Chapter 2

Random walks

Random walks are one of the basic objects studied in probability theory. The motivation comes from observations of various random motions in physical and biological sciences. The most well-known example is the erratic motion of pollen grains immersed in a fluid — observed by botanist Robert Brown in 1827 — caused, as we now know, by collisions with rapid molecules. The latter example serves just as well for the introduction of Brownian motion. As will be discussed in a parallel course, Brownian motion is a continuous analogue of random walk and, not surprisingly, there is a deep connection between both subjects.

2.1 Random walks and limit laws

The definition of a random walk uses the concept of independent random variables whose technical aspects are reviewed in Chapter 1. For now let us just think of independent random variables as outcomes of a sequence of random experiments where the result of one experiment is not at all influenced by the outcomes of the other experiments.

Definition 2.1 [Random walk] *Suppose that $X_1, X_2, \dots$ is a sequence of $\mathbb{R}^d$-valued independent and identically distributed random variables. A* random walk *started at $z \in \mathbb{R}^d$ is the sequence $(S_n)_{n\geq 0}$ where $S_0 = z$ and*

$$S_n = S_{n-1} + X_n, \qquad n \geq 1. \tag{2.1}$$

The quantities (X_n) are referred to as steps *of the random walk.*

Our interpretation of the above formula is as follows: The variable S_n marks the position of the walk at time n. At each time the walk chooses a step at random — with the same step distribution at each time — and adds the result to its current position. The above can also be written as

$$S_n = z + X_1 + \cdots + X_n \tag{2.2}$$

for each $n \geq 1$. Note that while the steps $X_1, X_2, \dots$ are independent as random variables, the actual positions of the walk $S_0, S_1, \dots$ are not.

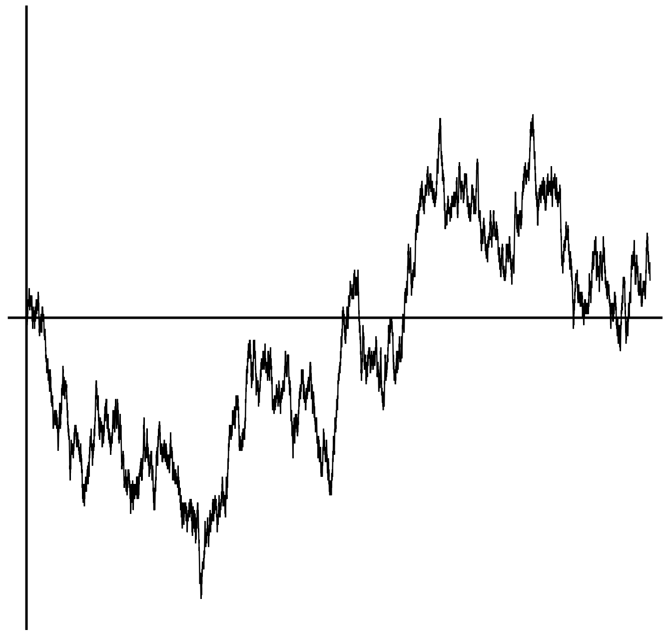

Figure 2.1: A path of length 10^4 of the simple random walk on $\mathbb{Z}$ drawn by interpolating linearly between the points with coordinates (n, S_n), $n = 0, \dots, 10^4$.

Exercise 2.2 Let $(S_n)_{n\geq 0}$ be a random walk. Show that $S_{2n} - S_n$ and S_n are independent and have the same distribution.

Here are some representative examples of random walks:

Example 2.3 *Simple random walk (SRW) on* $\mathbb{Z}$: This is the simplest of all random walks — hence the name. Here X_1 takes values in $\{+1, -1\}$ and the walk S_n started from 0 is thus confined to the set of all integers $\mathbb{Z}$. Often enough, X_1 takes both values with equal probabilities, i.e.,

$$\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2 \tag{2.3}$$

The walk then jumps left or right equally likely at each time. This case is more correctly referred to as the "simple *symmetric* random walk," but the adjective "symmetric" is almost invariably dropped. In the other cases, i.e., when

$$\mathbb{P}(X_1 = 1) = p \quad \text{and} \quad \mathbb{P}(X_1 = -1) = 1 - p \tag{2.4}$$

with $p \neq 1/2$, the walk is referred to as *biased*. The bias is to the right when $p > 1/2$ and to the left when $p < 1/2$.

Example 2.4 *Simple random walk on* $\mathbb{Z}^d$: This is a d-dimensional version of the first example. Here X_1 takes values in $\{\pm\hat{e}_1, \dots, \pm\hat{e}_d\}$ where $\hat{e}_k$ is the "coordinate vector" $(0, \dots, 0, 1, 0, \dots, 0)$ in $\mathbb{R}^d$ with the "1" appearing in the k-th position. This random walk is confined to the set of points in $\mathbb{R}^d$ with integer coordinates,

$$\mathbb{Z}^d = \big\{(n_1, \dots, n_d) \colon n_1, \dots, n_d \in \mathbb{Z}\big\}. \tag{2.5}$$

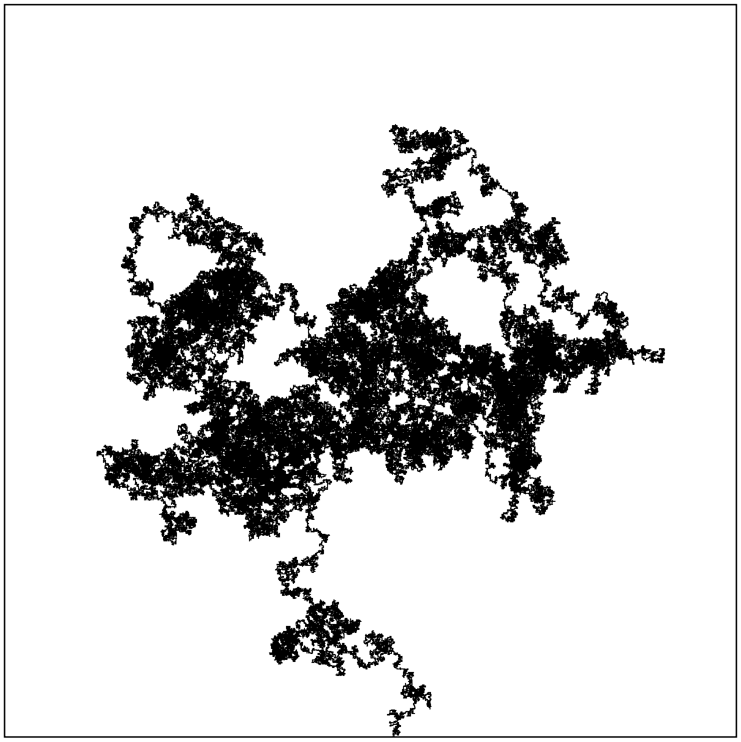

Figure 2.2: The set of vertices visited by a two-dimensional simple random walk before it exited a box of side 10^3. The walk was started at the center of the box and it took 682613 steps to reach the boundary.

The easiest example to visualize is the case of $d = 2$ where the set $\mathbb{Z}^2$ are the vertices of a square grid. Thinking of $\mathbb{Z}^2$ as a graph, the links between the neighboring vertices represent the allowed transitions of the walk. A majority of appearances of this random walk is in the symmetric case; i.e., when X_1 takes any of the $2d$ allowed values with equal probabilities.

Example 2.5 *"As the knight jumps" random walk on* $\mathbb{Z}^2$: This random walk takes steps allowed to the knight in the game of chess; i.e., there are 8 allowed jumps

$$2\hat{e}_1 + \hat{e}_2, \quad \hat{e}_1 + 2\hat{e}_2, \quad -2\hat{e}_1 - \hat{e}_2, \quad -\hat{e}_1 - 2\hat{e}_2, \tag{2.6}$$

$$2\hat{e}_1 - \hat{e}_2, \quad \hat{e}_1 - 2\hat{e}_2, \quad -2\hat{e}_1 + \hat{e}_2, \quad -\hat{e}_1 + 2\hat{e}_2. \tag{2.7}$$

Some experience with chess reveals that the random walk can reach every vertex of $\mathbb{Z}^2$ in a finite number of steps. This fails to be true if we further reduce the steps only to those in the top line; the random walk is then restricted to the fraction of $3/16$ of all vertices in $\mathbb{Z}^2$; see Fig. 2.3.

