THE BULLETIN OF SYMBOLIC LOGIC
Volume 5, Number 4, Dec. 1999

THE SYLLOGISM'S FINAL SOLUTION

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In 1883, while a student of C. S. Peirce at Johns Hopkins University, Christine Ladd-Franklin published a paper titled *On the Algebra of Logic*, in which she develops an elegant and powerful test for the validity of syllogisms that constitutes the most significant advance in syllogistic logic in two thousand years.[1] Sadly, her work has been all but forgotten by logicians and historians of logic. Ladd-Franklin's achievement has been overlooked, partly because it has been overshadowed by the work of other logicians of the nineteenth century renaissance in logic, but probably also because she was a woman.[2] Though neglected, the significance of her contribution to the field of symbolic logic has not been diminished by subsequent achievements of others.

In this paper, I bring to light the important work of Ladd-Franklin so that she is justly credited with having solved a problem over two millennia old. First, I give a brief survey of the history of syllogistic logic. In the second section, I discuss the logical systems called "algebras of logic". I then outline Ladd-Franklin's algebra of logic, discussing how it differs from others, and explain her test for the validity of the syllogism, both in her symbolic language and the more familiar language of modern logic. Finally,

Received April 30, 1998; revised July 23, 1999.

I am indebted to my dear friend, George Boolos, for encouraging me to read Christine Ladd-Franklin's work. I am grateful to the Bunting Institute of Radcliffe College and the American Association of University Women for funding during the 1997–1998 academic year. Thanks also to an anonymous referee for helpful comments on an earlier version of this paper.

[1] In an article on Ladd-Franklin, Eugene Shen [21] writes:

> ... a propos of [Ladd-Franklin's antilogism] the late Josiah Royce of Harvard was in the habit of saying to his classes: "there is no reason why this should not be accepted as the definitive solution of the problem of the reduction of syllogisms. It is rather remarkable that the crowning activity in a field worked over since the days of Aristotle should be the achievement of an American woman."

[2] In [8] it is noted that despite the fact that Ladd-Franklin had completed all the requirements for the Ph.D. at Johns Hopkins University by 1882, since the university did not admit women, she was not awarded the degree until 1926.

1079-8986/99/0504-0002/$2.90

I present a rigorous proof of her theorem. Ladd-Franklin developed her algebra of logic before the methods necessary for a rigorous proof were available to her. Thus, I do now what she could not have done then.

§1. Aristotle defined syllogism as:

> a discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so ... I mean by the last phrase that they produce the consequence, and by this, that no further term is required from without to make the consequence necessary.[3]

Although this definition does not distinguish the syllogism from other forms of inference, Aristotle's analysis focused on a specific kind of argument that involves what he called the categorical statements. There are four forms of such statements. Each consists of a quantifier, a subject term, and a predicate term. They can be represented as follows:

All *A* is *B*	Universal affirmative
No *A* is *B*	Universal negative
Some *A* is *B*	Particular affirmative
Some *A* is not *B*	Particular negative

Interestingly, Aristotle treated the syllogism as a conditional statement, the antecedent of which is a conjunction of two categoricals and the consequent, a third categorical.[4] I follow tradition and take a syllogism to be a set of three categorical statements, two of which are the premises and the third, the conclusion of the syllogism. A syllogism is valid when the conclusion follows necessarily from the premises. Thus, a valid syllogism may have false premises and a true conclusion, or false premises and a false conclusion, but it may not have true premises and a false conclusion. The classic example of a valid syllogism is:

All men are mortal.
All Greeks are men.
Therefore, all Greeks are mortal.

The problem that Aristotle posed and attempted to solve is to give a general characterization of the valid syllogisms. This is the problem that Ladd-Franklin finally solved in the late nineteenth century.

Aristotle recognized that the validity of a syllogistic argument is determined by the forms of the categorical statements that constitute its premises and conclusion, and has nothing to do with the particular subject and predicate terms of those statements. He noted that a syllogism has three *terms*,

[3] *Prior Analytics*: i.I (c).

[4] There is some controversy about this interpretation of Aristotelian logic. Łukasiewicz [17] reads Aristotle in this way, but Lear [15] does not.

each of which occurs in two of the statements, and also distinguished three *figures*, or arrangements of the terms in the three statements. They may be represented by the following patterns:

$$\begin{array}{ccc} \text{1st} & \text{2nd} & \text{3rd} \\ MP & MP & PM \\ \underline{SM} & \underline{MS} & \underline{SM} \\ SP & SP & SP. \end{array}$$

The predicate term of the conclusion, 'P', is called the *major* term, the conclusion's subject term, 'S', the *minor term*, and the term common to the two premises, 'M', the *middle term*. Aristotle observed that a categorical statement can follow from two others in which one links the major term to the middle term and the other links the middle term to the minor term. The middle term thus "mediates" the inference. The example above, then, is a first figure syllogism where *mortal* is the major term, *Greek*, the minor, and *man*, the middle, or mediating, term.

