THE BULLETIN OF SYMBOLIC LOGIC
Volume 4, Number 4, Dec. 1998

CUT ELIMINATION IN THE PRESENCE OF AXIOMS

SARA NEGRI AND JAN VON PLATO

For Oiva Ketonen's 85th birthday

Abstract. A way is found to add axioms to sequent calculi that maintains the eliminability of cut, through the representation of axioms as rules of inference of a suitable form. By this method, the structural analysis of proofs is extended from pure logic to free-variable theories, covering all classical theories, and a wide class of constructive theories. All results are proved for systems in which also the rules of weakening and contraction can be eliminated. Applications include a system of predicate logic with equality in which also cuts on the equality axioms are eliminated.

§1. Introduction. The use of sequent calculus for the analysis of proofs outside pure logic is considered rather problematic, as the addition of non-logical axioms usually destroys the eliminability of cut. A simple example is given by Girard [7, p. 125]: with $\Rightarrow$ standing for the sequent arrow, let the axioms have the forms $\Rightarrow A \supset B$ and $\Rightarrow A$. The sequent $\Rightarrow B$ is derived from these axioms by

$$\dfrac{\Rightarrow A \qquad \dfrac{\Rightarrow A \supset B \qquad \dfrac{A \Rightarrow A \quad B \Rightarrow B}{A, A \supset B \Rightarrow B}\,L\supset}{A \Rightarrow B}\,Cut}{\Rightarrow B}\,Cut$$

Inspection of sequent calculus rules shows that there is no cut-free derivation of $\Rightarrow B$, which leads Girard to conclude that "the *Hauptsatz* fails for systems with proper axioms." (Ibid.) More generally, the cut elimination theorem does not apply to sequent calculus derivations having premisses that are not logical axioms.

Another way of adding axioms, already used by Gentzen [5, sec. 1.4], is to add "mathematical basic sequents" which are (substitution instances of) sequents $P_1, \ldots, P_m \Rightarrow Q_1, \ldots, Q_n$. Here P_i, Q_j are atomic formulas (typically containing free parameters). By Gentzen's Hauptsatz, the use of the cut rule can be pushed into such basic sequents. Following this line of

Received January 24, 1998; revised September 30, 1998.

1079-8986/98/0404-0003/$2.80

thought in our example, let us take as basic sequents $P \Rightarrow Q$ and $\Rightarrow P$. It is to be expected that cut elimination will not extend to derivations with basic sequents as premisses: indeed, $\Rightarrow Q$ is derived by

$$\frac{\Rightarrow P \quad P \Rightarrow Q}{\Rightarrow Q}\, Cut$$

and there is no cut-free derivation of $\Rightarrow Q$ unless it was already assumed as a basic sequent.

A third way of adding axioms, first found in Gentzen's consistency proof of elementary arithmetic in [4, sec. IV.3], is to treat axioms as a context Γ, and to relativize all theorems into Γ, thus proving results of form $\Gamma \Rightarrow A$. Now the sequent calculus derivations have only logical axioms as premisses, and cut elimination applies. For our example, we derive $P, P \supset Q \Rightarrow Q$ without cut by

$$\frac{P \Rightarrow P \quad Q \Rightarrow Q}{P, P \supset Q \Rightarrow Q}\, L\supset$$

But structural analysis of proofs usually breaks down in this approach, as the main results of purely logical sequent calculus proof theory do not extend to sequents with a context.

We shall here propose a fourth way of adding axioms to sequent calculus, namely, in the form of *nonlogical rules of inference*. When formulated in a suitable way, cut elimination will not be lost by such addition. This was first realized in Negri [10] for the intuitionistic theories of apartness and order. By converting axioms into rules, it becomes possible to prove properties of systems by induction on the height of derivations.

Our method of extension by nonlogical rules works uniformly for systems based on classical logic. For constructive systems, there will be some special forms of axioms, notably $(P \supset Q) \supset R$, that cannot be treated through cut-free rules.

Gentzen's original subformula property is lost; instead, we have a subformula property stating that all formulas in a derivation are either subformulas of the endsequent or atomic formulas.

To give an idea of the method, consider again the above example. In general, axioms are converted into rules by inspecting the left rule that matches the logical form of the axiom. The rules added to the logical ones will have only atomic active formulas. With P and Q atomic formulas and C an arbitrary formula, $P \supset Q$ is rendered into a rule by stipulating that if $Q \Rightarrow C$, then $P \Rightarrow C$, and P is rendered into a rule by stipulating that if $P \Rightarrow C$, then $\Rightarrow C$:

$$\frac{Q \Rightarrow C}{P \Rightarrow C} \qquad \frac{P \Rightarrow C}{\Rightarrow C}$$

The sequent $\Rightarrow Q$ now has the cut-free derivation

$$\dfrac{\dfrac{Q \Rightarrow Q}{P \Rightarrow Q}}{\Rightarrow Q}$$

Observe that by putting Q and P for C in the two rules, the basic sequent $P \Rightarrow Q$ and the axiomatic sequents $\Rightarrow P \supset Q$ and $\Rightarrow P$ follow. But the other direction requires cuts, as is to be expected.

The rule of contraction permits to eliminate duplications of formulas in the antecedents of sequents, to conclude $A, \Gamma \Rightarrow C$ from $A, A, \Gamma \Rightarrow C$. This rule is often as "harmful" for structural proof analysis as the rule of cut, as it permits an unending bottom-up proof search. Contraction is sometimes a built-in feature of sequent calculi, through the treatment of contexts as sets, or else assumed as a rule. Here we shall treat contexts as multisets and prove applications of contraction to be eliminable.

§2. **From axioms to rules.** Using classical logic, we find that all free-variable axioms can be turned into rules of inference permitting cut elimination. The constructive case is more complicated, and we shall deal with it first.

2.1. Extension of constructive systems with nonlogical rules. We shall be using the intuitionistic multi-succedent sequent calculus for propositional logic of Dragalin [2]. We call it here **G3ipm**, in conformity with Troelstra and Schwichtenberg [12] whose notation we also follow, except that we use $\supset$ for implication. Atomic formulas will be denoted by $P, Q, R, \dots$, arbitrary formulas by $A, B, C, \dots$, and contexts (finite multisets of formulas) by Γ, $\Delta, \Theta, \dots$.