Example 2.6 *Gaussian random walk*: This random walk has steps that can take any value in $\mathbb{R}$. The probability distribution of X_1 is normal (or Gaussian) with mean

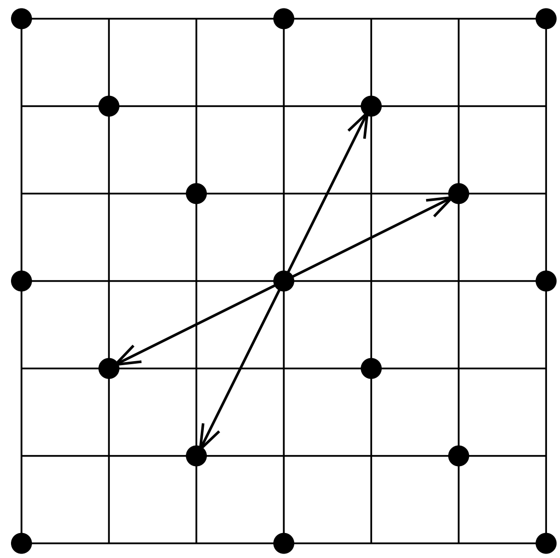

Figure 2.3: The set of allowed steps (arrows) and reachable vertices (dots) for the random walk discussed in Example 2.5.

zero and variance 1, i.e., $X_1 = \mathcal{N}(0,1)$ or, explicitly,

$$\mathbb{P}(X_1 \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, \mathrm{e}^{-x^2/2}\, \mathrm{d}x \tag{2.8}$$

A distinguished feature of this walk that the distribution of S_n is also normal with mean zero but variance $\sqrt{n}$. A typical displacement of this random walk after n steps is thus "order-$\sqrt{n}$" — a scale that, as we will see in Theorem 2.11, is quite typical for random walks with zero mean.

Example 2.7 *Heavy tailed random walk*: To provide contrast to the previous example, we can also take a random walk on $\mathbb{R}$ with a step distribution that is symmetric but has "heavy tails." (We discuss these briefly in Chapter 1.) For instance, take X_1 continuous with probability density

$$f(x) = \begin{cases} \dfrac{\alpha}{2}\, \dfrac{1}{|x|^{\alpha+1}}, & \text{if } |x| \geq 1, \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

where α is a parameter with $0 < \alpha < 2$. As is seen by comparing Fig. 2.1 and Fig. 2.4, a distinction between this random walk and the SRW is clear at first sight.

We finish our introduction to random walks by adapting standard limit theorems for sequences of i.i.d. random variables to the quantity $S_n = X_1 + \cdots + X_n$. Note the requirement of a particular moment condition in each theorem.

We begin by extracting the leading order (linear) scaling of S_n:

Theorem 2.8 [Strong Law of Large Numbers] *Suppose that $E|X_1| < \infty$. Then, with probability one,*

$$\lim_{n\to\infty} \frac{S_n}{n} \text{ exists and equals } \mathbb{E}X_1 \tag{2.10}$$

The expectation $\mathbb{E}X_1$ thus defines the *asymptotic velocity* of the walk. In particular, if $\mathbb{E}X_1 \neq 0$ then the walks moves away from the starting point at linear speed while for $\mathbb{E}X_1 = 0$ the speed is zero.

Exercise 2.9 Show that if $\mathbb{E}X_1 \neq 0$, the probability that the random walk with steps $X_1, X_2, \dots$ visits the starting point infinitely often is zero.

Problem 2.10 An example of a heavy tailed random walk is the *Cauchy random walk* where X_1 has Cauchy distribution characterized by the probability density

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}. \tag{2.11}$$

This example is analogous — or technically, in same "basin of attraction" — as the $\alpha = 1$ random walk discussed in Example 2.7. Show that if $X_1, \dots, X_n$ are independent Cauchy, then so is S_n/n for each n. In particular, the conclusion of Theorem 2.8 fails in this case.

Next we will describe the fluctutations of the position S_n around its mean:

Theorem 2.11 [Central Limit Theorem] *Consider a one-dimensional random walk with $\mathbb{E}(X_1^2) < \infty$. Then, as $n \to \infty$,*

$$\frac{S_n - n\mathbb{E}X_1}{\sqrt{n}} \tag{2.12}$$

has asymptotically normal distribution with mean zero and variance $\sigma^2 = \mathrm{Var}(X_1)$.

The crux of this result is that, for the walks with $\mathbb{E}X_1 = 0$, the distribution of the endpoint is asymptotically very close to that of the Gaussian random walk with a properly adjusted variance. This is a manifestation of a much more general *invariance principle* that deals with the distribution of the entire path of the random walk. The limiting object there is Brownian motion.

Problem 2.12 Consider the Gaussian random walk of length n. Show that the largest step is of size order $\sqrt{\log n}$ and that the difference between the first and second largest positive step tends to zero as $n \to \infty$.

Exercise 2.13 Suppose $1 < \alpha < 2$ and consider the first n steps of the heavy tailed random walk from Example 2.7. Show that the probability that the largest step is twice as large than any other step is bounded away from zero uniformly in n.

Problem 2.14 Suppose now that $0 < \alpha < 1$. Show that that with probability that is uniformly positive in n, the largest step of a heavy tailed random walk of length n is larger than the sum of the remaining steps. See Fig. 2.4.

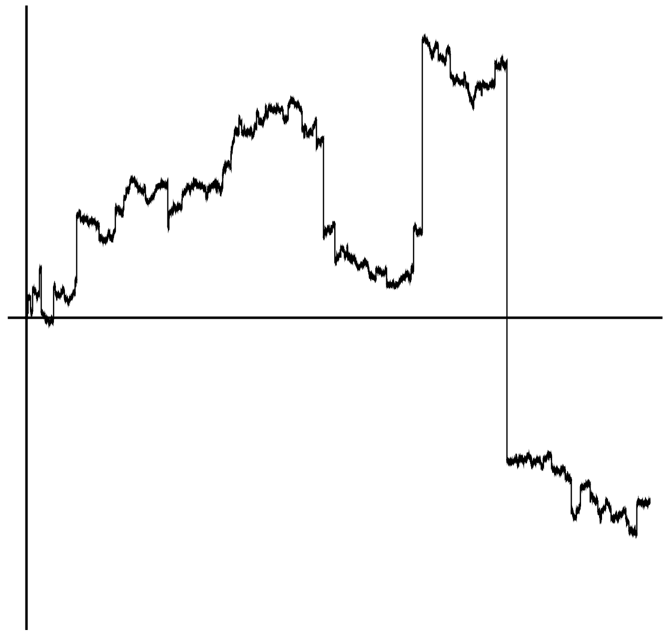

Figure 2.4: A plot of 25000 steps of the heavy tailed random walk from Example 2.7 with $\alpha = 1.2$. The defining feature of heavy tailed random walks is the presence of "macroscopic" jumps, i.e., those comparable with the typical distance of the walk from the starting point at the time of their occurrence. In particular, the Central Limit Theorem does not apply due to the lack of the second moment of X_1.

2.2 Transition in $d = 2$: Recurrence vs transience

In the previous section we introduced random walks in quite some generality. However, to make our discussion easier, we will henceforth assume that

$$\text{all random walks have step distribution concentrated on } \mathbb{Z}^d$$

Our next business is to try to address two basic questions:

(1) Under what conditions does a random walk come infinitely often back to its starting position?

(2) When do the paths of two independent copies of the same random walk intersect infinitely often?

The interest in these is bolstered by the fact that the answer depends sensitively on the dimension. Explicitly, for rather generic step distributions, the character of the answer changes as dimension goes from 2 to 3 for the first question and from 4 to 5 for the second question.

Throughout this section we will focus on the first question.

Definition 2.15 [Recurrence & transience] *We say that a random walk is* recurrent *if it visits its starting position infinitely often with probability one and* transient *if it visits its starting position finitely often with probability one.*

Our analysis begins by showing that every random walk is either recurrent or transient; no intermediate scenarios take place. Let N be the number of visits of (S_n) to its starting point S_0,

$$N = \sum_{n\geq 0} 1_{\{S_n=S_0\}}. \tag{2.13}$$

Recurrence then means $\mathbb{P}(N = \infty) = 1$ while transience means $\mathbb{P}(N < \infty) = 1$ and so absence of intermediate scenarios is equivalent to showing $P(N < \infty) \in \{0, 1\}$. Let τ denote the first time the walk is back to the starting point:

$$\tau = \inf\{n \geq 1\colon X_1 + \cdots + X_n = 0\} \tag{2.14}$$

If no such visit exists, then $\tau = \infty$. Note that

$$\mathbb{P}(N = 1) = \mathbb{P}(\tau = \infty). \tag{2.15}$$

Lemma 2.16 [Either recurrent or transient] *For each $n \geq 1$,*

$$\mathbb{P}(N = n) = \mathbb{P}(\tau = \infty)\mathbb{P}(\tau < \infty)^{n-1}. \tag{2.16}$$

Then either $\mathbb{P}(\tau = \infty) = 0$ which implies $\mathbb{P}(N < \infty) = 0$, or $\mathbb{P}(\tau = \infty) > 0$ which implies $\mathbb{P}(N < \infty) = 1$. In particular, every random walk is either recurrent or transient.