Aristotle believed that the three figures enabled him to look systematically at pairs of categoricals in order to determine which give rise to valid inferences. Working on the assumption that the universal categoricals have "existential import",[5] that is, that they entail their corresponding particular statements, Aristotle described fourteen valid forms of syllogistic arguments, four of the first figure, four of the second, and six of the third. He considered the first figure to be "perfect", obviously yielding valid inferences, and believed that every valid syllogism can be "reduced" to a logically equivalent syllogism of the first figure. Syllogisms of the first figure are chains of inclusions, or predications. In the example above, 'mortal' is predicated of men, and 'man' in turn is predicated of Greeks. Aristotle accordingly attempted to formulate rules for converting syllogisms of the second and third figures to those of the first, thereby demonstrating their validity. He reduced syllogisms either "directly" or "indirectly". Roughly, direct reduction of one syllogistic form to another involves showing that the conclusion of the first, or its equivalent, follows from premises of a syllogism of the second form. Indirect reduction of one form to another involves giving a proof by *reductio*, using a syllogism of the second syllogistic form, that the premises of the first are inconsistent with the contradictory of its conclusion. Though he did

[5]Note that the Aristotelian "square of opposition" and, in particular, the doctrine that universals entail their "subalterns" gives rise to some difficulties if we allow general terms with empty extensions. Kneale and Kneale write, in [11], that "the notion of existential import was introduced as something that was required especially for the understanding of universal statements within the Aristotelian scheme." It has been suggested that Aristotle assumed that all of the general terms used in his scheme had non-empty extensions, but this, of course, seems unnecessarily restrictive to the modern logician. See Kneale and Kneale [11] for an enlightening discussion of this issue. We will see that with the introduction of the null class by the algebraic treatment of the categoricals, these problems do not arise.

not succeed in providing a unified and complete treatment of the syllogistic argument, it is significant that Aristotle attempted to construct a general principle of the syllogism, for it indicates that he had an an intuition shared by subsequent logicians that all valid syllogisms have something fundamental in common. Ladd-Franklin manages to capture this common feature in her logical system two millennia later.

Followers of Aristotle made many attempts to refine, simplify, and reconstruct his theory. In fact, study of the syllogism remained a focus of logicians through the middle of the nineteenth century, when significant advances were made toward today's mathematical logic. Medieval logicians gave the valid syllogistic forms Latin mnemonic names, such as *Barbara*, ẽxtitCelarent, and *Darii*, and developed a method for coding the various forms. The four categoricals were named A, E, I, and O, for the universal affirmative, the universal negative, the particular affirmative, and the particular negative, respectively, and a trio of these letters partially specified a syllogism's form by indicating its *mood*. For example, mood AAA indicates a syllogism in which the two premises and the conclusion are all universal affirmatives, such as the example above. As Aristotle observed, a syllogism's form can be further specified by indicating its *figure*, and his successors distinguished *four* possible patterns, or *figures*:

1st	2nd	3rd	4th
MP	MP	PM	PM
SM	MS	SM	MS
SP	SP	SP	SP

With the addition of the fourth figure, there are twenty four valid syllogistic forms, and fifteen, if "existential import" is not given to universals.

Logicians attempted to give necessary and sufficient conditions for a mood to yield a valid syllogism. Rules of the syllogism were developed for showing that certain moods do not give rise to valid syllogisms in various figures, but it was not shown, in any systematic way, that the remaining syllogistic forms are valid. Although the medieval logicians refined Aristotle's method for testing syllogisms for validity, they made little real progress toward Aristotle's goal of a general treatment of syllogistic reasoning.

The next significant advance in the development of logic came with Leibniz in the seventeenth century. Like Aristotle, he was solely concerned with subject-predicate sentences, yet he did not believe that all inferences could be put into syllogistic form. He wanted to extend Aristotelian logic to a logical calculus that could be used to analyze all inferences uniformly and he recognized and stressed an analogy between reasoning and calculating. Leibniz thought that logic required a precise language regulated by formal rules. Thus, he attempted to formulate a "universal characteristic", that is, a language of rigorously defined symbols that could be used to code various relations between statements, and he sought to create a logical calculus that gives the means both for constructing rigorous proofs of known truths and for discovering new truths. Though he did not ultimately succeed, his notion that logical inference could be understood by means of a formal language and algebraic methods foreshadows the advances of the nineteenth century logicians.

§**2.** Following a period of stagnation in the field, there was a revolution in logic in the nineteenth century that started with work by George Boole and Augustus De Morgan. De Morgan and others enlarged syllogistic logic to include different forms of inference while Boole developed some of the ideas of Leibniz and formulated the first algebraic language for logic. Boole's insight was that mathematical methods are applicable to the study of reasoning in general. He saw a parallel between algebraic operations and logical operations and developed the first algebra of logic. His theory was revised by De Morgan and Jevons, and then further by C. S. Peirce and Ernst Schröder. Ladd-Franklin's algebra of logic is a variation of the Boole-Peirce-Schröder system.[6] It is interesting that many of the developments in logic during the nineteenth century resulted from attempts to provide an accurate statement of the doctrine of the syllogism. Eventually, however, the syllogism was incorporated into the algebra and was no longer the focus of logical inquiry.