G3ipm

Axioms:

$P, \Gamma \Rightarrow \Delta, P \qquad\qquad \bot, \Gamma \Rightarrow \Delta$

Rules:

$$\frac{A, B, \Gamma \Rightarrow \Delta}{A\&B, \Gamma \Rightarrow \Delta}\,L\& \qquad \frac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A\&B}\,R\&$$

$$\frac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}\,L\vee \qquad \frac{\Gamma \Rightarrow \Delta, A, B}{\Gamma \Rightarrow \Delta, A \vee B}\,R\vee$$

$$\frac{A \supset B, \Gamma \Rightarrow A \quad B, \Gamma \Rightarrow \Delta}{A \supset B, \Gamma \Rightarrow \Delta}\,L\supset \qquad \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow \Delta, A \supset B}\,R\supset$$

Remarks. Axioms are to be considered rules with zero premisses. The first axiom is restricted to atomic formulas, the second instead applies to arbitrary formulas. It is essential that $\bot$ is not considered an atomic formula, but a zero-place logical operation. None of the usual structural rules of sequent calculus need be assumed in the above calculus. Exchange rules are absent due to properties of multisets, and the other structural rules, those of weakening, contraction, and cut, can be eliminated. The left implication rule has a repetition of the principal formula $A \supset B$ in the left premiss; this is a crucial ingredient in the proof of admissibility of contraction.

The rules of weakening, contraction and cut are formulated as follows:

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LW \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ RW \qquad \frac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ LC$$

$$\frac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}\ RC \qquad \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ Cut$$

In general, a rule with the sequents $S_1, \dots, S_n$ as premisses and the sequent S as conclusion is *admissible* if, whenever an instance of $S_1, \dots, S_n$ is derivable, the corresponding instance of S is derivable. The rule is *invertible* if each of $\frac{S}{S_i}$ is admissible. All rules of **G3ipm** except those for implication are invertible, but $R\supset$ is invertible in case Δ is empty. Finally, $L\supset$ is "semi-invertible": inference from $A \supset B, \Gamma \Rightarrow \Delta$ to the first premiss $A \supset B, \Gamma \Rightarrow A$ is not admissible, inference to the second premiss $B, \Gamma \Rightarrow \Delta$ instead is admissible. For more details, see [12].

In a proof of admissibility it is assumed that in a derivation there is only one instance of the rule in question, the last one. Such a proof establishes also the eliminability of the rule in any derivation, by induction on the number of occurrences of the rule. The inductive step consists in the application of admissibility to the topmost instance of the rule.

In adding nonlogical rules representing axioms, we will be guided by the following

PRINCIPLE 2.1. *In nonlogical rules, the premisses and conclusion are sequents that have atoms as active and principal formulas in the antecedent, and an arbitrary context in the succedent.*

The most general scheme corresponding to this principle is

$$\frac{Q_1, \Gamma \Rightarrow \Delta \quad \dots \quad Q_n, \Gamma \Rightarrow \Delta}{P_1, \dots, P_m, \Gamma, \Rightarrow \Delta}\ Reg$$

where Γ, Δ are arbitrary multisets and $P_1, \dots, P_m, Q_1, \dots, Q_n$ are fixed atoms, and the number of premisses n can be zero.

Once we have shown structural rules admissible, we can conclude that a rule admitting several atoms in the antecedents of the premisses reduces to as many rules with one atom, for example, the rule

$$\frac{Q_1, Q_2, \Gamma \Rightarrow \Delta \quad R, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta}$$

reduces to

$$\frac{Q_1, \Gamma \Rightarrow \Delta \quad R, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta} \qquad \frac{Q_2, \Gamma \Rightarrow \Delta \quad R, \Gamma \Rightarrow \Delta}{P, \Gamma \Rightarrow \Delta}$$

The second and third rule follow from the first by weakening of the left premiss. In the other direction, weakening $R, \Gamma \Rightarrow \Delta$ to $R, Q_2, \Gamma \Rightarrow \Delta$ we obtain the conclusion $P, Q_2, \Gamma \Rightarrow \Delta$ from $Q_1, Q_2, \Gamma \Rightarrow \Delta$ by the second rule, and weakening again $R, \Gamma \Rightarrow \Delta$ to $R, P, \Gamma \Rightarrow \Delta$, we obtain by the third rule $P, P, \Gamma \Rightarrow \Delta$ which contracts to $P, \Gamma \Rightarrow \Delta$. This argument generalizes, so we do not need to consider premisses with several atoms.

The full rule corresponds to the formula $P_1 \,\&\, \dots \,\&\, P_m \supset Q_1 \vee \dots \vee Q_n$. In order to better see what forms of axioms the rule-scheme covers, we write out a few cases, together with their corresponding axiomatic statements in Hilbert-type calculus. We also write a suggestive identifier for each rule. Omitting the contexts, the rules for $Q \,\&\, R$ (Et), $Q \vee R$ (Vel) and $P \supset Q$ (Si) become

$$\frac{Q \Rightarrow \Delta}{\Rightarrow \Delta}, \frac{R \Rightarrow \Delta}{\Rightarrow \Delta}\, {\scriptstyle Et} \qquad \frac{Q \Rightarrow \Delta \quad R \Rightarrow \Delta}{\Rightarrow \Delta}\, {\scriptstyle Vel} \qquad \frac{Q \Rightarrow \Delta}{P \Rightarrow \Delta}\, {\scriptstyle Si}$$

The rules for Q $(Atom)$, $\sim P$ (Non) and $\sim(P_1 \,\&\, P_2)$ $(Asym)$ are:

$$\frac{Q \Rightarrow \Delta}{\Rightarrow \Delta}\, {\scriptstyle Atom} \qquad \frac{}{P \Rightarrow \Delta}\, {\scriptstyle Non} \qquad \frac{}{P_1, P_2 \Rightarrow \Delta}\, {\scriptstyle Asym}$$

We define the class of *regular sequents* by the following

DEFINITION 2.2. A sequent is *regular* if it is of form

$$P_1, \dots, P_m \Rightarrow Q_1, \dots, Q_n, \perp, \dots, \perp$$

where the number of $\perp$'s, m and n can be 0, and $P_i \neq Q_j$ for all i, j.

