Homework Assignment #5

(Note: This homework will not be collected, and will not be graded. Think of this as a Practise Midterm (midterm will be shorter, but this is the difficulty you should expect!)

Problem 1

show that the Region of Absolute Stability for the trapezoidal method is (-infinity,0).

Problem 2:

Consider an initial value problem of the form

$dy/dt = F(y)$
$y(t_0) = y_0$

This is called an autonomous differential equation, because F(y) does not depend explicitly on t. Give an expression for the leading term of the truncation error for the Trapezoidal method

Problem 3:

Consider the initial value problem

$dy/dt = t*\sin(y)*\cos(2*y)$
$0 <= t <= 2.51$
$y(0) = 1$

Estimate a restriction on the timestep you can use applying a Runge Kutta scheme of order 3, such that the solution exhibits qualitatively the correct behavior.
The region of absolute stability for a third order Runge Kutta scheme is (-2.51,0).

Problem 4:

Consider applying the Trapezoidal method

$w_{k+1} = w_k + h/2*f(w_k , t_k) + h/2*f(w_{k+1}\ t_{k+1})$

To the differential equation $dy/dt = 4t^2\cos(y)$.

(a) Write out the equation that must be solved at each timestep to advance the solution from w_k to w_{k+1}.
(b) Write out the Newton's method iteration that could be used to solve the equation in (a).
(c) Newton's method requires an initial guess; what would you choose as an initial guess for

the iteration in (b) ?

Problem 5:
Consider the third order differential equation

d^3y/dt^3 - 3 dy/dt = 6
y(0) = 1
dy/dt(0) = 2
$d^2y/dt^2(0)$ = 3

(a) Put this equation into an equivalent first order system form.
(b) Express the system in (a) using matrix/vector notation i.e. give an expression for **du**/dt if your unknowns are the components of a vector **u**.

Problem 6:
Assume that the numerical approximation of a certain method produces the results given below at a time t=2.0. The methods was used with different timesteps, as indicated. Can you estimate the order of the method ? (Hint: Assume that the result for the small timestep h=0.0001 "is almost exact").

h	Approximation
0.0001	1.00000
.1	0.999439
.05	0.999849
.025	0.9999605
.0125	0.99999

HW5, 1)

Consider Trapezoidal method:

$$w_{i+1} = w_i + \frac{h}{2}\left[f(t_{i+1}, w_{i+1}) + f(t_i, w_i) \right]$$

Model problem: $\frac{dy}{dt} = \lambda y = f(t, y)$

$\Rightarrow$

$$w_{i+1} = w_i + \frac{h\lambda}{2}\left(w_{i+1} + w_i \right)$$

$$= \left(1 + \frac{h\lambda}{2}\right) w_i + \frac{h\lambda}{2} w_{i+1}$$

$\Leftrightarrow$

$$w_{i+1} = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} w_i = \frac{2 + h\lambda}{2 - h\lambda} w_i$$

$$= \left(\frac{2 + h\lambda}{2 - h\lambda} \right)^{i+1} w_0 = \left(Q(h\lambda) \right)^{i+1} w_0$$

Clearly, for any $\lambda > 0$, $Q > 1$

for any $\lambda < 0$ $Q < 1$

$\Rightarrow$ $R = (-\infty, 0)$

Homework 5. Problem 2

The truncation error is:

$$\tau_{k+1} = \frac{1}{h}\Big[\underbrace{y_{k+1}}_{\text{exact}} - \underbrace{y_k - h\,\phi(t_k, y_k)}_{\text{approximation}}\Big]$$

$$\tau_{k+1} = \frac{y_{k+1} - y_k}{h} - \phi(y_k)$$

for Trapezoid
$$= \frac{y_{k+1} - y_k}{h} - \frac{1}{2}\Big[f(y_k) + f(y_{k+1})\Big]$$

Now, Taylor expand y_{k+1}:

$$y_{k+1} = y_k + h y' + \frac{h^2}{2} y'' + \frac{h^3}{6} y''' + O(h^4)$$

$$y' = \frac{dy}{dt} = f$$

$$y'' = \frac{d}{dt} f(y) = \frac{df}{dy}\frac{dy}{dt} = \frac{df}{dy}\cdot f$$

$$y''' = \frac{d}{dt}\left(\frac{df}{dy} f\right) = f\frac{d}{dt}\frac{df}{dy} + \frac{df}{dy}\frac{df}{dt}$$

$$= f\frac{d^2f}{dy^2}\frac{dy}{dt} + \frac{df}{dy}\frac{df}{dy}\frac{dy}{dt} = f^2\frac{d^2f}{dy^2} + f\left(\frac{df}{dy}\right)$$

Note: more complicated if not autonomous cont.