This algebra of logic was developed as an abstract system that admits various interpretations. The two interpretations of interest here yield a calculus of classes and a calculus of propositions. When the calculus is interpreted as a theory about classes, classes are treated in extension. The domain of all objects under consideration is called the *universe of discourse*, or the *universe*, and is symbolized as *1*.[7] The empty or null class is symbolized

[6]Although the Boolean algebra of logic and its role in the ultimate development of modern mathematical logic is of great significance, my primary interest in this paper is the syllogism and its final treatment by Ladd-Franklin.

[7]The notion of a universe of discourse was first introduced by De Morgan. In his *Formal Logic* [5] he writes:

> Let us take a pair of contrary names, as man and not-man. It is plain that between them they represent everything, imaginable or real, in the universe. But the contraries of common language embrace, not the whole universe, but some one general idea. Thus, of men, Briton and alien are contraries: every man must be one of the two, no man can be both .... The same may be said of integer and fraction among numbers, peer and commoner among subjects of a realm, male and female among animals, and so on. In order to express this let us say that the whole idea under consideration is the *universe* (meaning merely the whole of which we are considering parts) and let names which have nothing in common, but which between them contain the whole of the idea under consideration, be called contraries in, or with respect to, that universe.

as *0*. There are three operations, negation, logical addition, and logical multiplication, and two relations, inclusion and equality. Operations on classes yield classes, and relations on classes yield statements about classes.

The *negation* of a class is the class of everything in the universe that is not in the class negated. The negation of class a is represented as $-a$. The *logical sum* of classes a and b is represented as $a + b$ and has as members both the elements of a and the elements of b. The *logical product* product of a and b is represented as $a \times b$, or ab and contains those things that are in both a and b.[8]

The fundamental relation is *class inclusion*. That class a is included in class b is represented as $a < b$, which means that every member of a is a member of b. The relation $<$ is transitive and not symmetrical. Equality of two classes is defined as mutual inclusion. The equality of two classes a and b is represented as $a = b$ and means $a < b$ and $b < a$. The inequality of two classes a and b can be represented as $a \neq b$.

The postulates, or fundamental principles, of the system can be stated as follows,[9] where a, b, and c are any classes:

1.	Principle of Identity	$a < a$
2.	Principle of Contradiction	$a - a = 0$
3.	Principle of Excluded Middle	$a + -a = 1$
4.	Principle of Commutation	$ab = ba$ and $a + b = b + a$
5.	Principle of Association	$(ab)c = a(bc)$ and $(a + b) + c = a + (b + c)$
6.	Principle of Distribution	$(a + b)c = ac + bc$
7.	Principle of Tautology	$aa = a$ and $a + a = a$
8.	Principle of Absorption	$a + ab = a$ and $a(a + b) = a$
9.	Principle of Simplification	$ab < a$ and $a < a + b$
10.	Principle of Composition	If $a < b$ and $c < d$ then $(a + c) < (b + d)$ and If $a < b$ and $c < d$ then $ac < bd$
11.	Principle of the Syllogism	If $a < b$ and $b < c$ then $a < c$.

A second interpretation of the algebra takes $a, b, c, \ldots$ to be propositions and yields a theory about propositions. The relation of inclusion is taken to be material implication, and equality, the relation of logical equivalence. Multiplication is conjunction and addition is disjunction. The negation of a proposition is its contradictory. If a is a proposition, then $a = 1$ is taken to mean *a is true*. Similarly, $a = 0$ is interpreted as *a is false*.

[8]The contemporary terms for *logical product* and *logical sum* are *intersection* and *union*, respectively.

[9]These postulates follow the system outlined by Cohen and Nagel in [4] The system can be specified somewhat differently, but other formulations are equivalent. See also Baldwin's *Dictionary*, "Symbolic Logic, algebra of logic", written by Christine Ladd-Franklin. The expression of the categoricals also follows that of Cohen and Nagel in [4].

The four categorical statements can be expressed in the language of the algebra as follows:

A	All a's are b.	$a < b$	$a - b = 0$
E	No a's are b.	$a < -b$	$ab = 0$
I	Some a's are b.	$(a < -b)'$	$ab \neq 0$
O	Some a's are not b.	$(a < b)'$	$a - b \neq 0$.

The algebra clarifies various issues raised in traditional logic. It is obvious that the pairs, **A** and **O**, and **E** and **I**, are contradictories in this formulation. The **A** and **E** statements were traditionally taken to be "contraries", that is, they may both be false but cannot both be true. This holds in the algebra except in those cases where the subject of the two statements is the null class, since both are then true. Again, traditionally, the **I** was taken to follow from the **A** and the **O** from the **E**. But, of course, this fails when the subject of an **A** or **E** statement denotes the null class. It does hold when the subject denotes a non-empty class. In general, the traditional relationships between the four categorical statements hold in the algebra when the subjects of the statements denote non-empty classes. But when the subjects of the universals do denote the null class, the **A** and **E** are both true while the corresponding **I** and **O** are false. Since it is possible that a class is empty, the universals do not have existential import in the algebra.[10]