Regular sequents are grouped into four types, each with a corresponding *formula trace*:

1. $m > 0, n > 0$, *trace* $P_1 \,\&\, \dots \,\&\, P_m \supset Q_1 \vee \dots \vee Q_n$,
2. $m = 0, n > 0$, *trace* $Q_1 \vee \dots \vee Q_n$,
3. $m > 0, n = 0$, *trace* $\sim(P_1 \,\&\, \dots \,\&\, P_m)$,
4. $m = 0, n = 0$, *trace* $\perp$.

Regular sequents are precisely the sequents that correspond to rules (lat. *regulae*) following our rule-scheme.

Given a sequent $\Rightarrow A$, we can perform a bottom-up decomposition by means of rules of **G3ipm**. If the decomposition terminates, we reach leaves that are either logical axioms or regular sequents. Among such leaves, we distinguish those that are reached from $\Rightarrow A$ by "invertible paths," i.e., paths that never pass via the left premiss of a $L\supset$-rule, nor via an instance of $R\supset$ with nonempty context Δ in its conclusion, and call them *invertible leaves*. The other leaves are called *noninvertible*.

We now define the class of *regular formulas*:

DEFINITION 2.3. A formula A is *regular* if it has a decomposition that leads to invertible leaves that are either logical axioms or regular sequents and noninvertible leaves that are logical axioms.

We observe that the invertible leaves in a decomposition of $\Rightarrow A$ are independent of the order of decomposition chosen, since any two rules among $L\&$, $R\&$, $L\vee$, $R\vee$, and $R\supset$ with empty right context Δ, commute with each other and each of them commutes with the right premiss of $L\supset$. This uniqueness justifies the following

DEFINITION 2.4. For a regular formula A, its *regular decomposition* is the set $\{A_1, \ldots, A_k\}$, where the A_i are the formula traces of the regular sequents among the invertible leaves of A. The *regular normal form* of a regular formula A is $A_1 \,\&\, \ldots \,\&\, A_k$.

Note that the regular decomposition of a regular formula A is unique, and A is equivalent to its regular normal form. Thus, regular formulas are those that permit a constructive version of a conjunctive normal form, one where each conjunct is an implication of form $P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$, instead of the classically equivalent disjunctive form $\sim P_1 \vee \ldots \vee \sim P_m \vee Q_1 \vee \ldots \vee Q_n$. The class of formulas constructively equivalent to usual conjunctive normal form is strictly smaller than the class of formulas having regular normal form. The following proposition shows some closure properties of the latter class of formulas:

PROPOSITION 2.5.

(i) *If A has no $\supset$, then A is regular,*
(ii) *If A, B are regular, then $A \,\&\, B$ is regular,*
(iii) *If A has no $\supset$ and B is regular, then $A \supset B$ is regular.*

PROOF. (i) By invertibility of the rules for $\&$ and $\vee$. (ii) Obvious. (iii) Starting with $R\supset$, a decomposition of $\Rightarrow A \supset B$ has invertible leaves of the form $P_1, \ldots, P_m, \Gamma \Rightarrow \Delta$, where $P_1, \ldots, P_m$ are atoms (from the decomposition of A) and $\Gamma \Rightarrow \Delta$ is either a logical axiom or a regular sequent. Thus also $P_1, \ldots, P_m, \Gamma \Rightarrow \Delta$ is either a logical axiom or a regular sequent. ⊣

From the two cases of noninvertible rules we see that typical formulas that need not be regular are disjunctions that contain an implication, and implications that contain an implication in the antecedent. But sometimes even these are regular, such as the formula $(A \supset B) \supset (A \supset C)$.

In the next section we show that the class of regular formulas consists precisely of the formulas the corresponding rules of which commute with the cut rule. The reason for adopting Principle 2.1 will then be clear.

2.2. Extension of classical systems with nonlogical rules. We use the classical multisuccedent sequent calculus **G3cp** in which all structural rules are built in. It is obtained from **G3ipm**, in the notation of the table above, by permitting in the $R\supset$ rule a premiss with succedent Δ, B, thus removing the intuitionistic restriction to one formula, and by having in the first premiss of the $L\supset$ rule Δ, A as succedent.

All rules of **G3cp** are invertible, but instead of analysing regularity of formulas through decomposability, we can use the existence of conjunctive normal form in classical propositional logic: each formula is equivalent to a conjunction of disjunctions of atoms and negations of atoms. Each conjunct can be converted into the classically equivalent form $P_1 \& \ldots \& P_m \supset Q_1 \vee \ldots \vee Q_n$ which is representable as a rule of inference. We therefore have

PROPOSITION 2.6. *All classical quantifier-free axioms can be represented by formulas in regular normal form.*

Thus, to every classical quantifier-free theory, there is a corresponding sequent calculus with structural rules admissible.

2.3. Conversion of axiom systems into systems with rules. Conversion of a Hilbert-type axiomatic system into a Gentzen-type sequent system proceeds, after quantifier-elimination, by first finding the regular decomposition of each axiom, and then converting each conjunct into a corresponding rule following Principle 2.1. Right contraction is unproblematic due to the arbitrary context Δ in the succedents of the rule scheme. In order to handle left contraction, we have to augment this scheme. So assume we have a derivation of $A, A, \Gamma \Rightarrow \Delta$, and assume the last rule is nonlogical. Then the derivation of $A, A, \Gamma \Rightarrow \Delta$ can be of three different forms. First, neither occurrence of A is principal in the rule; second, one is principal; third, both are principal. The first case is handled by a straightforward induction, and the second case by the method, familiar from the work of Kleene and exemplified by the $L\supset$ rule above, of repeating the principal formulas of the conclusion in the premisses. Thus, the general rule-scheme becomes

$$\frac{P_1, \ldots, P_m, Q_1, \Gamma \Rightarrow \Delta \quad \ldots \quad P_1, \ldots, P_m, Q_n, \Gamma \Rightarrow \Delta}{P_1, \ldots, P_m, \Gamma \Rightarrow \Delta} \; Reg$$

