Homework 2

HW 2, heavy ~ (5.4.6) Problem : Show that Heun's Method is order 2, not 3 !

$$T^{(3)} = f(t,y) + \frac{h}{2} f'(t,y) + \frac{h^2}{6} f''(t,y)$$

$$f'(t,y) = \frac{df}{dt}(t,y) = \frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\, f(t,y)$$

$$f''(t,y) = \frac{d}{dt}\left(\frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\, f(t,y)\right)$$

$$= \frac{d}{dt}\frac{\partial f}{\partial t}(t,y) + f(t,y)\frac{d}{dt}\frac{\partial f}{\partial y}(t,y) + \frac{\partial f}{\partial y}(t,y)\frac{df}{dt}(t,y)$$

$$= \frac{\partial^2 f}{\partial t^2}(t,y) + \frac{\partial^2 f}{\partial y \partial t}(t,y)\frac{dy}{dt} + f(t,y)\left(\frac{\partial^2 f}{\partial y \partial t}(t,y) + \frac{\partial^2 f}{\partial y^2}(t,y)\cdot\frac{dy}{dt}\right)$$

$$+ \frac{\partial f}{\partial y}(t,y)\left(\frac{\partial f}{\partial t}(t,y) + \frac{\partial f}{\partial y}(t,y)\, f(t,y)\right)$$

$$= \frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial y \partial t}\cdot f + f\frac{\partial^2 f}{\partial y \partial t} + f^2\frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial f}{\partial t}\right) + \left(\frac{\partial f}{\partial y}\right)^2\cdot f$$

$$= \frac{\partial^2 f}{\partial t^2} + 2f\frac{\partial^2 f}{\partial y \partial t} + f^2\frac{\partial^2 f}{\partial y^2} + \left(\frac{\partial f}{\partial y}\right)\left(\frac{\partial f}{\partial t}\right) + f\left(\frac{\partial f}{\partial y}\right)^2$$

$\Rightarrow$

$$T^{(3)} = f + \frac{h}{2}\frac{\partial f}{\partial t} + \frac{h}{2}\frac{\partial f}{\partial y}f + \frac{h^2}{6}\frac{\partial^2 f}{\partial t^2} + \frac{h^2}{3}f\frac{\partial^2 f}{\partial y \partial t} + \frac{h^2}{6}f^2\frac{\partial^2 f}{\partial y^2} + \frac{h^2}{6}\frac{\partial f}{\partial y}\frac{\partial f}{\partial t} + \frac{h^2}{6}\left(\frac{\partial f}{\partial y}\right)^2 f$$

Expand

$a_1 f(t,y) + a_2 f(t+\alpha_2, y + \delta_2 f(t,y))$:

$a_1 f(t,y) + a_2 f(t,y) + a_2\alpha_2 \frac{\partial f}{\partial t}(t,y) + a_2\delta_2 f(t,y)\frac{\partial f}{\partial y}(t,y)$

$+ \frac{1}{2} a_2 \alpha_2^2 \frac{\partial^2 f}{\partial t^2} + a_2\alpha_2\delta_2 f \frac{\partial^2 f}{\partial t \partial y} + \frac{1}{2} a_2 \delta^2 f^2 \frac{\partial^2 f}{\partial y^2}$

$(a_1+a_2) f + a_2\alpha_2 \frac{\partial f}{\partial t} + a_2\delta_2 f \frac{\partial f}{\partial y} + \frac{a_2\alpha_2^2}{2}\frac{\partial^2 f}{\partial t^2} + a_2\alpha_2\delta_2 f \frac{\partial^2 f}{\partial t\partial y} + \frac{a_2\delta^2 f^2}{2}\frac{\partial^2 f}{\partial y^2}$

$f: (a_1 + a_2) = 1$

$\frac{\partial f}{\partial t}: a_2\alpha_2 = \frac{h}{2}$

$\frac{\partial f}{\partial y}: a_2\delta_2 f = \frac{h}{2} f$

$\frac{\partial^2 f}{\partial t^2}: \frac{a_2\alpha_2^2}{2} = \frac{h^2}{6}$

$\frac{\partial^2 f}{\partial y^2}: \frac{a_2\delta^2}{2} f^2 = \frac{h^2}{6} f^2$

$\frac{\partial^2 f}{\partial y \partial t}: a_2\alpha_2\delta_2 f = \frac{h^2}{3} f$

6 equations,
4 parameters

The parameters for modified Euler don't work !!

These for Heun's Method do !

But : we can't match $\frac{\partial f}{\partial y}\frac{\partial f}{\partial t}$ and $\left(\frac{\partial f}{\partial y}\right)^2$ terms.

Thus: truncation error is of order h^2.

q.e.d.

2) $y' = 1 + \frac{y}{t}$ $\qquad 1 \le t \le 2 \quad ; y(1) = 2$

Well posed ?

$|f(t, y_1) - f(t, y_2)|$

$= \left| 1 + \frac{y_1}{t} - 1 - \frac{y_2}{t} \right|$

$= \left| \frac{1}{t} (y_1 - y_2) \right|$

$= \frac{1}{t} |y_1 - y_2| \le 1 \cdot |y_1 - y_2|$

Lipschitz constant