The algebra answers other questions concerning inferences between pairs of categoricals. The universal negative, *No a is b*, entails its simple converse, *No b is a*, for, given the commutativity of logical multiplication, $ab = 0$ is equivalent to $ba = 0$. Similarly, the particular affirmative is equivalent to its converse, since $ab \neq 0$ is equivalent to $ba \neq 0$, and furthermore, the particular negative can be "converted" for $a - b \neq 0$ is equivalent to $-ba \neq 0$. Although the "converse" of the universal affirmative, $a - b = 0$, was traditionally taken to be the corresponding particular affirmative with

[10]This, of course, differs from the traditional treatment. See page 408 of [11] where Kneale and Kneale comment:

> ... the introduction of the notions of the universe class and the null class involves an interesting novelty. Aristotle, as we have seen, confined his attention to general terms which were neither universal in the sense of applying to everything nor null in the sense of applying to nothing. When Boole wrote of the universe class and the null class, he made an important extension of the ordinary use of the word 'class'.

See also Baldwin's *Dictionary*, "Proposition", by Ladd-Franklin, in which she comments, in response to the question of whether universals have existential import:

> For the most part we should regard it as waste of time to speak much about things which do not exist, yet we can say *All disobedience is punished* without in the least asserting that disobedience ever occurs. But in formal logic, where terms have become *a* and *b* and we know nothing about the meaning of our concepts, it is necessary to adopt some fixed conventions in this matter ....

the terms transposed, $ba \neq 0$, this inference does not hold in the algebra, since universals have no existential import.[11]

This classical algebra provides a treatment of syllogistic reasoning. The premises of any syllogism concern three classes, corresponding to the major, minor, and middle terms of the propositions. A valid syllogism has as conclusion a categorical proposition that has no occurrence of the middle term. Thus, syllogistic reasoning can be seen as consisting in an "elimination" of the middle term. The formula used for this elimination can be derived from the postulates and definitions of the algebra:[12]

from

$$ax + b - x = 0,$$

it follows that

$$ab = 0.$$

This is clear because everything is either in x or its negation, and if nothing is either in the product of a and x or in the product of b and the negation of x, then there is nothing in the product of a and b.

If the conclusion of a syllogism is universal, one can combine the two premises into one equation, and eliminate the middle term in accordance with the elimination formula given above. Thus, for example, consider the two premises,

No a is b
All c is a,

which can be represented in the algebra as

$$ab = 0$$

and

$$c - a = 0,$$

respectively. Since

$$a + b = 0$$

can be shown to be equivalent to

$$a = 0 \text{ and } b = 0,$$

take the combination of the two premises,

$$ab + c - a = 0.$$

The result of eliminating a is

$$bc = 0,$$

[11] See Cohen and Nagel [4] p. 58 for a discussion of conversion.
[12] See Lewis [16] p. 152.

which is the algebraic representation of

No b is c.

The algebra treats syllogisms with particular conclusions differently. Such a syllogism will have one particular premise, which can be represented in the algebra as an inequality with 0, and one universal premise, which can be represented as an equality with 0. Consider, for example, the premises,

All b is a

and

Some b is c,

which can be represented as

$$b - a = 0$$

and

$$bc \neq 0,$$

respectively. The following is a theorem of the algebra:

$$bc = bc(a + -a) = bca + bc - a.$$

Thus, by introducing the term a into the second premise, we get,

$$bca + bc - a \neq 0$$

is true. Now, if the first premise,

$$b - a = 0,$$

is true, then so is

$$bc - a = 0.$$

Given the truth of both '$bca + bc - a \neq 0$' and

$$bc - a = 0$$

we can infer that

$$bca \neq 0$$

is true, too. From this follows the truth of

$$ca \neq 0,$$

which represents

Some c is a.

Thus, the syllogism,

All b is a
Some b is c
Therefore, some c is a

is valid.

The conclusion of any syllogism is either particular or universal, and these examples illustrate two corresponding algebraic methods for deriving the conclusion from the premises of a valid syllogism.[13]

Another treatment of syllogistic reasoning, developed by John Venn, is suggested by a third interpretation of the algebra of logic. This interpretation takes $a, b, c, \ldots$ to be regions in space. In this interpretation, the product of a and b is interpreted as the region common to regions a and b, and the sum of a and b is the set of all points that belong to either a or b. The formula, $a < b$, represents the proposition that region a is contained in region b. The null region is represented by 0 and the universal space is represented by 1. This interpretation leads to a diagrammatic method for testing the validity of syllogisms that remains well-known to logic students and is included in many contemporary texts on symbolic logic.[14]

§**3.** Ladd-Franklin's treatment of the syllogism differs from all of the foregoing. I initially detail her treatment of the syllogism using the notation of her variation of the algebra. Rather than the relation of inclusion, symbolized as $<$ in the algebra given above, the fundamental relation for Ladd-Franklin's system is that of *exclusion* symbolized by $\bar{\vee}$. The formula

$$a \vee b$$

represents

$$a \text{ is partly } b \quad (a \text{ is not wholly excluded from } b)$$

and its negation,

$$a \mathbin{\bar{\vee}} b$$

represents

$$a \text{ is excluded from } b.$$

These two relations can be used to express both inclusion and equality: $a < b$ is expressed as $a \mathbin{\bar{\vee}} -b$, and $a = b$ is expressed as $(a \mathbin{\bar{\vee}} -b)(-a \mathbin{\bar{\vee}} b)$.[15] The four categoricals, then, are expressed as follows:

A	All a is b	$a \mathbin{\bar{\vee}} -b$
E	No a is b	$a \mathbin{\bar{\vee}} b$
I	Some a is b	$a \vee b$
O	Some a is not b	$a \vee -b$.