Proof. We first prove the identity

$$\mathbb{P}(N = n + 1) = \mathbb{P}(N = n)\mathbb{P}(\tau < \infty), \qquad n \geq 1. \tag{2.17}$$

Consider the first visit back to the origin and suppose it occurred at time $\tau = k$. Then $N = n + 1$ implies that the walk $S'_m = S_{k+m} - S_k$ — namely, the part of the walk (S_n) after time k — makes n visits back to its starting point, $S'_0 = 0$. But the walk S'_m is independent of the event $\{\tau = k\}$ because $\tau = k$ is determined by $X_1, \dots, X_k$ while S'_m is a function of only $X_{k+1}, X_{k+2}, \dots$. This implies

$$\begin{aligned}\mathbb{P}(N = n + 1 \ \&\ \tau = k) &= \mathbb{P}\Big(\sum_{m\geq 0} 1_{\{S'_m=0\}} = n \ \&\ \tau = k\Big) \\ &= \mathbb{P}\Big(\sum_{m\geq 0} 1_{\{S'_m=0\}} = n\Big)\,\mathbb{P}(\tau = k) = \mathbb{P}(N = n)\mathbb{P}(\tau = k)\end{aligned} \tag{2.18}$$

where we used that the walk S'_m has the same distribution as S_m. Summing

$$\mathbb{P}(N = n + 1 \ \&\ \tau = k) = \mathbb{P}(N = n)\mathbb{P}(\tau = k) \tag{2.19}$$

over k in the range $1 \leq k < \infty$ gives (2.17).

To get (2.16), plug (2.15) in (2.17) and solve recursively for $\mathbb{P}(N = n)$. For the rest of the claim, we note that if $\mathbb{P}(\tau = \infty) = 0$, then $\mathbb{P}(N = n) = 0$ for all $n < \infty$

implying $\mathbb{P}(N < \infty) = 0$. If, on the other hand, $\mathbb{P}(\tau = \infty) > 0$ then $\mathbb{P}(\tau < \infty) < 1$ and, by (2.17), the probabilities $\mathbb{P}(N = n)$ form a geometric sequence. Summing over all n in the range $1 \le n < \infty$ gives

$$\mathbb{P}(N < \infty) = \frac{\mathbb{P}(\tau = \infty)}{1 - \mathbb{P}(\tau < \infty)} = 1 \tag{2.20}$$

as desired. □

Problem 2.17 Suppose (S_n) is a random walk and let x be such that $\mathbb{P}(S_n = x) > 0$ for some $n \ge 0$. Prove that with probability one (S_n) visits x only finitely often if (S_n) is transient and infinitely often if (S_n) is recurrent.

The main technical point of the previous derivations is that transience can be characterized in terms of finiteness of $\mathbb{E}N$:

Lemma 2.18 *A random walk is transient if $\mathbb{E}N < \infty$ and recurrent if $\mathbb{E}N = \infty$.*

Proof. If $\mathbb{E}N < \infty$ then $\mathbb{P}(N < \infty) = 1$ and the walk is transient. However, the other implication is more subtle. Assume $\mathbb{P}(N < \infty) = 1$ and note that then also $\mathbb{P}(\tau = \infty) > 0$. Then sequence $\mathbb{P}(N = n)$ thus decays exponentially and so

$$\begin{aligned} \mathbb{E}N = \sum_{n=1}^{\infty} n\mathbb{P}(N = n) &= \mathbb{P}(\tau = \infty) \sum_{n=1}^{\infty} n\mathbb{P}(\tau < \infty)^{n-1} \\ &= \frac{\mathbb{P}(\tau = \infty)}{[1 - \mathbb{P}(\tau < \infty)]^2} = \frac{1}{\mathbb{P}(\tau = \infty)} \end{aligned} \tag{2.21}$$

Hence $\mathbb{E}N < \infty$ as we intended to show. □

Exercise 2.19 As noted in the proof, the fact that $\mathbb{E}N < \infty$ implies $\mathbb{P}(N < \infty) = 1$ is special for the context under consideration. To see this is not true in general, find an example of an integer valued random variable $Z \ge 0$ such that $\mathbb{P}(Z < \infty) = 1$ but $\mathbb{E}Z = \infty$.

Exercise 2.20 Show that the probability $\mathbb{P}(S_n = 0)$ for the simple symmetric random walk in $d = 1$ decays like $n^{-1/2}$. Conclude that the walk is recurrent.

A practical advantage of the characterization using the finiteness of $\mathbb{E}N$ is that the expectation can be explicitly computed:

Lemma 2.21 [Expectation formula] *Consider a random walk on $\mathbb{Z}^d$ with steps denoted by $X_1, X_2, \dots$ and let*

$$\varphi(k) = \mathbb{E}(e^{ik\cdot X_1}) := \mathbb{E}\cos(k \cdot X_1) + i\,\mathbb{E}\sin(k \cdot X_2). \tag{2.22}$$

Then

$$\mathbb{E}N = \lim_{t \uparrow 1} \int_{[-\pi,\pi]^d} \frac{dk}{(2\pi)^d} \frac{1}{1 - t\varphi(k)} \tag{2.23}$$

Proof. The proof is based on the formula

$$1_{\{S_n=0\}} = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}k\cdot S_n}, \tag{2.24}$$

which is a consequence of d-fold application of the Fourier identity

$$\int_{[-\pi,\pi]} \frac{\mathrm{d}\theta}{2\pi}\, \mathrm{e}^{\mathrm{i}n\theta} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n \in \mathbb{Z} \setminus \{0\}. \end{cases} \tag{2.25}$$

(Here is where we used the fact that the walk is confined to *integer* lattice.) Taking expectation in (2.24), we thus get

$$\mathbb{P}(S_n = 0) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d}\, \mathbb{E}(\mathrm{e}^{\mathrm{i}k\cdot S_n}). \tag{2.26}$$

Since S_n is the sum $X_1 + \cdots + X_n$, we have $\mathrm{e}^{\mathrm{i}S_n} = \mathrm{e}^{\mathrm{i}k\cdot X_1} \cdots \mathrm{e}^{\mathrm{i}k\cdot X_n}$. Moreover, as $X_1, \dots, X_n$ are independent, then so $\mathrm{e}^{\mathrm{i}k\cdot X_1}, \cdots, \mathrm{e}^{\mathrm{i}k\cdot X_n}$. Using that expectation of a product of independent random variables is a product of expectations,

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}k\cdot S_n}) = \mathbb{E}(\mathrm{e}^{\mathrm{i}k\cdot X_1}) \cdots \mathbb{E}(\mathrm{e}^{\mathrm{i}k\cdot X_n}) = \varphi(k)^n \tag{2.27}$$

Next multiply (2.26) by t^n for some $t \in [0,1)$ and sum on $n \geq 0$. This gives

$$\begin{aligned} \sum_{n=0}^{\infty} t^n \mathbb{P}(S_n = 0) &= \sum_{n\geq 0} t^n \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \varphi(k)^n \\ &= \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \sum_{n\geq 0} \big[t\varphi(k)\big]^n \\ &= \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{1}{1 - t\varphi(k)} \end{aligned} \tag{2.28}$$

where we used that $|t\varphi(k)| \leq t < 1$ to see that the sum and integral can be interchanged in the second line. Taking the limit $t \uparrow 1$ makes the left-hand side tend to $\sum_{n\geq 0} \mathbb{P}(S_n = 0) = \mathbb{E}N$. □

These observation allow us to characterize when the simple random walk is recurrent and when it is transient:

Theorem 2.22 *The simple symmetric random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$.*

Proof. To apply the previous lemma, we need to calculate φ for the SRW. Using that the walk makes steps only in (positive or negative) coordinate directions, we get

$$\begin{aligned} \varphi(k) &= \frac{1}{2d}\mathrm{e}^{\mathrm{i}k_1} + \frac{1}{2d}\mathrm{e}^{-\mathrm{i}k_1} + \cdots + \frac{1}{2d}\mathrm{e}^{\mathrm{i}k_d} + \frac{1}{2d}\mathrm{e}^{-\mathrm{i}k_d} \\ &= \frac{1}{d}\cos(k_1) + \cdots + \frac{1}{d}\cos(k_d). \end{aligned} \tag{2.29}$$

This shows that $\varphi(k) = 1$ on $[-\pi,\pi]^d$ if an only if $k_1 = \cdots = k_d = 0$ and so $k = 0$ is the only point that could make the integral diverge in the limit as $t \uparrow 1$. To find out what happens precisely, we will need to control the behavior of the function $1 - t\varphi(k)$ around $k = 0$ for t close to one. First we note that

$$1 - \cos(x) = 2\sin^2(x/2) \quad \text{and} \quad \frac{2x}{\pi} \le \sin(x) \le x \tag{2.30}$$

yield

$$2\frac{k_i^2}{\pi^2} \le 1 - \cos(k_i) \le \frac{k_i^2}{2} \tag{2.31}$$

Plugging this in the definition of $\varphi(k)$ shows that

$$1 - t + 2t\frac{|k|^2}{\pi^2 d} \le 1 - t\varphi(k) \le 1 - t + \frac{|k|^2}{2d}. \tag{2.32}$$

Taking the limit we find that the function $k \mapsto 1 - t\varphi(k)$ is uniformly integrable around $k = 0$ if and only if the function $k \mapsto |k|^2$ is integrable, i.e.,

$$\mathbb{E}N < \infty \quad \text{if and only if} \quad \int_{|k|<1} \frac{\mathrm{d}k}{|k|^2} < \infty \tag{2.33}$$

The integral is finite if and only if $d \ge 3$. □

A famous quote sums up the previous theorem as follows: "A drunken man will always find his way home but a drunken bird may get lost forever." This, of course, assumes that the spontaneous motion of intoxicated biological material is described by a random walk with similar properties as the SRW.