Now, expand $f(y_{k+1})$:

$$f(y_{k+1}) = f\left(y_k + hy' + \frac{h^2}{2}y'' + O(h^3)\right)$$

$$= f(y_k) + \left(hy' + \frac{h^2}{2}y''\right)\frac{df}{dy} + \frac{1}{2}\left(hy' + \frac{h^2}{2}y''\right)^2\frac{d^2f}{dy^2} + O(h^3)$$

$$= f(y_k) + hy'\frac{df}{dy} + \frac{h^2}{2}y''\frac{df}{dy} + \frac{1}{2}h^2y'^2\frac{d^2f}{dy^2} + O(h^3)$$

where all h^3, h^4 terms are neglected

Now, put all in τ_{k+1}:

$$\tau_{k+1} = \frac{1}{h}\left(y_k + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' - y_k\right) - \frac{1}{2}\Big[f(y_k) + f(y_k)$$
$$+ hy'\frac{df}{dy} + \frac{h^2}{2}y''\frac{df}{dy} + \frac{1}{2}h^2y'^2\frac{d^2f}{dy^2}\Big] + O($$

$$= y' + \frac{h}{2}y'' + \frac{h^2}{6}y''' - \frac{1}{2}\left[2f(y_k) + hy'\frac{df}{dy} + \frac{h^2}{2}y''\frac{df}{dy} + \frac{1}{2}h^2y'^2\frac{d^2f}{dy^2}\right]$$

now, use y', y'', y''' from previous page:

$$= f + \frac{h}{2}\frac{df}{dy}f + \frac{h^2}{6}f^2\frac{d^2f}{dy^2} + \frac{h^2}{6}f\left(\frac{df}{dy}\right)^2 - f - \frac{h}{2}f\frac{df}{dy} - \frac{h^2}{4}f\left(\frac{df}{dy}\right)^2$$
$$- \frac{h^2}{4}f^2\frac{d^2f}{dy^2}$$

all h^0, h^1 terms cancel!

$$\tau_{k+1} = h^2\left(\frac{1}{6} - \frac{1}{4}\right)f\left(\frac{df}{dy}\right)^2 + h^2\left(\frac{1}{6} - \frac{1}{4}\right)f^2\frac{d^2f}{dy^2} \neq 0$$

$$\Rightarrow \quad \tau_{k+1} = O(h^2)$$

HW 5, Problem 3

$$\frac{dy}{dt} = t \sin y \cos 2y \qquad \begin{array}{l} 0 \le t \le 2.51 \\ y(0) = 1 \end{array}$$

We want:

$h \cdot \frac{\partial f}{\partial y} \notin R$ for $\frac{\partial f}{\partial y} > 0$

$h \cdot \frac{\partial f}{\partial y} \in R$ for $\frac{\partial f}{\partial y} < 0$

$f = t \sin y \cos 2y$

$\Rightarrow \frac{\partial f}{\partial y} = t\left(\cos y \cdot \cos 2y + 2 \sin y(-\sin 2y)\right)$

For 3rd order Runge Kutta: $R = (-2.51, 0)$

$\Rightarrow$ for $\frac{\partial f}{\partial y} > 0$: $h \frac{\partial f}{\partial y} \notin R$ is guaranteed

but: for $\frac{\partial f}{\partial y} < 0$:

rough estimate: $(\cos y \cos 2y + 2 \sin y(-\sin 2y)) > -3$

$\Rightarrow \frac{\partial f}{\partial y} > -7.53$ for all $0 \le t \le 2.51$

$\Rightarrow$ timestep restriction: $-7.53h \in R \Rightarrow h < \frac{1}{3}$

4) $\frac{dy}{dt} = 4t^2 \cos(y) = f(t, y)$

Trapezoidal:

a) $w_{k+1} = w_k + \frac{h}{2}\left[4t_k^2 \cos(w_k) + 4t_{k+1}^2 \cos(w_{k+1})\right]$

$\Leftrightarrow F(w) = w_{k+1} - w_k - \frac{h}{2}\left[4t_k^2(\cos(w_k) + 4t_{k+1}^2 \cos(w)\right]$

b) Newton:

$$w_{k+1}^{(i)} = w_{k+1}^{(i-1)} - \frac{F(w_{k+1}^{(i-1)})}{F'(w_{k+1}^{(i-1)})}$$

$$F' = 1 + \frac{h}{2} 4t_{k+1}^2 \sin(w)$$

$\Rightarrow$

$$w_{k+1}^{(i)} = w_{k+1}^{(i-1)} - \frac{w_{k+1}^{(i-1)} - w_k - \frac{h}{2}\left[4t_k^2 \cos w_k + 4t_{k+1}^2 \cos w_k^{i}\right.}{1 + \frac{h}{2} 4t_{k+1}^2 \sin(w_{k+1}^{i-1})}$$

$$= w_{k+1}^{(i-1)} - \frac{w_{k+1}^{(i-1)} - w_k - 2h\left[t_k^2 \cos w_k + t_{k+1}^2 \cos w_{k+1}^{i-1}\right.}{1 + 2h t_{k+1}^2 \sin(w_{k+1}^{i-1})}$$

c) Explicit !

Problem # 5

$$\frac{d^3 y}{dt^3} - 3\frac{dy}{dt} = 6$$

define : $y = u_1$

$$\frac{dy}{dt} = u_2$$

$$\frac{d^2 y}{dt^2} = u_3$$

$$\Rightarrow \frac{du_3}{dt} = 3u_2 + 6$$

$$\frac{du_2}{dt} = u_3$$

$$\frac{du_1}{dt} = u_2$$

$$\Rightarrow \frac{d\vec{u}}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 3 & 0 \end{pmatrix} \vec{u} + \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}$$

with $\vec{u}(0) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Problem 5)

h	Approx	$\Rightarrow$ error
0.0001	1.00000	0
0.1	0.999439	0.00056
0.05	0.999849	0.00015
0.025	0.9999605	0.00004
0.0125	0.99999	0.00001

Clearly, doubling h (going bottom to top) leads to an increase of the error by approximately a factor of 4.

$\Rightarrow$

order $\sim$ $\underline{2}$