[13]See Lewis [16] pp. 194–195. He asserts, without proof, that all valid syllogisms can be treated in this way.

[14]W. V. Quine [19], for instance, endorses this method for testing the validity of syllogisms.

[15]Note that in Ladd-Franklin's system, symbols are used in various ways, and at times in the same formula. Here logical multiplication is used to conjoin propositions. We will see below that she uses her symbol '*vēe*' in two ways in her principle of the syllogism, for when a and b are classes, $a \mathbin{\bar{\vee}} b$ means no a is b, and when a and b are propositions, $a \mathbin{\bar{\vee}} b$ means *a and b are inconsistent*.

Note that in Ladd-Franklin's system, universals are expressed with the negated copula and the particulars, with the positive. Also, unlike the relation of inclusion, Ladd-Franklin's fundamental relation is symmetrical. Thus, the algebra eliminates the formal distinction between the subject and predicate of propositions, since the proposition, $av\bar{e}eb$ is equivalent to $bv\bar{e}ea$.

If a and b are taken to be propositions, then $a \vee b$ is interpreted as meaning *a and b are consistent*, and $a \mathbin{\bar{\vee}} b$ means *a and b are inconsistent*. Ladd-Franklin uses the symbol ∞ where we have used 1 above. Thus, in the class calculus, $x \mathbin{\bar{\vee}} \infty$ is interpreted as *class x is empty* and $x \vee \infty$ is interpreted as *x is not empty*. Further, she abbreviates $x \vee \infty$ as $x\vee$, and $ab \vee \infty$ as $ab\vee$. Hence, the system allows that the copula $\vee$ or $\bar{\vee}$ to be moved through its terms without changing meaning. Thus, $a \vee b$ is equivalent to $ab\vee$ and $a \mathbin{\bar{\vee}} b$ is equivalent to $ab\bar{\vee}$. Note then that in the propositional interpretation, $ab\bar{\vee}$ means *propositions a and b are inconsistent*.

Ladd-Franklin gives the following formula, which she interprets as a statement of inconsistency between three propositions:

$$\text{(I)} \qquad (a \mathbin{\bar{\vee}} b)(c \mathbin{\bar{\vee}} d) \mathbin{\bar{\vee}} (ac \vee b + d).$$

In other words, if no a is b and no c is d, then it cannot be the case that some things that are both a and c are either b or d. This is clearly true, for it is not possible that an object that is common to two classes should have some quality that is excluded from one of them. She writes that (I)

> is the most general form of that mode of reasoning in which a conclusion is drawn from two premises, by throwing away part of the information which they convey and uniting in one proposition that part which it is desired to retain. It will be shown that it includes syllogism as a particular case. The essential character of the syllogism is that it effects the elimination of a middle term, and in this argument there is no middle term to be eliminated.[16]

Here I take her to mean that (I) is what we now call a *schema* and that a, b, c, ... can be treated as variables that may be replaced by terms of any complexity, for she argues that inconsistency, (I), holds, regardless of the number of terms involved. Given exclusions with any number of terms, we can construct a corresponding proposition with which they are inconsistent by taking any number of terms out of each exclusion, forming their logical sum and uniting it with the product of the remaining terms. For instance, given the two exclusions

$$abc\bar{\vee}, \qquad def\bar{\vee}$$

we can construct sixteen different propositions that also cannot be true by taking none, one, two, or three of the terms from each exclusion and forming

[16] Ladd-Franklin [13] p. 33.

the sum of those terms with none, one, two, or three of the terms from the other conclusion. Thus,

$$\begin{aligned}
&abcdef\\
&(a+d) \vee bcef\\
&(b+d) \vee ace\\
&\dots\\
&abc + def.
\end{aligned}$$

This result leads to Ladd-Franklin's treatment of elimination in syllogistic reasoning, for (I) yields the elimination formula of the classical algebra given above. When d is $-b$, (I) becomes

$$\text{(II)} \qquad (a \mathbin{\bar{\vee}} b)(c \mathbin{\bar{\vee}} -b)(ac\vee) \mathbin{\bar{\vee}}\,.$$

The third proposition, $ac\vee$, lacks the term, b, which is common to the first two. Ladd-Franklin observes that since (II) states an inconsistency between three propositions, it yields three valid arguments, for any two of the statements jointly entail the negation of the third. She also points out that in the standard classical treatment, there are two distinct forms of elimination, as we saw above, one for syllogisms with two universal premises and one for those with one universal premise and one particular premise. When the three terms, a, b, c, are complex, these two forms are the sole means the algebra provides for elimination of the third "middle" term. Yet Ladd-Franklin claims that when the terms are simple, her formula provides a single form to which all valid syllogisms can be reduced, regardless of their form. Hence it provides a simple test for the validity of all syllogisms. She states her Rule of Syllogism as follows:

> Take the contradictory of the conclusion, and see that universal propositions are expressed with a negative copula and particular propositions with an affirmative copula. If two of the propositions are universal and the other particular, and if that term only which is common to the two universal propositions has unlike signs, then, and only then, the syllogism is valid.[17]

This, then, is Ladd-Franklin's test. Given any syllogism, it can be checked to determine if its corresponding "triad" has the described form. Thus, given any valid syllogism, its "triad", or the set that contains the premises and the negation of the conclusion, is an inconsistent set of propositions. For, given that any two of the three are true, the other must be false. Hence, such a set actually yields three valid syllogisms.

[17] *Ibid*, p. 41.

As an illustration, consider, again, the valid syllogism,

All men are mortal
All Greeks are men
Therefore, all Greeks are mortal.

The corresponding triad is

All men are mortal
All Greeks are men
Not all Greeks are mortal,

which clearly form an inconsistent set. An inconsistent triad gives rise to three different valid syllogisms. In this case, in addition to the one above, we get both

All Greeks are men
Not all Greeks are mortal
Therefore, not all men are mortal,

and

All men are mortal
Not all Greeks are mortal
Therefore, not all Greeks are men.

Returning to Ladd-Franklin's algebra of logic, the triad that results from taking the premises along with the negation of the conclusion of a syllogism will conform to her formula,

$$\text{(II)} \qquad (a \mathbin{\bar{\veebar}} b)(-b \mathbin{\bar{\veebar}} c)(c \vee a) \mathbin{\bar{\veebar}}$$

if and only is the syllogism is valid. That is, in the resulting triad, two of the propositions are universal, one is particular, and the term common to the universal propositions is negated in one proposition and not in the other, and each of the other two terms has the same sign at both of its occurrences.

This elegant treatment of the syllogism succeeds in uncovering a general feature shared by all valid syllogisms. Aristotle thought that all valid syllogisms were reducible to what he took to be the perfect form of a chain of inclusions, yet Ladd-Franklin's treatment captures something different.

§**4.** Although she presents her *Rule of Syllogism* as a theorem, Ladd-Franklin does not give a rigorous proof of the correctness of her result. She is claiming that *all* inconsistent triads, or "antilogisms", as she called them, share a certain form yet the results necessary for a proof were unknown at the time she did this work. Although it is obvious that all triads with the form she describes are inconsistent, it is not at all obvious that every inconsistent triad has that form. It has been unrecognized by those who have written about Ladd-Franklin's work that not only did she give no proof of her theorem, but she could not have done so. Moreover, those who have mentioned a proof seem unaware that it is not a trivial one.[18]

To offer the proof of her result, I switch now to the more familiar language of modern logic. Let A, B, and C be three different predicates. If 'W' and 'V' are any predicates, let

$$\text{'}W^{*}\text{' be '}\neg W\text{'}$$
$$\text{'}WV\text{' be '}W \;\&\; V\text{'}$$
$$\text{'}W_i\text{' be '}W\text{' or '}W^{*}\text{'.}$$

Let quantifier, 'Q_i', be '$\exists$' or '$\neg\exists$'. We identify

$$\text{'}W\text{' and '}W^{**}\text{',}$$
$$\text{'}WV\text{' and '}VW\text{',}$$

and

$$\text{'}\exists\text{' and '}\neg\neg\exists\text{'.}$$

A *triad* is defined as a set of statements of the form:

$$\{Q_1A_1B_1, Q_2A_2C_2, Q_3B_3C_3\}.$$

Note that if

$$\begin{array}{c} Q_1M_1P_1 \\ \underline{Q_2S_2M_2} \\ Q_3S_3P_3 \end{array}$$

is a syllogism, then $\{Q_1M_1P_1, Q_2S_2M_2, \neg Q_3S_3P_3\}$ is a triad. Ladd-Franklin observes that the syllogism is valid if and only if the triad is inconsistent. Her work gives rise to the question: Under what conditions is a triad consistent? Her response may be expressed as:

Christine Ladd-Franklin's Theorem. *Let T be the triad*

$$\{Q_1A_1B_1, Q_2A_2C_2, Q_3B_3C_3\}.$$

---

[18]C. I. Lewis [16] offers no proof, but writes of Ladd-Franklin's rule, "A few experiments will make this matter clear to the reader".

R. M. Sabre [20] asserts that Ladd-Franklin "proves" her result (and he takes her first name to be 'Elizabeth'!).

Another author, R. H. Dotterer [6] claims to give a proof that "every group of propositions which is constructed according to these four rules is an antilogism", but he only proves sufficiency. In a later reply to a critic [7] he emends his formulation of the rules and then states that "the rules proposed can be proved necessary as well as sufficient". He fails, however, to provide the proof of necessity.