Exercise 2.23 Show that a biased simple random walk on $\mathbb{Z}$ — i.e., the walk on $\mathbb{Z}$ with $\mathbb{P}(X_1 = +1) = p = 1 - \mathbb{P}(X_1 = -1)$ — is transient for all $p \neq 1/2$.

Problem 2.24 Use the above techniques to show that the random walk described in Fig. 2.3 is recurrent in dimensions $d = 1, 2$ and transient otherwise.

Problem 2.25 Consider a random walk on $\mathbb{Z}$ with step distribution

$$\mathbb{P}(X_1 = n) = \frac{1}{2}\Big(\frac{1}{|n|^\alpha} - \frac{1}{(|n|+1)^\alpha}\Big), \qquad n \neq 0. \tag{2.34}$$

Characterize the values of $\alpha > 0$ for which the walk is recurrent.

2.3 Transition in $d = 4$ & Loop-erased random walk

In this section will be devoted to the second question from Section 2.2 which concerns the *non-intersection of the paths* of independent copies of the same random walk. Consider two independent copies (S_n) and $(\tilde{S}_n)$ of the same random walk. We are interested in the cardinality of the set

$$\mathfrak{I}(S, \tilde{S}) := \{S_n \colon n \ge 0\} \cap \{\tilde{S}_n \colon n \ge 0\}. \tag{2.35}$$

First we note that in some cases the question can be answered directly:

Exercise 2.26 Use Problem 2.17 to show that paths of two independent copies of a (non-constant) recurrent random walk meet at infinitely many distinct points.

This allows us to focus, as we will do from now on, on transient random walks only. Some of these can be still handled by geometric arguments:

Problem 2.27 Show that the paths of two independent copies of a simple random walk on $\mathbb{Z}$, biased or symmetric, intersect infinitely often with probability one.

To address the general case, instead of $|\mathfrak{I}(S,\tilde{S})|$ we will work with the number

$$N^{(2)} = \sum_{m,n\geq 0} 1_{\{S_m = S_n\}} \tag{2.36}$$

that counts the number of pairs of times when the walks collided. To see this comes at no loss, we note that

$$N^{(2)} < \infty \quad \Rightarrow \quad \big|\mathfrak{I}(S,\tilde{S})\big| < \infty \tag{2.37}$$

To get the opposite implication, we note:

Lemma 2.28 *Suppose the random walks S and $\tilde{S}$ are transient. Then*

$$\mathbb{P}(N^{(2)} = \infty) = 1 \quad \textit{if and only if} \quad \mathbb{P}\big(|\mathfrak{I}(S,\tilde{S})| = \infty\big) = 1 \tag{2.38}$$

Proof. Let n_x be the number of visits of (S_n) to x,

$$n_x = \sum_{n\geq 0} 1_{\{S_n = x\}} \tag{2.39}$$

and let $\tilde{n}_x$ be the corresponding quantity for $\tilde{S}_n$. By the assumption of transience, $n_x < \infty$ and $\tilde{n}_x < \infty$ for every x with probability one. Next we note

$$N^{(2)} = \sum_{m,n\geq 0} \sum_{x\in\mathfrak{I}(S,\tilde{S})} 1_{\{S_n = x\}} 1_{\{\tilde{S}_m = x\}} = \sum_{x\in\mathfrak{I}(S,\tilde{S})} n_x \tilde{n}_x \tag{2.40}$$

If $|\mathfrak{I}(S,\tilde{S})| < \infty$, then the sum would be finite implying $N^{(2)} < \infty$. Thus, if $\mathbb{P}(N^{(2)} = \infty) = 1$ then we must have $|\mathfrak{I}(S,\tilde{S})| = \infty$ with probability one. □

We now proceed to characterize the transient random walks which $N^{(2)}$ is finite with probability one. The analysis is analogous to the question of recurrence vs transience but some steps are more tedious and so will be a bit sketchy at times.

Using arguments that we omit for brevity, one can again show that $\mathbb{P}(N^{(2)} < \infty)$ takes only values zero and one and

$$\mathbb{P}(N^{(2)} < \infty) = 1 \quad \text{if and only if} \quad \mathbb{E}N^{(2)} < \infty. \tag{2.41}$$

Next we prove:

Lemma 2.29 *Consider a random walk on $\mathbb{Z}^d$ with steps $X_1, X_2, \dots$ and let, as before, $\varphi(k) = \mathbb{E}(\mathrm{e}^{\mathrm{i}k\cdot X_1})$. Then*

$$\mathbb{E}N^{(2)} = \lim_{t\uparrow 1} \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d} \frac{1}{|1 - t\varphi(k)|^2} \tag{2.42}$$

Proof. A variant of the formula (2.24) gives

$$1_{\{S_n=\tilde{S}_m\}} = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d}\, \mathrm{e}^{\mathrm{i}k\cdot(S_n-\tilde{S}_m)}. \tag{2.43}$$

Applying (2.27) we have

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}k\cdot(S_n-\tilde{S}_m)}) = \varphi(k)^n\overline{\varphi(k)}^m \tag{2.44}$$

and so taking expectations on both sides of (2.43) leads to

$$\mathbb{P}(S_n=\tilde{S}_m) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d}\, \varphi(k)^n\overline{\varphi(k)}^m \tag{2.45}$$

Multiplying by t^{n+m} and summing over $n,m \ge 0$ we get

$$\sum_{m,n\ge 0} t^{m+n}\mathbb{P}(S_n=\tilde{S}_m) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}k}{(2\pi)^d}\frac{1}{|1-t\varphi(k)|^2}. \tag{2.46}$$

As before, from here the claim follows by taking the limit $t\uparrow 1$. □

The principal conclusion is now as follows:

Theorem 2.30 *The paths of two independent simple symmetric random walks on* $\mathbb{Z}^d$ *intersect infinitely often with probability one in dimensions* $d\le 4$ *and only finitely often with probability one in dimensions* $d\ge 5$.

Proof. Using the same estimates as in Theorem 2.22, we get

$$\mathbb{E}N^{(2)}<\infty \quad \text{if and only if} \quad \int_{|k|<1}\frac{\mathrm{d}k}{|k|^4}<\infty. \tag{2.47}$$

The integral is finite if and only if $d\ge 5$. □

Problem 2.31 For what values of $\alpha>0$ do the paths of independent copies of the walk described in Problem 2.25 intersect infinitely often.

There is a heuristic explanation of the above phenomena: The fact that the random walk is recurrent in $d=2$, but just barely, means that the path of the walk is *two dimensional*. (This is actually a theorem if we interpret the dimension in the sense of Hausdorff dimension.) Now it is a fact from geometry two generic two-dimensional subspaces of $\mathbb{R}^d$ do not intersects in dimension $d\ge 5$ and they do in dimensions $d\le 4$. Hence we *should* expect that Theorem 2.30 is true, except perhaps for the subtle boundary case $d=4$.

Problem 2.32 To verify the above heuristics, let us investigate the intersections of m paths of SRW. Explicitly, let $S^{(1)},\dots,S^{(m)}$ be m independent SRW and define

$$N^{(m)} = \sum_{\ell_1,\dots,\ell_m\ge 0} 1_{\{S_{\ell_1}=\dots=S_{\ell_m}\}} \tag{2.48}$$

Find in what dimensions we have $\mathbb{E}N^{(m)}<\infty$.