Here $P_1, \ldots, P_m$ in the conclusion are *principal* in the rule, and $P_1, \ldots, P_m$ and $Q_1, \ldots, Q_n$ in the premisses are *active* in the rule. Repetitions in the premisses will make left contractions commute with rules following the scheme. For the remaining case, with both occurrences of formula A principal in the last rule, consider the situation with a Hilbert-type axiomatization. We have some axiom, say $\sim(a < b \;\&\; b < a)$ in the theory of strict linear order, and substitution of b with a produces $\sim(a < a \;\&\; a < a)$ that we routinely abbreviate to $\sim a < a$, irreflexivity of strict linear order. This is in fact a contraction. For systems with rules, the case where a substitution produces two identical formulas that are both principal in a nonlogical rule, is taken care of by the

CLOSURE CONDITION 2.7. *Given a system with nonlogical rules, if it has a rule where a substitution instance in the atoms produces a rule of form*

$$\frac{P_1, \dots, P_{m-2}, P, P, Q_1, \Gamma \Rightarrow \Delta \quad \dots \quad P_1, \dots, P_{m-2}, P, P, Q_n, \Gamma \Rightarrow \Delta}{P_1, \dots, P_{m-2}, P, P, \Gamma \Rightarrow \Delta}\ \mathit{Reg}$$

then it also has to contain the rule

$$\frac{P_1, \dots, P_{m-2}, P, Q_1, \Gamma \Rightarrow \Delta \quad \dots \quad P_1, \dots, P_{m-2}, P, Q_n, \Gamma \Rightarrow \Delta}{P_1, \dots, P_{m-2}, P, \Gamma \Rightarrow \Delta}\ \mathit{Reg}$$

This condition is unproblematic, since the number of rules to be added to a given system of nonlogical rules is bounded. Often the closure condition is superfluous; for example, the rule expressing irreflexivity in the constructive theory of strict linear order is derivable from the other rules.

§3. Admissibility of cut for sequent systems with rules.

In this section we shall prove the admissibility of the structural rules of weakening, contraction and cut for extensions of logical systems with nonlogical rules of inference. We shall deal in detail with constructive systems, and just note that the proofs go through for classical systems with inessential modifications.

We shall denote by **G3ipm*** any extension of the system **G3ipm** with rules following our general rule-scheme. Starting from the proof of admissibility of structural rules for **G3ipm** in the style of Dragalin (see [2], also [3] for a more detailed exposition), we then prove admissibility of the structural rules for **G3ipm***. This proof is a generalization of the method, found in Negri [10], of extending admissibility of structural rules to the calculi of apartness and constructive order.

We say that an admissible rule is *height-preserving* if whenever the premisses are derivable with derivation of height $\leqslant n$ then the conclusion is derivable with the same bound on the derivation height.

PROPOSITION 3.1. *The rules of weakening*

$$\frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}\ \mathit{LW} \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}\ \mathit{RW}$$

are admissible and height preserving in **G3ipm***.

PROOF. For left weakening, since the two axioms and all rules have an arbitrary context in the antecedent, adding the weakening formula to the antecedent of each sequent will give a derivation of $A, \Gamma \Rightarrow \Delta$. For right weakening, adding the weakening formula to the succedents of all sequents that are not followed by an instance of the $R\supset$ rule will give a derivation of $\Gamma \Rightarrow \Delta, A$. ⊣

The proof of admissibility of contraction rules and the cut rule for **G3ipm** requires the use of *inversion lemmas* (see [12, p. 66]). We observe that all the inversion lemmas holding for **G3ipm** hold for **G3ipm*** as well. This is achieved by having only atomic formulas as principal in nonlogical rules, a property guaranteed by the restriction given in Principle 2.1.

THEOREM 3.2. *The rules of contraction*

$$\frac{\Gamma, A, A \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}\ LC \qquad \frac{\Gamma \Rightarrow A, A, \Delta}{\Gamma \Rightarrow A, \Delta}\ RC$$

are admissible and height-preserving in **G3ipm***.

PROOF. For left contraction, the proof is by induction on the height of the derivation of the premiss. If it is an axiom, the conclusion is also an axiom.

If A is not principal in the last rule (either logical or nonlogical), apply inductive hypothesis to the premisses and then the rule.

If A is principal and the last rule is logical, for $L\&$ and $L\vee$ apply height-preserving invertibility, inductive hypothesis and then the rule. For $L\supset$ apply inductive hypothesis to the left premiss, invertibility and inductive hypothesis to the right premiss, and then the rule. If the last rule is nonlogical, A is an atomic formula P and there are two cases. In the first case one occurrence of A belongs to the context, another is principal in the rule, say $A = P_m$ $(= P)$. The derivation ends with

$$\frac{P_1, \ldots, P_{m-1}, P, Q_1, P, \Gamma' \Rightarrow \Delta \quad \ldots \quad P_1, \ldots, P_{m-1}, P, Q_n, P, \Gamma' \Rightarrow \Delta}{P_1, \ldots, P_{m-1}, P, P, \Gamma' \Rightarrow \Delta}\ Reg$$

and we obtain

$$\frac{\dfrac{P_1, \ldots, P_{m-1}, P, Q_1, P, \Gamma' \Rightarrow \Delta}{P_1, \ldots, P_{m-1}, P, Q_1, \Gamma' \Rightarrow \Delta}\ Ind \quad \ldots \quad \dfrac{P_1, \ldots, P_{m-1}, P, Q_n, P, \Gamma' \Rightarrow \Delta}{P_1, \ldots, P_{m-1}, P, Q_n, \Gamma' \Rightarrow \Delta}\ Ind}{P_1, \ldots, P_{m-1}, P, \Gamma' \Rightarrow \Delta}\ Reg$$

In the second case both occurrences of A are principal in the rule, say $A = P_{m-1} = P_m = P$, thus the derivation ends with

$$\frac{P_1, \ldots, P_{m-2}, P, P, Q_1, \Gamma' \Rightarrow \Delta \quad \ldots \quad P_1, \ldots, P_{m-2}, P, P, Q_n, \Gamma' \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, P, \Gamma' \Rightarrow \Delta}\ Reg$$

and we obtain

$$\frac{\dfrac{P_1, \ldots, P_{m-2}, P, P, Q_1, \Gamma' \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, Q_1, \Gamma' \Rightarrow \Delta}\ Ind \quad \ldots \quad \dfrac{P_1, \ldots, P_{m-2}, P, P, Q_n, \Gamma' \Rightarrow \Delta}{P_1, \ldots, P_{m-2}, P, Q_n, \Gamma' \Rightarrow \Delta}\ Ind}{P_1, \ldots, P_{m-2}, P, \Gamma' \Rightarrow \Delta}\ Reg$$

with the last rule given by Closure condition 2.7.

