.

Homework 7, bonus problems

<u>Corollary 7.20, part 2:</u>

If $\|T\| < 1$ for any natural norm, then the following error bounds hold for sequence $\{x^{k}\}$, defined by $x^{(k)} = Tx^{(k-1)} + c$:

i) $\|x - x^{(k)}\| \le \|T\|^{k} \|x^{(0)} - x\|$

ii) $\|x - x^{(k)}\| \le \dfrac{\|T\|^{k}}{1-\|T\|} \|x^{(1)} - x^{(0)}\|$

Proof:

i) $$\|x^{(k)} - x\| = \|(Tx^{(k-1)} + c) - (Tx + c)\| = \|Tx^{(k-1)} - Tx\|$$

$$= \|T(x^{(k-1)} - x)\|$$

$$\le \|T\| \, \|x^{(k-1)} - x\|$$

Corollary 7.10 repeat argument

....

$$\le \|T\|^{k} \|x^{(0)} - x\|$$

ii) $$\|x^{(k+1)} - x^{(k)}\| = \|Tx^{(k)} - Tx^{(k-1)}\|$$

$$= \|T(x^{(k)} - x^{(k-1)})\|$$

$$\le \|T\| \, \|x^{(k)} - x^{(k-1)}\|$$

...

$$\le \|T\|^{k} \|x^{(1)} - x^{(0)}\|$$

Now, for $m > k > 1$:

$$\|x^{(m)} - x^{(k)}\| = \|x^{(m)} - x^{(m-1)} + x^{(m-1)} - x^{(m-2)} + x^{(m-2)} - \dots - x^{(k)}\|$$

$$\leq \|x^{(m)} - x^{(m-1)}\| + \|x^{(m-1)} - x^{(m-2)}\| + \dots + \|x^{(k+1)} - x^{(k)}\|$$

$$\leq \|T\|^{m-1} \|x^{(1)} - x^{(0)}\| + \|T\|^{m-2} \|x^{(1)} - x^{(0)}\| + \dots + \|T\|^{k} \|x^{(1)} - x^{(0)}\|$$

$$= \left(\|T\|^{m-1} + \|T\|^{m-2} + \dots + \|T\|^{k}\right) \cdot \|x^{(1)} - x^{(0)}\|$$

for $m \to \infty$: $x^{(m)} \to x$

$\Rightarrow$

$$\|x - x^{(k)}\| \leq \|T\|^{k} \cdot \underbrace{\sum_{i=0}^{\infty} \|T\|^{i}}_{\substack{\text{for } \|T\| < 1: \\ \text{geometric series,} \\ \frac{1}{1 - \|T\|}}} \cdot \|x^{(1)} - x^{(0)}\|$$

$$\|x - x^{(k)}\| \leq \frac{\|T\|^{k}}{1 - \|T\|} \|x^{(1)} - x^{(0)}\|$$