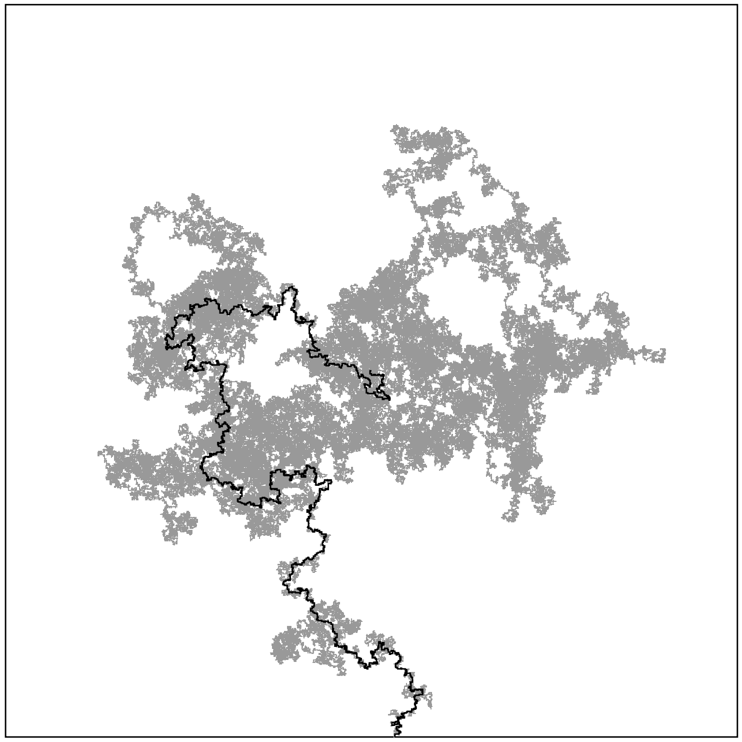

Figure 2.5: A path of the loop erased random walk obtained by loop-erasure from the SRW from Fig. 2.2. The trace of the SRW is depicted in light gray. While the SRW needed 682613 steps to exit the box, its loop erasure took only 3765 steps.

The non-intersection property of the simple random walk above 4 dimensions plays a crucial role in the understanding of a walk derived from the SRW called the *loop-erased random walk* (LERW). Informally, the LERW is extracted from a finite path of the SRW by sequentially erasing all cycles on the path of the SRW in the order they were created. The main point of doing this is that the resulting LERW has *self-avoiding paths* — i.e., paths that visit each point at most once.

Definition 2.33 [Loop erasure of SRW path] *Let $S_0, S_1, \dots, S_n$ be a finite path of the SRW. Define the sequence of times $T_0 < T_1 < \dots$ by setting $T_0 = 0$ and*

$$T_{k+1} = 1 + \sup\{m\colon T_k \le m \le n \ \&\ S_m = S_{T_k}\} \tag{2.49}$$

The loop erasure *of (S_k) is then the sequence (Z_m) where*

$$Z_m = S_{T_m}, \qquad T_m \le n. \tag{2.50}$$

The subject of the LERW goes way beyond the level and scope of these notes. (Indeed, it has only been proved recently that, in all dimensions, the LERW has a well defined scaling limit which is understood in $d = 2$ — see Fig. 2.5 — and $d \ge 4$, but not in $d = 3$.) However, the analysis of the path-avoiding property of the SRW allows us to catch at least a glimpse of what is going on in dimensions $d \ge 5$.

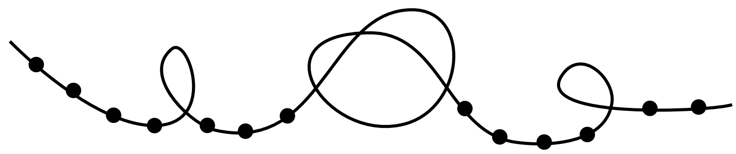

Figure 2.6: A schematic picture of the path of a two sided random walk which, in high dimension, we may think of as a chain of little tangles or knots separated by cutpoints (marked by the bullets).

The key notion to be studied in high dimension is that of a *cut point*. The cleanest way to define this notion is for the *two sided* random walk which is a sequence of random variables $(S_n)_{n\in\mathbb{Z}}$ indexed by (both positive and negative) integers, where S_n is defined by (2.1) even for negative n. (We assume, of course, that the sequence (X_n) of steps is doubly infinite as well.)

Definition 2.34 [Cut point] *Consider a two sided random walk on $\mathbb{Z}^d$. Then $x \in \mathbb{Z}^d$ is said to be a* cut point *if there exists k such that $S_k = x$ and the paths of one sided walks*

$$S'_n = S_{n+k} \quad \text{and} \quad S''_n = S_{k-n}, \qquad n \ge 0, \tag{2.51}$$

intersect only at their starting point. The time k is then referred to as the cut time *of the random walk (S_n).*

Lemma 2.35 *Consider the two sided random walk (S_n) and let (Z_n) be the loop erasure of the $n \ge 0$ portion of the path. Then the sequence (Z_n) visits all cutpoints (of the two-sided path) on the $n \ge 0$ portion of the path (S_n) in chronological order.*

Proof. The loop erasure removes only vertices on the path that are inside cycles. Cut points are never part of a cycle and so they will never be loop-erased. □

The fact that the SRW and the LERW agree on all cutpoints has profound consequences provided we can control the frequency of occurrence of cutpoints. We state a very weak claim to this extent:

Lemma 2.36 *Let R_n be the number of cut times — in the sense of Definition 2.34 — in the set of times $\{1, \dots, n\}$. Then*

$$\mathbb{E}R_n = n\mathbb{P}(N^{(2)} = 1). \tag{2.52}$$

In particular, for each $\epsilon \in [0, 1)$,

$$\mathbb{P}(R_n \ge \epsilon n) \ge \frac{\mathbb{P}(N^{(2)} = 1) - \epsilon}{1 - \epsilon}. \tag{2.53}$$

Proof. We have

$$R_n = \sum_{k=1}^{n} 1_{\{k \text{ is a cut time}\}} \tag{2.54}$$

Taking expectation we get

$$\mathbb{E}R_n = \sum_{k=1}^{n} \mathbb{P}(k \text{ is a cut time}). \tag{2.55}$$

But the path of the two-sided random walk looks the same from every time and so $\mathbb{P}(k \text{ is a cut time})$ equals the probability that 0 is a cut time. That probability in turn equals $\mathbb{P}(N^{(2)} = 1)$. This proves (2.52). To get also (2.53) we note

$$\mathbb{E}R_n \le \epsilon n\big[1 - \mathbb{P}(R_n \ge \epsilon n)\big] + n\mathbb{P}(R_n \ge \epsilon n). \tag{2.56}$$

Then (2.53) follows from (2.52) and some simple algebra. □

Of course having a positive density of points where the SRW and the LERW agree is not sufficient to push the path correspondence through. However, if we can show that the "tangles" between the cutpoints have negligible diameter and that none of them consumes a macroscopic amount of time, then on a large scale the paths of the LERW and the SRW will be hardly distinguishable.

2.4 Harmonic analysis and electric networks

Random walks have a surprising connection to *electric* or, more specifically, *resistor networks*. This connection provides very efficient means to estimate various hitting probabilities and other important characteristics of random walks. The underlying common ground is the subject of *harmonic analysis*.

We begin by a definition of a resistor network:

Definition 2.37 [Resistor network] *A* resistor network *is an unoriented (locally finite) graph $G = (V, E)$ endowed with a collection $(c_{xy})_{(x,y)\in E}$ of positive and finite numbers — called* conductances *— that obey the symmetry*

$$c_{xy} = c_{yx}, \qquad (x, y) \in E, \tag{2.57}$$

and local boundedness

$$\sum_{y\in V} c_{xy} < \infty, \qquad x \in V, \tag{2.58}$$

conditions. The reciprocal values, $r_e = 1/c_e$ are referred to as resistances.

The above definition builds on applications of electric networks in engineering where one often considers circuits with *nodes* and *links*. The links transport *electric current* between the nodes and the resistance of the link characterizes energy dissipation — generally due to heat production — of the link. The nodes, on the other hand, are characterized by the value of the *electric potential*. The currents and voltages are in one-to-one correspondence via Ohm's Law, which we will regard as a definition of the currents:

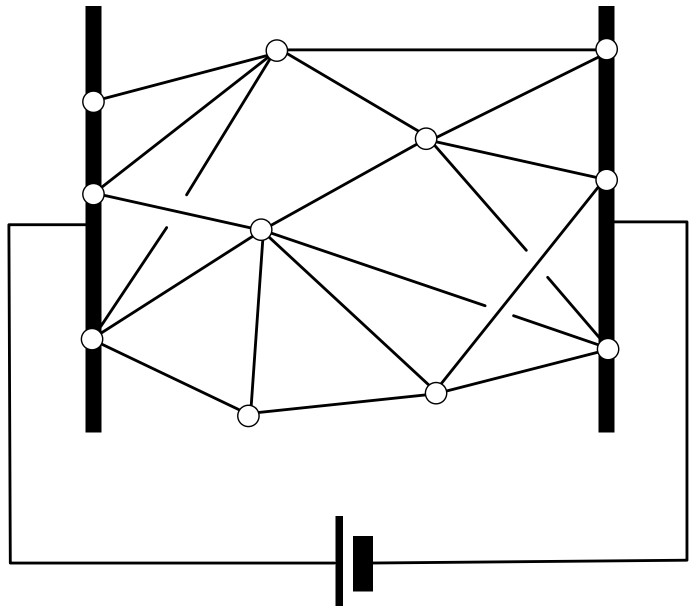

Figure 2.7: A circuit demonstrating the setting in the first electrostatic problem mentioned above. Here vertices on the extreme left and right are placed on conducting plates that, with the help of a battery, keep them at a constant electrostatic potential. The problem is to determine the potential at the "internal" vertices.