The proof of admissibility of right contraction in **G3ipm*** does not present any additional difficulty with respect to the proof of admissibility in **G3ipm** since in nonlogical rules the succedent in both the premisses and the conclusion is an arbitrary multiset Δ. So in case the last rule in a derivation of $\Gamma \Rightarrow \Delta, A, A$ is a nonlogical rule, one simply proceeds by applying the inductive hypothesis to the premisses, and then the rule. $\dashv$

THEOREM 3.3. *The cut rule*

$$\frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \mathit{Cut}$$

is admissible in **G3ipm***.

PROOF. The proof is by induction on the length of A with subinduction on the sum of the heights of the derivations of $\Gamma \Rightarrow \Delta$, A and A, $\Gamma' \Rightarrow \Delta'$. We consider here in detail only the cases arising from the addition of nonlogical rules and refer to [2] and its elaboration in [3] for the remaining cases.

1. If the left premiss is a nonlogical axiom, then also the conclusion is a nonlogical axiom, since nonlogical axioms have an arbitrary context as succedent.

2. If the right premiss is a nonlogical axiom with A not principal in it, the conclusion is a nonlogical axiom for the same reason as in 1.

3. If the right premiss is a nonlogical axiom with A principal in it, A is atomic and we consider the left premiss. The case that it is a nonlogical axiom is covered by 1. If it is a logical axiom with A not principal, the conclusion is a logical axiom; else Γ contains the atom A and the conclusion follows from the right premiss by weakening. In the remaining cases we consider the last rule in the derivation of $\Gamma \Rightarrow \Delta$, A. Since A is atomic, A is not principal in the rule. Let us consider the case of a nonlogical rule (the others being dealt with similarly, except $R\supset$ that is covered in 4). We transform the derivation, where $\mathbf{P}_m$ stands for $P_1, \ldots, P_m$,

$$\frac{\dfrac{\mathbf{P}_m, Q_1, \Gamma'' \Rightarrow \Delta, A \quad \ldots \quad \mathbf{P}_m, Q_n, \Gamma'' \Rightarrow \Delta, A}{\mathbf{P}_m, \Gamma'' \Rightarrow \Delta, A}\ \mathit{Reg} \qquad A, \Gamma' \Rightarrow \Delta'}{\mathbf{P}_m, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\ \mathit{Cut}$$

into

$$\frac{\dfrac{\mathbf{P}_m, Q_1, \Gamma'' \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\mathbf{P}_m, Q_1, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\ \mathit{Cut} \quad \ldots \quad \dfrac{\mathbf{P}_m, Q_n, \Gamma'' \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\mathbf{P}_m, Q_n, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\ \mathit{Cut}}{\mathbf{P}_m, \Gamma', \Gamma'' \Rightarrow \Delta, \Delta'}\ \mathit{Reg}$$

where the cut has been replaced by n cuts with left premiss with derivation of lower height.

Let us now consider the cases in which neither premiss is an axiom.

4. A is not principal in the left premiss. These are dealt with as above, with cut permuted upwards to the premisses of the last rule used in the derivation of the left premiss, except for $R\supset$. By the intuitionistic restriction in this rule, A does not appear in the premiss, and the conclusion is obtained without cut by $R\supset$ and weakening.

5. A is principal in the left premiss only. Then A has to be a compound formula. Therefore, if the last rule of the right premiss is a nonlogical rule, A

cannot be principal in the rule, because only atomic formulas are principal in nonlogical rules. In this case cut is permuted to the premisses of the right premiss. If the right rule is a logical one with A not principal in it, the usual reductions are applied (as in [3]).

6. A is principal in both premisses. This case can only involve logical rules, and is dealt with as in the usual proof for pure logic. ⊣

The conversions used in the proof of admissibility of cut show why it is necessary to formulate the nonlogical rules so that they have an arbitrary context in the succedent, both in the premisses and in the conclusion. Besides, as already observed, active and principal formulas have to be atomic and appear in the antecedent. Thus nonlogical rules have the form of left rules.

Admissibility of all structural rules holds also for extensions of the classical calculus **G3cp** with nonlogical rules.

§**4. Extension of predicate logic.** We show that admissibility of structural rules is maintained in extensions of predicate logic, and then show how this result can be applied to obtain a structural proof theory of predicate logic with equality.

4.1. Admissibility of structural rules for extension of predicate logic. We add rules for quantifiers to the propositional calculi **G3ipm** and **G3cp**, to obtain two calculi denoted **G3im** and **G3c**. The rules for the classical calculus are, with repetition of the principal formula in $L\forall$ and $R\exists$ to obtain admissibility of contraction,

$$\frac{A(t), \forall x A(x), \Gamma \Rightarrow \Delta}{\forall x A(x), \Gamma \Rightarrow \Delta}\ L\forall \qquad \frac{\Gamma \Rightarrow \Delta, A(x)}{\Gamma \Rightarrow \Delta, \forall x A(x)}\ R\forall$$

$$\frac{A(x), \Gamma \Rightarrow \Delta}{\exists x A(x), \Gamma \Rightarrow \Delta}\ L\exists \qquad \frac{\Gamma \Rightarrow \Delta, \exists x A(x), A(t)}{\Gamma \Rightarrow \Delta, \exists x A(x)}\ R\exists$$

The restriction in $R\forall$ and $L\exists$ is that x must not occur free in Γ, Δ. The rules for intuitionistic predicate logic are the same, except that in the succedent of the premiss of the $R\forall$ rule only one formula is permitted and the restriction is that x must not occur free in Γ.

Similarly to the propositional case, extensions of the two calculi by rules following the rule-scheme and satisfying the closure condition are denoted by **G3im*** and **G3c***. The proofs of admissibility of structural rules in such extensions are extensions of the corresponding proofs for the purely logical calculi, similarly to the propositional case in Section 3. We shall therefore not dwell more on these proofs, but just note the results:

THEOREM 4.1. *The rules of weakening, contraction and cut are admissible in* **G3im*** *and* **G3c***.