E. Shen [21] writes, "The derivation of the formula is very simple."

Cohen and Nagel [4] do not attribute a proof to Ladd-Franklin, nor do they give one themselves, but they do misrepresent the case when they write, "It can be shown without difficulty that these three conditions are present in every antilogism, and the reader should not hesitate to prove that this is so."

T is inconsistent if and only if for some predicates, 'X', 'Y', 'Z',

$$T = \{\exists XY, \neg\exists XZ, \neg\exists YZ^*\}.$$

PROOF. Sufficiency is obvious , for it is clear that any triad with the specified form is inconsistent. For necessity, suppose that T is inconsistent. Note that given three predicates, there are eight types of objects that may be represented as follows:

$$\begin{array}{l} ABC \\ ABC^* \\ AB^*C \\ \ldots \\ \ldots \\ A^*B^*C^*. \end{array}$$

First, we show that two members of T are universal and one is particular. It is clear that *at least one statement in T must be particular*. That is, one statement in T must begin with $\exists$. For suppose that

$$Q_1 = Q_2 = Q_3 = \neg\exists,$$

in which case, T is $\{\neg\exists A_1B_1, \neg\exists A_2C_2, \neg\exists B_3C_3\}$. Then T is consistent, for the statements of T are all true under an interpretation in which the universe of discourse contains just one object of type $A_1^*B_3^*C_2^*$. Moreover, *at least one statement in T must be universal*. That is, one statement must begin with $\neg\exists$. For suppose that

$$Q_1 = Q_2 = Q_3 = \exists$$

in which case, T is $\{\exists A_1B_1, \exists A_2C_2, \exists B_3C_3\}$. But then it is clear that T is a consistent set, for each member of the set is true in any interpretation in which the universe of discourse contains objects of types $A_1B_1C_2$, $A_2B_3C_2$, and $A_1B_3C_3$.

Finally, *at most one member of T is particular*. That is, at most one member of T begins with $\exists$. For suppose that T is $\{\exists A_1B_1, \neg\exists AC, \exists B_3C_3\}$. Then each member of T is true in an interpretation in which the universe of discourse has just two members, one of type $A_1B_1C^*$ and one of type $A^*B_3C_3$. Therefore, *exactly one statement in T must be particular*, that is, begin with $\exists$.

Now, let the particular statement in T be '$\exists XY$', and let the other two statements in T be '$\neg\exists X_1Z_1$' and '$\neg\exists Y_2Z_2$'. We need to show that

(i) $X_1 = X$
(ii) $Y_2 = Y$

and

(iii) $Z_2 = Z^*$ and $Z_1 = Z$. (In other words, $Z_2^* = Z = Z_1$; $Z_1 \neq Z_2$.)

First, suppose that $X \neq X_1$. That is, suppose that $X_1 = X^*$. In that case, $T = \{\exists XY, \neg\exists X^* Z_1, \neg\exists Y_2 Z_2\}$. But then each member of T is true in an interpretation with a universe of discourse with just one member of type XYZ_2^*.

Second, suppose that $Y_2 = Y^*$. Hence, $T = \{\exists XY, \neg\exists XZ_1, \neg\exists Y^* Z_2\}$. But then each member of T is true in an interpretation with a universe of discourse with one member of type XYZ_1^*.

We have now shown that $T = \{\exists XY, \neg\exists XZ_1, \neg\exists YZ_2\}$. We must now show that $Z_1 \neq Z_2$. Suppose that $Z_1 = Z_2$. It follows that $T = \{\exists XY, \neg\exists XZ_1, \neg\exists YZ_1\}$. But then T is consistent, for each member of T is true in an interpretation with one member of type XYZ_1^*. Therefore, $Z_1 \neq Z_2$ and $Z_1 = Z_2^*$. We may take Z_1 to be Z. Now, we have shown that if T is inconsistent, $T = \{\exists XY, \neg\exists XZ, \neg\exists YZ^*\}$. $\dashv$

The notions of consistency and inconsistency were used by Ladd-Franklin and her contemporaries. She writes that "an inconsistency ... simply denies the possible co-existence of two propositions."[19] For Ladd-Franklin, to say that a and b are inconsistent ($a \bar{\vee} b$, in her notation) is to say that if a is true, b is false and if b is true, a is false. To say that two propositions, a and b are consistent ($a \vee b$, in her notation) is to say that the truth of a does not imply that b is false and the truth of b does not imply that a is false. But we now define *inconsistency* and *consistency* of sets of statements in terms of *possible interpretations*. That is, a set of statements is inconsistent if and only if there is no possible interpretation under which each member of the set is true. The proof I have given above depends on demonstrating the consistency of triads by constructing interpretations with different domains in which the members of various triads of statements are all true. Neither the semantic notion of *truth under an interpretation* nor the idea of varying domains was developed until after the time of Ladd-Franklin's work.[20] Although de Morgan introduced the concept of *universe of discourse* in the middle of the nineteenth century, and Ladd-Franklin does discuss varying the "universe"[21], we have seen that their notion is different from the contemporary one.

[19] Ladd-Franklin [14] p. 532.