Definition 2.38 [Ohm's Law] *Suppose $G = (V, E)$ is an resistor network with conductances (c_{xy}). Let $u\colon V \to \mathbb{R}$ be an electric potential at the nodes. Then the* electric current *i_{xy} from x to y is given by*

$$i_{xy} = c_{xy}\big[u(y) - u(x)\big]. \tag{2.59}$$

For ease of exposition, we also introduce the notation $i(x)$ for the total current,

$$i(x) := \sum_{y \in V} i_{xy} \tag{2.60}$$

out of vertex x. There are two basic engineering questions that one may ask about resistor networks:

(1) Suppose the values of the potential u are fixed on a set $A \subset V$. Find the potential at the remaining nodes.

(2) Suppose that we are given the total current $i(x)$ out of the vertices in $A \subset V$. Find the potential at the nodes of V that is consistent with these currents.

The context underlying these questions is sketched in Figs. 2.7 and 2.10.

Of course, the above questions would not come even close to having a unique solution without imposing an additional physical principle:

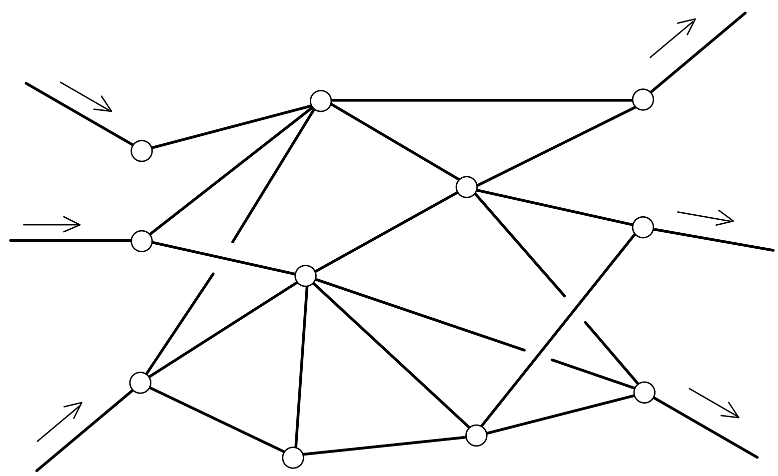

Figure 2.8: A circuit demonstrating the setting in the second electrostatic problem above. The topology of the circuit is as in Fig. 2.7, but now the vertices on the sides have a prescribed current flowing in/out of them. The problem is again to determine the electrostatic potential consistent with these currents.

Definition 2.39 [Kirchhoff's Law of Currents] *We say that a collection of currents (i_{xy}) obeys Kirchhoff's law of currents in the set $W \subset V$ if the total current out of any vertex in W is conserved, i.e.,*

$$i(x) = 0, \qquad x \in W. \tag{2.61}$$

The imposition of Kirchhoff's law has the following simple consequence:

Lemma 2.40 *For a function $f\colon V \to \mathbb{R}$, define $(\mathcal{L}f)\colon V \to \mathbb{R}$ by*

$$(\mathcal{L}f)(x) = \sum_{y \in V} c_{xy}\big[f(y) - f(x)\big] \tag{2.62}$$

Let $W \subset V$ and suppose u is an electric potential for which the currents defined by Ohm's law satisfy Kirchhoff's law of currents in W. Then

$$(\mathcal{L}u)(x) = 0, \qquad x \in W. \tag{2.63}$$

Proof. Using Ohm's Law, the formula for the current out of x becomes

$$i(x) = \sum_{y \in V} i_{xy} = \sum_{y \in V} c_{xy}\big[u(y) - u(x)\big] = (\mathcal{L}u)(x) \tag{2.64}$$

Thus if $i(x) = 0$, then $\mathcal{L}u$ vanishes at x. □

Definition 2.41 [Harmonic function] *We say that $f\colon V \to \mathbb{R}$ is* harmonic in W with respect to $\mathcal{L}$ *if $(\mathcal{L}f)(x) = 0$ for each x in a subset $W \subset V$.*

Note that while the definition of harmonicity of f speaks only about the vertices in W, vertices outside W may get involved due to the non-local nature of $\mathcal{L}$. Harmonic functions are special in that they satisfy the Maximum Principle. Given a set $W \subset V$, we use

$$\partial W = \big\{y \in V \setminus W \colon \exists x \in W \text{ such that } (x,y) \in E\big\} \tag{2.65}$$

to denote its outer boundary. Then we have:

Theorem 2.42 [Maximum Principle] *Let $W \subset V$ be finite and connected and suppose $f\colon V \to \mathbb{R}$ is harmonic on W with respect to $\mathcal{L}$. Then*

$$\inf_{y\in\partial W} f(y) \le \min_{x\in W} f(y) \le \max_{x\in W} f(x) \le \sup_{y\in\partial W} f(y), \tag{2.66}$$

Proof. Let $x \in W$ be a local maximum of f on $W \cup \partial W$. We claim that then

$$f(y) = f(x) \text{ for all } y \text{ with } (x,y) \in E \tag{2.67}$$

Indeed, if $f(y) \le f(x)$ for all neighbors of x with at least one inequality strict, then

$$\sum_{y\in V} c_{xy} f(x) > \sum_{y\in V} c_{xy} f(y). \tag{2.68}$$

But that is impossible because the difference of the left and right-hand side equals $(\mathcal{L}f)(x)$ which is zero because $x \in W$ and because f is harmonic at x.

Now suppose that the right-inequality on (2.66) does not hold. Then the maximum of f over $W \cup \partial W$ occurs on W. We claim that then f is constant on $W \cup \partial W$. Indeed, if $x \in W \cup \partial W$ were a vertex where f is not equal its maximum but that has a neighbor where it is, then we would run into a contradiction with the first part of the proof by which f must be constant on the neighborhood of any local maxima. Hence, f is constant on $W \cup \partial W$. But then the inequality on the right of (2.66) *does* hold and so we have a contradiction anyway. The inequality on the left is equivalent to that on the right by passing to $-f$. □

Corollary 2.43 [Rayleigh's Principle] *Let $W \subset V$ be finite and let $u_0\colon V \setminus W \to \mathbb{R}$ be bounded. Then there exists a unique $u\colon V \to \mathbb{R}$ which is harmonic on W with respect to $\mathcal{L}$ and satisfies*

$$u(x) = u_0(x), \qquad x \in V \setminus W. \tag{2.69}$$

Moreover, this function is the unique minimizer of the Dirichlet energy functional,

$$\mathcal{E}(u) = \frac{1}{2} \sum_{x,y\in V} c_{xy} \big[u(y) - u(x)\big]^2 \tag{2.70}$$

subject to the condition (2.69).

Proof. First we will establish uniqueness. Suppose u and $\tilde{u}$ are two distinct functions which are harmonic on W with respect to $\mathcal{L}$ and both of which obey (2.69).

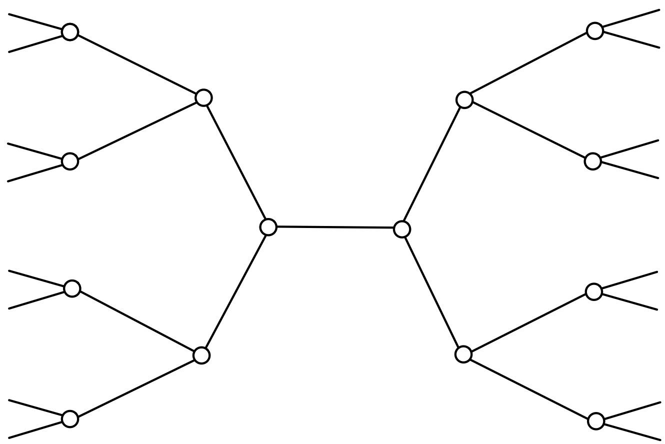

Figure 2.9: A regular ternary tree.

Then $v = u - \tilde{u}$ is harmonic on W and vanishes on $V \setminus W$. But the Maximum Principle implies

$$0 = \inf_{y \in \partial W} v(y) \le \min_{x \in W} v(x) \le \max_{x \in W} v(x) \le \sup_{y \in \partial W} v(y) = 0 \tag{2.71}$$

and so $v \equiv 0$. It follows that $u \equiv \tilde{u}$.