4.2. Cut-free predicate logic with equality. Axiomatic presentations of predicate logic with equality assume a primitive relation $a = b$ with the axiom of *reflexivity*, $a = a$, and the *replacement scheme*, $a = b \,\&\, A(a) \supset A(b)$. In sequent calculus, the usual way of treating equality is to add regular sequents with which derivations can start (as in [12, p. 98]). These are of form $\Rightarrow a = a$ and $a = b,\ P(a) \Rightarrow P(b)$, with P atomic, and Gentzen's "extended Hauptsatz" says that cuts can be reduced to cuts on these equality axioms. For example, symmetry of equality is derived by letting $P(x)$ be $x = a$. Then the second axiom gives $a = b,\ a = a \Rightarrow b = a$, and a cut with the first axiom $\Rightarrow a = a$ gives $a = b \Rightarrow b = a$. But there is no cut-free derivation of symmetry. Note also that in this approach, the rules of weakening and contraction must be assumed, and only then can cuts be reduced to cuts on axioms. (Weakening could be made admissible by letting arbitrary contexts appear on both sides of the regular sequents, but contraction not.)

By our method, cuts on equality axioms are avoided. We first restrict the replacement scheme to atomic predicates $P, Q, R, \ldots$ and then convert the axioms into rules,

$$\frac{a = a, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,{\scriptstyle Ref} \qquad \frac{a = b, P(a), P(b), \Gamma \Rightarrow \Delta}{a = b, P(a), \Gamma \Rightarrow \Delta}\,{\scriptstyle Repl}$$

There is a separate replacement rule for each predicate P, and $a = b,\ P(a)$ are repeated in the premiss to obtain admissibility of contraction. By the restriction to atomic predicates, both forms of rules follow the rule-scheme. A case of duplication is produced in the conclusion of the replacement rule in case $P(x)$ is $x = b$. The replacement rule concludes $a = b,\ a = b,\ \Gamma \Rightarrow \Delta$ from the premiss $a = b,\ a = b,\ b = b,\ \Gamma \Rightarrow \Delta$. We note that the rule where both duplications are contracted is an instance of the reflexivity rule so that the closure condition is satisfied. We therefore have the

THEOREM 4.2. *The rules of weakening, contraction and cut are admissible in predicate logic with equality.*

Next we have to show the replacement rule admissible for arbitrary predicates.

LEMMA 4.3. *The replacement axiom* $a = b,\ A(a) \Rightarrow A(b)$ *is derivable for arbitrary* A.

PROOF. The proof is by induction on length of A. If $A = \bot$, the sequent is an axiom, and if A is an atom, it follows from the replacement rule. If $A = B \,\&\, C$ or $A = B \vee C$, we apply inductive hypothesis to B and C and then left and right rules. If $A = B \supset C$, we have the derivation

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{b = a, B(b) \Rightarrow B(a)}{a = b, a = a, b = a, B(b) \Rightarrow B(a)}\,{\scriptstyle W,W}}{a = b, a = a, B(b) \Rightarrow B(a)}\,{\scriptstyle Repl}}{a = b, B(b) \Rightarrow B(a)}\,{\scriptstyle Ref}}{a = b, B(a) \supset C(a), B(b) \Rightarrow B(a)}\,{\scriptstyle W} \qquad \dfrac{a = b, C(a) \Rightarrow C(b)}{a = b, C(a), B(b) \Rightarrow C(b)}\,{\scriptstyle W}}{a = b, B(a) \supset C(a), B(b) \Rightarrow C(b)}\,{\scriptstyle L\supset}}{a = b, B(a) \supset C(a) \Rightarrow B(b) \supset C(b)}\,{\scriptstyle R\supset}
$$

If $A = \forall x B(x)$, the sequent $a = b, \forall x\ B(x, a) \Rightarrow \forall x B(x, b)$ is derived from $a = b,\ B(x, a) \Rightarrow B(x, b)$ by applying first $L\forall$ and then $R\forall$. Finally, the sequent $a = b, \exists x\ B(x, a) \Rightarrow \exists x B(x, b)$ is derived by applying first $R\exists$ and then $L\exists$. ⊣

THEOREM 4.4. *The replacement rule*

$$
\dfrac{a = b, A(a), A(b), \Gamma \Rightarrow \Delta}{a = b, A(a), \Gamma \Rightarrow \Delta}\,{\scriptstyle Repl}
$$

is admissible for arbitrary predicates A.

PROOF. By the lemma, $a = b,\ A(a) \Rightarrow A(b)$ is derivable. A cut with the premiss of the replacement rule and contractions lead to $a = b, A(a), \Gamma \Rightarrow \Delta$. Therefore, by admissibility of contraction and cut in the calculus of predicate logic with equality, admissibility of the replacement rule follows. ⊣

Our cut- and contraction-free calculus is equivalent to the usual calculi: the sequents $\Rightarrow a = a$ and $a = b,\ P(a) \Rightarrow P(b)$ follow at once from the reflexivity rule and the replacement rule. In the other direction, the two rules are easily derived from $\Rightarrow a = a$ and $a = b, P(a) \Rightarrow P(b)$ using cut and contraction.

§5. Application to axiomatic systems. All classical systems permitting quantifier-elimination, and most intuitionistic ones, can be converted into systems of cut-free nonlogical rules of inference. Theories of equality, apartness, and order, as well as theories with operations, such as lattices and Heyting algebras are presented in Negri [10] as cut-free intuitionistic systems. On the other hand, it is noted that the intuitionistic theory of negative equality does not admit of a good structural proof theory under the present approach: this theory has a primitive relation $a \neq b$ and the two axioms $\sim a \neq a$ and $\sim a \neq c \,\&\, \sim b \neq c \supset\ \sim a \neq b$ expressing reflexivity and transitivity of negative equality.