[20] In his "On possibilities in the calculus of relatives" (1915) in [22], Löwenheim discusses the validity of formulas of first-order predicate calculus in different domains. van Heijenoort writes that "... these topics had remained alien to the trend that had by then become dominant in logic, that of Frege-Peano-Russell." ([22] p. 228). van Heijenoort [22] also includes Alessandro Padoa's "Logical introduction to any deductive theory" (1900), which deals with the relation between a system and its interpretations, and writes in his preface to the Source Book that Padoa's paper is one of the first to deal with semantic questions.

[21] She defines *universe of discourse* as follows:

> It may be the universe of conceivable things, or of actual things, or any limited portion of either. It may include non-Euclidian n-dimensional space, or it may be limited to the surface of the earth, or to the field of a microscope. It may exclude things and be restricted to qualities, or it may be made coextensive with

It should be noted, then, that Ladd-Franklin had the insight to formulate the theorem I attribute to her here without having had the logical tools to prove it in a rigorous way. Since the number of syllogistic forms is finite, she could, of course, have demonstrated the invalidity of the invalid forms by giving appropriately constructed instances in ordinary language. In fact, she frequently shows the invalidity of an inference in this way. However, it is clear that she could not have proved her theorem without our notion of interpretation. She does observe the connection between the notions of *valid* and *inconsistent*, and demonstrates the validity of an argument by showing that its premises are inconsistent with the negation of its conclusion. And she seems to recognize the corresponding connection between *invalid* and *consistent* when she states that in a valid inference, that "the premise and denial of the conclusion cannot go together"[22] and then goes on to write that if the premise of an argument is consistent with the denial of the conclusion this means that "both the premise and the negative of the conclusion must, at some time, be true."[23] Here, her use of *at some time* seems to hint at the contemporary notion of *interpretation*.

Yet despite the limitations of her time, Ladd-Franklin succeeded in finally giving a treatment of the syllogism that captures the generality Aristotle sought. The traditional treatments of syllogistic reasoning take the subject and predicate of the categorical statements to be non-transposable. Thus, the various moods and rules resulted rather than the single general rule of the antilogism. However, as Ladd-Franklin's work shows, it is unnecessary to distinguish the various moods and figures of the syllogism. Her symmetrical negative *copula*, $\bar{\vee}$, enables us to reduce any valid syllogism to the single formula of the "antilogism". She writes,

> The view of logic which I have based upon the antilogism is that to make use of the syllogism is a great mistake when a so much better form of reasoning lies at hand.[24]

Elsewhere, she explains,

> Two premises and a conclusion taken together constitute a syllogism; the following three propositions taken together—
>
> 'None who are discontented are happy,'
> 'But some reformers are happy
> and no reformers are contented,'

fictions of any kind. In any proposition of formal logic, ∞ represents what is logically possible; in a material proposition it represents what exists. ([13] p. 19.)

[22] *Ibid*, p. 28.
[23] *Ibid*.
[24] Ladd-Franklin [14] p. 532.

> form an argument,—not it is true an argument in which there is a sequence but an argument in which there is a rebuttal. In this argument the implication contained in the word *but* is that the statements made cannot be all three true together; if the first and either of the others are true, the remaining one is not true. In other words, the three propositions taken together constitute an inconsistency, or incompatibility, or, as it may perhaps be called, to distinguish it from the syllogism, an antilogism.[25]

The antilogism highlights that the two premises of a valid syllogism are inconsistent with the negation of the conclusion, whereas in the traditional treatment of the syllogism the premises were viewed as entailing the conclusion. Any syllogism is thus easily tested for validity, and given two premises of any syllogism we can determine which, if any, conclusion follows by merely consulting Ladd-Franklin's rule. Moreover, any triad of the prescribed form, or antilogism, yields three valid syllogisms.

Ladd-Franklin points out that not only is the antilogism an elegant way to treat the syllogism, but it mirrors a "natural" form of argumentation—the rebuttal. She illustrates the rebuttal with various examples, some delightful:

> A little girl of four years of age was making, at her dinner, the interesting example of eating her soup with a fork. Her nurse said to her, "Nobody eats soup with a fork, Emily," and Emily replied, "But I do, and I am somebody."[26]

If we let 'f' stand for the class of those who eat soup with a fork, and 'e' stand for the class of those identical with Emily, then we can easily see that the argument conforms to the form of the antilogism hence that the statements are inconsistent:

$$(e \mathbin{\bar{\vee}} -f)(e \vee \infty) \mathbin{\bar{\vee}} (f \mathbin{\bar{\vee}} \infty).$$

Martin Gardner[27] captures a view shared by many contemporary logicians when he writes:

> We now know that Aristotle's syllogism is only one of an infinite variety of forms of inference, but within its own domain it does exactly what it is supposed to do.

He continues,

> Leibniz thought it was "one of the most beautiful inventions of the human spirit," and there is no reason why a logician today need disagree, even though he finds the syllogism's structure no longer a field for further exploration.

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[25] Baldwin's *Dictionary*, "Syllogism", by Ladd-Franklin.

[26] Ladd-Franklin [14] p. 532.

[27] Gardner [9] pp. 38–39.

Given Ladd-Franklin's lovely treatment of this ancient form of reasoning, we can easily agree.

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