To see that the desired harmonic function in W exists for each "boundary condition" u_0, we will use the characterization by means of the minimum of $\mathcal{E}(u)$. The function $u \mapsto \mathcal{E}(u)$, regarded as a function of variables $\{u(x) \colon x \in W\}$ — with u_0 substituted for u outside W — is continuous and bounded from below, and so it achieves its minimum. The minimizer is characterized by the vanishing of partial derivatives,

$$\frac{\partial}{\partial u(x)} \mathcal{E}(u) = -2 \sum_{y \in V} c_{xy} \big[u(y) - u(x) \big] = -2(\mathcal{L}u)(x) \tag{2.72}$$

for all $x \in W$. This means that it is harmonic on W. □

An important consequence of the characterization in terms of a minimizer of the Dirichlet energy is the monotonicity in c_{xy}. Indeed, we note:

Lemma 2.44 *Let $W \subset V$ be finite or infinite. The (value of the) minimum of $u \mapsto \mathcal{E}(u)$ subject to $u = u_0$ on $V \setminus W$ is non-decreasing in all variables c_{xy}.*

Proof. This is an immediate consequence of the fact that the minimum of a family of non-decreasing functions is non-decreasing. □

Let us go back to the two questions we posed above and work them out a little more quantitatively. Suppose A and B are disjoint sets in V and suppose that A is kept at potential $u = 0$ and B at a constant potential $u = U > 0$. A current I will then flow from A to B. Thinking of the whole network as just one resistor, the natural question is what is its *effective resistance* $R_{\text{eff}} = U/I$. A formal definition of this quantity is as follows:

Definition 2.45 [Effective resistance] *Let $A, B \subset V$ be disjoint. The* effective resistance $R_{\text{eff}}(A,B)$ *is a number in* $[0,\infty]$ *defined by*

$$R_{\text{eff}}(A,B)^{-1} = \inf\bigl\{\mathcal{E}(u)\colon 0 \le u \le 1,\, u \equiv 0 \text{ on } A,\, u \equiv 1 \text{ on } B\bigr\}. \tag{2.73}$$

Problem 2.46 Show that adding or removing the condition $0 \le u \le 1$ does not change the value of the infimum.

A consequence of Lemma 2.44 is that the effective resistance $R_{\text{eff}}(A,B)$ increases when the individual resistances r_{xy} are increased.

Exercise 2.47 Show that adding an extra link of positive conductance to the graph G *decreases* the effective resistance $R_{\text{eff}}(A,B)$ between any two disjoint sets A and B.

Problem 2.48 Abusing the notation slightly, we write $R_{\text{eff}}(x,y)$ for $R_{\text{eff}}(\{x\},\{y\})$. Show that $(x,y) \mapsto R_{\text{eff}}(x,y)$ defines a *metric distance* on G, e.g., a non-negative function which is symmetric,

$$R_{\text{eff}}(x,y) = R_{\text{eff}}(y,x), \qquad x,y \in V, \tag{2.74}$$

and obeys the triangle inequality,

$$R_{\text{eff}}(x,y) \le R_{\text{eff}}(x,z) + R_{\text{eff}}(y,z), \qquad x,y,z \in V. \tag{2.75}$$

As it turns out, the most important instance of effective resistance $R_{\text{eff}}(x,y)$ is when one of the points is "at infinity." The precise definition is as follows:

Definition 2.49 [Resistance to infinity] *Consider an infinite resistor network and let B_R be a sequence of balls of radius R centered at a designated point* 0. *The* resistance $R_{\text{eff}}(x,\infty)$ from x to ∞ *is then defined by the monotone limit*

$$R_{\text{eff}}(x,\infty) = \lim_{R\to\infty} R_{\text{eff}}\bigl(\{x\}, B_R^{\mathrm{c}}\bigr). \tag{2.76}$$

Exercise 2.50 Show that the value of $R_{\text{eff}}(x,\infty)$ does not depend on the choice of the designated point 0.

Problem 2.51 Consider the tree as in Fig. 2.9. Use the aforementioned facts about effective resistance to show that, for any vertex v, we have $R_{\text{eff}}(v,\infty) < \infty$.

Apart from monotonicity, the resistor networks have the convenient property that certain parts of the network can be modified without changing effective resistances between sets non-intersecting the modified part. The most well known examples of these are the *parallel* and *serial* laws.

For the sake of stating these laws without annoying provisos, we will temporarily assume that vertices of G may have multiple edges between them each of which has its own private conductance. In graph theory, this means that we allow G to be an unoriented *multigraph*. The parallel and serial laws tell us how to reinterpret such networks back in terms of graphs.

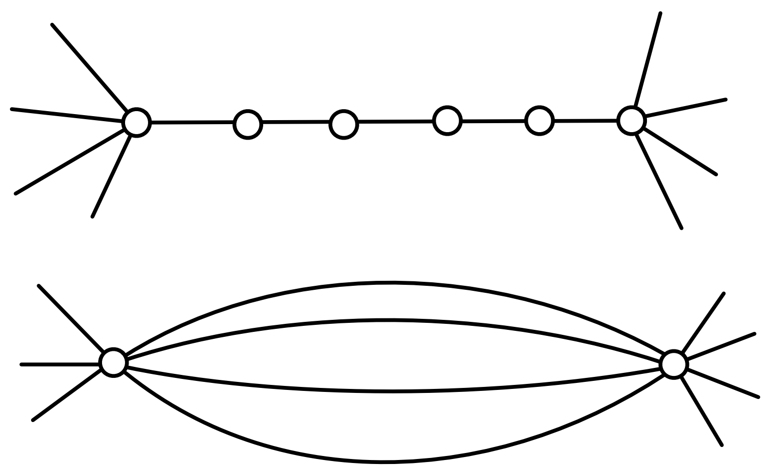

Figure 2.10: The setting for the application of the serial law (top) and parallel law (bottom). In the top picture the sequence of nodes is replaced by a single link whose resistance is the sum of the individual resistances. In the bottom picture, the cluster of parallel links can be replaced by a single link whose conductatance is the sum of the individual conductances.

Lemma 2.52 [Serial Law] *Suppose a resistor network contains a sequence of incident edges* $e_1, \dots, e_\ell$ *of the form* $e_j = (x_{j-1}, x_j)$ *such that the vertices* x_j, $j = 1, \dots, \ell - 1$, *are all of degree 2. Then the effective resistance* $R_{\text{eff}}(A, B)$ *between any sets* A, B *not containing* $x_1, \dots, x_{\ell-1}$ *does not change if we replace the edges* $e_1, \dots, e_\ell$ *by a single edge* e *with resistance*

$$r_e = r_{e_1} + \dots + r_{e_\ell} \tag{2.77}$$

Lemma 2.53 [Parallel Law] *Suppose two vertices* x, y *have multiple edges* $e_1, \dots, e_n$ *between them. Then the effective resistance* $R_{\text{eff}}(A, B)$ *between any sets* A, B *does not change if we replace these by a single edge* e *with conductance*

$$c_e = c_{e_1} + \dots + c_{e_n} \tag{2.78}$$

Problem 2.54 Prove the parallel and serial laws.

2.5 Random walks on resistor networks

To demonstrate the utility of resistor networks for the study of random walks, we will now define a *random walk on a resistor network*. Strictly speaking, this will *not* be a random walk in the sense of Definition 2.1 because resistor networks generally do not have any underlying (let alone Euclidean) geometry. However, the definition will fall into the class of *Markov chains* that are natural generalizations of random walks to non-geometric setting.

Definition 2.55 [Random walk on resistor network] *Suppose we have a resistor network — i.e., a connected graph* $G = (V, E)$ *and a collection of conductances* c_e, $e \in E$.

A random walk *on this network is a collection of random variables $Z_0, Z_1, \dots$ such that for all $n \geq 1$ and all $z_1, \dots, z_n \in V$,*

$$\mathbb{P}(Z_1 = z_1, \dots, Z_n = z_n) = \mathsf{P}(z, z_1)\mathsf{P}(z_1, z_2) \cdots \mathsf{P}(z_{n-1}, z_n) \tag{2.79}$$

where $\mathsf{P}\colon V \times V \to [0,1]$ is a symmetric matrix defined by

$$\mathsf{P}(x,y) = \frac{c_{xy}}{\pi(x)} \quad \text{with} \quad \pi(x) = \sum_{y \in V} c_{xy}. \tag{2.80}$$

We say that the random walk starts at z if

$$\mathbb{P}(Z_0 = z) = 1. \tag{2.81}$$

To mark the initial condition explicitly, we will write $\mathbb{P}^z$ for the distribution of the walks subject to the initial condition (2.81).