The properties of sequent systems representing axiomatic systems are based on the subformula principle for systems with nonlogical rules:

THEOREM 5.1. *If $\Gamma \Rightarrow \Delta$ is derivable in* **G3im*** *or* **G3c***, *then all formulas in the derivation are either subformulas of the endsequent or atomic formulas.*

PROOF. Only nonlogical rules can make formulas disappear in a derivation, and all such formulas are atomic. ⊣

The subformula principle is weaker than that for purely logical systems, but sufficient for structural proof-analysis. Some general consequences are obtained: consider a theory having as axioms a finite set $\mathcal{D}$ of regular formulas. Define $\mathcal{D}$ to be *inconsistent* if $\Rightarrow \bot$ is derivable in the corresponding extension, and *consistent* if it is not inconsistent. For a theory $\mathcal{D}$, inconsistency surfaces with the axioms through regular decomposition, with no consideration of the logical rules:

THEOREM 5.2. *Let $\mathcal{D}$ be inconsistent. Then*

(i) *All rules in the derivation of $\Rightarrow \bot$ are nonlogical,*

(ii) *All sequents in the derivation have $\bot$ as succedent,*

(iii) *Each branch in the derivation begins with a nonlogical rule of form*

$$\overline{P_1, \dots, P_m \Rightarrow \bot}$$

(iv) *The last step in the derivation is a rule of form*

$$\frac{Q_1 \Rightarrow \bot \quad \dots \quad Q_n \Rightarrow \bot}{\Rightarrow \bot}$$

PROOF. (i) By Theorem 5.1, no logical constants except $\bot$ can occur in the derivation. (ii) If the conclusion of a nonlogical rule has Δ as succedent, the premisses of the rule also have. Since the endsequent is $\Rightarrow \bot$, (ii) follows. (iii) By (ii) and by $\bot$ not being atomic, no derivation begins with $P, \Gamma \Rightarrow P$. Since only atoms can disappear from antecedents in a nonlogical rule, no derivation begins with $\bot, \Gamma \Rightarrow \bot$. This leaves only zero-premiss nonlogical rules. (iv) By observing that the endsequent has an empty antecedent. ⊣

It follows that if an axiom system is inconsistent, its formula traces contain negations, and atoms or disjunctions. Therefore, if there are neither atoms nor disjunctions, the axioms are consistent, and similarly if there are no negations.

By our method, the logical structure in axioms as they are usually expressed, is converted into combinatorial properties of derivation trees, and completely separated from steps of logical inference. This is especially clear in the classical quantifier-free case, where theorems to be proved can be converted into a finite number of regular sequents $\Gamma \Rightarrow \Delta$. By the subformula principle, derivations of these sequents use only the nonlogical rules and axioms of the corresponding sequent calculus. As applications of this result, we can use proof theory for syntactic proofs of mutual independence of axiom systems, as follows. Let the axiom to be proved independent be expressed by the logic-free sequent $\Gamma \Rightarrow \Delta$. When the rule corresponding to the axiom is left out from the system of nonlogical rules, underivability of $\Gamma \Rightarrow \Delta$ is usually very easily seen. For example, it is possible to prove the independence of Euclid's fifth postulate in affine plane geometry in this way. Another application of the fact that logical rules can be dispensed with is proof search. We can start bottom-up from the logic-free sequent $\Gamma \Rightarrow \Delta$ to be derived; the succedent will be the same throughout in derivations with nonlogical rules, and in typical cases very few nonlogical rules match the sequent to be derived.

Extension of the above to theorems with quantifiers is straightforward in the classical case: first convert the theorem to be proved into prenex form, then the propositional matrix into the variant of conjunctive normal form used above. Each conjunct corresponds to a regular sequent, without logical structure, and the overall structure of the derivation is as follows: first the regular sequents are derived by nonlogical rules only, then the conjuncts by $L\&$, $R\vee$ and $R\supset$. Now $R\&$ collects all these into the propositional matrix, and right quantifier rules lead into the theorem. The nonlogical rules typically contain function constants resulting from quantifier elimination. In the constructive case, these methods apply to formulas in the prenex fragment admitting a propositional part in regular normal form.

A simplified example from elementary geometry may illustrate the above structure of derivations: let $x \neq y$ express that points x and y are distinct, and let $Inc(x, z)$ express that point x is incident on line z. For any two points, there is a line on which the points are incident,

$$\forall x \forall y (x \neq y \supset \exists z (Inc(x, z) \,\&\, Inc(y, z))).$$

In prenex normal form, with the propositional matrix in implicational conjunctive normal form, this is equivalent to

$$\forall x \forall y \exists z ((x \neq y \supset Inc(x, z)) \,\&\, (x \neq y \supset Inc(y, z))).$$

In a quantifier-free approach, we start instead from a connecting line construction, a function constant $ln(a, b)$ giving for any two distinct points a, b as value a line. The nonlogical rules

$$\frac{a \neq b, Inc(a, ln(a, b)), \Gamma \Rightarrow \Delta}{a \neq b, \Gamma \Rightarrow \Delta}\,{\scriptstyle Inc} \qquad \frac{a \neq b, Inc(b, ln(a, b)), \Gamma \Rightarrow \Delta}{a \neq b, \Gamma \Rightarrow \Delta}\,{\scriptstyle Inc}$$

express the incidence properties of such constructed lines in a quantifier-free form. We have the following derivation, with repetition of $x \neq y$ left out in the premisses of the incidence rule to narrow down the derivation tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Inc(x, ln(x, y)) \Rightarrow Inc(x, ln(x, y))}{x \neq y \Rightarrow Inc(x, ln(x, y))}\,{\scriptstyle Inc}}{\Rightarrow x \neq y \supset Inc(x, ln(x, y))}\,{\scriptstyle R\supset} \qquad \dfrac{\dfrac{Inc(y, ln(x, y)) \Rightarrow Inc(y, ln(x, y))}{x \neq y \Rightarrow Inc(y, ln(x, y))}\,{\scriptstyle Inc}}{\Rightarrow x \neq y \supset Inc(y, ln(x, y))}\,{\scriptstyle R\supset}}{\Rightarrow (x \neq y \supset Inc(x, ln(x, y)))\ \&\ (x \neq y \supset Inc(y, ln(x, y)))}\,{\scriptstyle R\&}}{\Rightarrow \exists z((x \neq y \supset Inc(x, z))\ \&\ (x \neq y \supset Inc(y, z)))}\,{\scriptstyle R\exists}}{\Rightarrow \forall x \forall y \exists z((x \neq y \supset Inc(x, z))\ \&\ (x \neq y \supset Inc(y, z)))}\,{\scriptstyle R\forall R\forall}$$

Derivations with nonlogical rules and all but two of the logical rules do not show whether a system is classical or constructive. The difference only appears if classical logic is needed in the conversion of axioms into rules.

§**6. Concluding remarks.** We have shown how classical first-order theories admitting quantifier-elimination can be turned into cut-free systems of nonlogical rules, and determined what class of constructive systems has the same property. All methods used, in particular those for proving the admissibility of cut, are purely syntactical.

When logical sequent calculi are extended by rules corresponding to mathematical axioms, the strict subformula property cannot be maintained. We have shown, partly by general results, and partly by small examples, how structural proof analysis proceeds under the weak subformula property. In Negri [10], similar arguments are applied to much more complicated problems of conservativity between elementary intuitionistic theories.

In the introduction, we listed previous attempts at extending cut elimination to axiomatic systems. To these, the attempt by Uesu [13] we recently discovered has to be added. This work contains the correct way of presenting atomic axioms as rules of inference. As to the use of conjunctive normal form in sequent calculus, we owe it to Ketonen's thesis [9] of 1944, in which the invertible sequent calculus for classical propositional logic was discovered. (An accessible summary of Ketonen's main results is given in Bernays [1].)

For uniformity and ease of exposition, we have chosen for intuitionistic propositional logic a multi-succedent sequent calculus. However, all the results presented here hold *mutatis mutandis* if a single-succedent calculus is used. In addition, by using a single-succedent calculus, we easily obtain an extension of the disjunction property for constructive systems with rules. Specifically, if the formula traces of the regular decomposition of a set of regular formulas have no atoms or disjunctions, then no rules like *Atom* or *Vel* are added to the system, and therefore the only way to derive $\Rightarrow A \vee B$ is by either deriving $\Rightarrow A$ or $\Rightarrow B$. This result goes beyond the disjunction property under assumptions that are Harrop formulas. Negri [10] gives a single-succedent calculus for the intuitionistic theories of apartness and order, and proves the admissibility of structural rules for it.

To translate our results into natural deduction, a single-succedent sequent calculus needs to be used. The rule-scheme is restricted to one formula in the succedent, and the single-succedent intuitionistic calculus **G3i** is then extended by rules following the modified scheme. The cut-free derivations in these extensions translate into normal natural deductions, where the natural deduction rules are obtained from the translation of the rule-scheme for sequents,

$$\frac{P_1 \dots P_m \quad \begin{matrix}[Q_1]\\ \vdots \\ C\end{matrix} \;\; \dots \;\; \begin{matrix}[Q_n]\\ \vdots \\ C\end{matrix}}{C}$$

This is the natural deduction scheme for *nonlogical elimination rules*.

In Hallnäs and Schroeder-Heister [8], regular sequents $P_1, \dots, P_m \Rightarrow Q$ are translated, for the purposes of logic programming, into natural deduction *nonlogical introduction rules*, of the form

$$\frac{P_1 \dots P_m}{Q}$$

The two kinds of nonlogical natural deduction rules are interderivable. The relation of these two ways of extending natural deduction is analogous to the situation in sequent calculus: extension of sequent systems with regular sequents does not in general permit cut-free derivations, whereas extension with nonlogical rules does. This is seen clearly in the example of predicate logic with equality.

In the case of classical systems, extensions of natural deduction can be given through the cut-free single-succedent sequent calculus for classical propositional logic of von Plato [11]. This calculus is obtained by adding to the intuitionistic calculus **G3ip** a left rule of excluded middle for atomic formulas. Translation of this rule into natural deduction gives a generalization of the principle of indirect proof with the property that also inferences on disjunctions by the new rule convert to ones on atoms, whereas with indirect proof, this is not always the case. Thus, there is a uniform method for obtaining extensions of natural deduction permitting normal form, for any classical free-variable theory.

REFERENCES

[1] P. Bernays, Review of Ketonen [9], this Journal, vol. 10 (1945), pp. 127–130.

[2] A. Dragalin, ***Mathematical intuitionism: Introduction to proof theory***, American Mathematical Society, Providence, Rhode Island, 1988.

[3] R. Dyckhoff, *Dragalin's proof of cut-admissibility for the intuitionistic sequent calculi* **G3i** *and* **G3i′**, ***Research Report CS/97/8***, Computer Science Division, St. Andrews University, 1997.

[4] G. GENTZEN, *Untersuchungen über das logische Schliessen*, ***Mathematische Zeitschrift***, vol. 39 (1934–1935), pp. 176–210 and 405–431.

[5] ———, *Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie*, ***Forschungen zur Logik und zur Grundlegung der exakten Wissenschaften***, N. S., vol. 4 (1938), pp. 19–44.

[6] ———, ***The collected papers of Gerhard Gentzen***, North-Holland, Amsterdam, 1969.

[7] J.-Y. GIRARD, ***Proof theory and logical complexity***, Bibliopolis, Naples, 1987.

[8] L. HALLNÄS and P. SCHROEDER-HEISTER, *A proof-theoretic approach to logic programming. I. Clauses as rules*, ***Journal of Logic and Computation***, vol. 1 (1990), pp. 261–283.

[9] O. KETONEN, *Untersuchungen zum Prädikatenkalkül*, ***Annales Academiae Scientiarum Fennicae, Series A***, vol. I (1944), no. 23.

[10] S. NEGRI, *Sequent calculus proof theory of intuitionistic apartness and order relations*, ***Archive for Mathematical Logic***, in press.

[11] J. VON PLATO, *On the proof theory of classical logic*, submitted, 1998.

[12] A. TROELSTRA and H. SCHWICHTENBERG, ***Basic proof theory***, Cambridge University Press, Cambridge, 1996.

[13] T. UESU, *An axiomatization of the apartness fragment of the theory DLO^+ of dense linear order*, ***Logic colloquium '84***, Lecture Notes in Mathematics, no. 1104, 1984, pp. 453–475.

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF HELSINKI
HELSINKI, FINLAND
E-mail: negri@helsinki.fi

DEPARTMENT OF PHILOSOPHY
UNIVERSITY OF HELSINKI
HELSINKI, FINLAND
E-mail: vonplato@helsinki.fi