Example 2.56 Any *symmetric* random walk on $\mathbb{Z}^d$ is a random walk on the resistor network with nodes $\mathbb{Z}^d$ and an edge between any pair of vertices that can be reached in one step of the random walk. Indeed, if $X_1, X_2, \dots$ denote the steps of the random walk (S_n) with $S_0 = z$, then

$$\mathbb{P}^z(S_1 = z_1, \dots, S_n = z_n) = \mathbb{P}(X_1 = z_1 - z) \cdots \mathbb{P}(X_n = z_n - z_{n-1}). \tag{2.82}$$

To see that this is of the form (2.79–2.80), we define the conductance c_{xy} by

$$c_{xy} = \mathbb{P}(X_1 = y - x) \tag{2.83}$$

and note that symmetry of the step distribution implies $c_{xy} = c_{yx}$ while the normalization gives $\pi(x) = 1$.

The symmetry assumption is crucial for having $\mathsf{P}(x,y)$ of the form (2.80). If one is content with just the *Markov property* (2.79), then any random walk on $\mathbb{Z}^d$ will do. The simplest example of a symmetric random walk is the simple random walk, which just chooses a neighbor at random and passes to it. This "dynamical rule" generalizes to arbitrary graphs:

Example 2.57 *Random walk on a graph*: Consider a locally finite unoriented graph G and let $d(x)$ denote the degree of vertex x. Define

$$c_{xy} = 1, \qquad (x,y) \in E. \tag{2.84}$$

This defines a resistor network; the random walk on this network is often referred to as *random walk on G* because the probability to jump from x to neighbor y is

$$\mathsf{P}(x,y) = \frac{1}{d(x)}, \qquad (x,y) \in E, \tag{2.85}$$

which corresponds to choosing a neighbor at random. In this case $\pi(x) = d(x)$.

We proceed by a characterization of the electrostatic problems for resistor networks by means of the random walk $Z_0, Z_2, \dots$ on the network:

Lemma 2.58 [Dirichlet problem] *Let $W \subset V$ be a finite subset of the resistor network and let*

$$\tau_{W^c} = \inf\{n \ge 0 \colon Z_n \in W^c\} \tag{2.86}$$

be the first time the walk (Z_n) visits W^c. Then

$$\mathbb{P}^z(\tau_{W^c} < \infty) = 1, \qquad z \in V. \tag{2.87}$$

In addition, if $u_0 \colon W^c \to \mathbb{R}$ is a bounded function then

$$u(x) = \mathbb{E}^x\big(u_0(X_{\tau_{W^c}})\big), \qquad x \in V, \tag{2.88}$$

defines the unique function $u \colon V \to \mathbb{R}$ that is harmonic on W with respect to $\mathcal{L}$ defined in (2.62) *and that coincides with u_0 on W^c.*

Proof. Let u be given by (2.88). We clearly have $\mathbb{P}^z(\tau_{W^c} = 0) = 1$ when $z \in W^c$ and so $u = u_0$ outside W. Since there is only one harmonic function for each boundary condition u_0, we just need to show that u defined above is harmonic on W with respect to $\mathcal{L}$. This is quite easy: If the walk started at $x \in W$, then $\tau_{W^c} \ge 1$. Fixing explicitly the value of Z_1 yields

$$u(x) = \sum_{y \in V} \mathbb{E}^x\big(u_0(X_{\tau_{W^c}})1_{\{Z_1 = y\}}\big). \tag{2.89}$$

Since the probability of each path factors into a product (2.79), we have

$$\mathbb{E}^x\big(u_0(X_{\tau_{W^c}})1_{\{Z_1=y\}}\big) = \mathsf{P}(x,y)\mathbb{E}^y\big(u_0(X_{\tau_{W^c}})\big) = \mathsf{P}(x,y)u(y) \tag{2.90}$$

and so

$$u(x) = \sum_{y \in V} \mathsf{P}(x,y)u(y). \tag{2.91}$$

In explicit terms,

$$\pi(x)u(x) = \sum_{y \in V} c_{xy}u(y). \tag{2.92}$$

But $\pi(x)$ is the sum of c_{xy} over all y and so we can write the difference of the right and left-hand side as $(\mathcal{L}u)(x) = 0$. □

The probabilistic interpretation of the solution allows us to rewrite the formula for effective resistance as follows:

Lemma 2.59 *Let $W \subset V$ be a finite set and let $x \in W$. Let T_x denote the first return time of the walk (Z_n) to x,*

$$T_x = \inf\{n \ge 1 \colon Z_n = x\}. \tag{2.93}$$

Then

$$R_{\text{eff}}\big(\{x\}, W^c\big)^{-1} = \pi(x)\mathbb{P}^x\big(T_x \ge \tau_{W^c}\big). \tag{2.94}$$

Proof. Consider the function

$$u(z) = \begin{cases} 1, & z = x, \\ \mathbb{P}^z\big(T_x < \tau_{W^c}\big), & x \in W \setminus \{x\}, \\ 0, & x \in W^c. \end{cases} \tag{2.95}$$

Then u is a solution to the Dirichlet problem in $W \setminus \{x\}$ with boundary condition $u = 1$ on $\{x\}$ and 0 on W^c. In particular,

$$R_{\text{eff}}\big(\{x\}, W^c\big)^{-1} = \mathcal{E}(u). \tag{2.96}$$

We thus have to show that $\mathcal{E}(u)$ equals the RHS of (2.94). For that we insert

$$\big[u(y) - u(z)\big]^2 = u(z)\big[u(y) - u(z)\big] + u(y)\big[u(z) - u(y)\big] \tag{2.97}$$

into the definition of $\mathcal{E}(u)$ to get

$$\mathcal{E}(u) = \sum_{z \in V} u(x) \sum_{y \in V} c_{yz}\big[u(y) - u(z)\big] = \sum_{z \in V} u(z)(\mathcal{L}u)(z), \tag{2.98}$$

where we used that $c_{yz} = c_{zy}$ to write the contribution of each term on the right of (2.97) using the same expression. But $\mathcal{L}u(z) = 0$ for $z \in W \setminus \{x\}$ and $u(z) = 0$ for $z \in W^c$. At $z = x$ we have $u(z) = 1$ and

$$\begin{aligned} (\mathcal{L}u)(x) &= \sum_{y \in V} c_{xy}\big[1 - \mathbb{P}^z(T_x < \tau_{W^c})\big] \\ &= \pi(x) - \pi(x) \sum_{y \in V} P(x,y)\mathbb{P}^y(T_x < \tau_{W^c}) \\ &= \pi(x) - \pi(x)\mathbb{P}^x(T_x < \tau_{W^c}) = \pi(x)\mathbb{P}^x(T_x \geq \tau_{W^c}) \end{aligned} \tag{2.99}$$

Plugging this in (2.98) yields $\mathcal{E}(u) = \pi(x)\mathbb{P}^x(T_x \geq \tau_{W^c})$. □

Theorem 2.60 [Effective resistance and recurrence vs transience] *Recall the notation* (2.93) *for* T_x. *Then*

$$R_{\text{eff}}(x, \infty) = \infty \quad \Leftrightarrow \quad \mathbb{P}^x(T_x < \infty) = 1 \tag{2.100}$$

and

$$R_{\text{eff}}(x, \infty) < \infty \quad \Leftrightarrow \quad \mathbb{P}^x(T_x = \infty) > 0. \tag{2.101}$$

Proof. It clearly suffices to prove only (2.101). By Lemma 2.59 and Exercise 2.50, for the sequence of balls B_R of radius R centered at any designated point,

$$\pi(x) \lim_{R \to \infty} \mathbb{P}^x(T_x > \tau_{B_R^c}) = R_{\text{eff}}(x, \infty)^{-1}. \tag{2.102}$$

But $\tau_{B_R^c} \geq R$ and so

$$\lim_{R \to \infty} \mathbb{P}^x(T_x > \tau_{B_R^c}) \leq \lim_{R \to \infty} \mathbb{P}^x(T_x \geq R) = \mathbb{P}^x(T_x = \infty) \tag{2.103}$$

On the other hand,
$$\mathbb{P}^x(T_x > \tau_{B_R^c}) \geq \mathbb{P}^x(T_x = \infty) \tag{2.104}$$
and so
$$\pi(x)\mathbb{P}^x(T_x = \infty) = R_{\text{eff}}(x, \infty)^{-1}. \tag{2.105}$$
As $\pi(x) > 0$, the claim now directly follows. □

Corollary 2.61 [Monotonicity of recurrence/transience] *Suppose that the random walk (Z_n) on a resistor network with conductances (c_{xy}) is recurrent. Then so is the random walk on the network with conductances $(\tilde{c}_{xy})$ provided $c_{xy} \leq \tilde{c}_{xy}$ for all $(x, y) \in E$. Similarly, if the random walk is transient for conductances (c_{xy}) then it is transient also for any conductances $(\tilde{c}_{xy})$ provided $\tilde{c}_{xy} \leq c_{xy}$ for all $(x, y) \in E$.*

Proof. This follows because the effective resistance, $R_{\text{eff}}(x, \infty)$, is a decreasing function of the conductances. □