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NEW DICHOTOMIES FOR BOREL EQUIVALENCE RELATIONS

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We announce two new dichotomy theorems for Borel equivalence relations, and present the results in context by giving an overview of related recent developments.

§1. Introduction. For X a Polish (i.e., separable, completely metrizable) space and E a Borel equivalence relation on X, a (complete) classification of X up to E-equivalence consists of finding a set of invariants I and a map $c : X \to I$ such that $xEy \Leftrightarrow c(x) = c(y)$. To be of any value we would expect I and c to be "explicit" or "definable". The theory of Borel equivalence relations investigates the nature of possible invariants and provides a hierarchy of notions of classification.

The following partial (pre-)ordering is fundamental in organizing this study. Given equivalence relations E and F on X and Y, resp., we say that E can be *Borel reduced* to F, in symbols

$$E \leq_B F,$$

if there is a Borel map $f : X \to Y$ with $xEy \Leftrightarrow f(x)Ff(y)$. Then if $\tilde{f}([x]_E) = [f(x)]_F$, $\tilde{f} : X/E \to Y/F$ is an embedding of X/E into Y/F, which is "Borel" (in the sense that it has a Borel lifting).

Intuitively, $E \leq_B F$ might be interpreted in any one of the following ways:

(i) The classification problem for E is simpler than (or can be reduced to) that of F: any invariants for F work as well for E (after composing by an f as above).

(ii) One can classify E by using as invariants F-equivalence classes.

(iii) The quotient space X/E has "Borel cardinality" less than or equal to that of Y/F, in the sense that there is a "Borel" embedding of X/E into Y/F.

We let

$$E \sim_B F \Leftrightarrow E \leq_B F \,\&\, F \leq_B E.$$

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This means that E, F have equivalent classification problems, or X/E, Y/F have the same "Borel cardinality". Finally, let

$$E <_B F \Leftrightarrow E \leq_B F \,\&\, F \not\leq_B E.$$

§2. The Silver and Glimm-Effros dichotomies. For each Polish space X, we also denote by X the equality relation $\Delta(X) = \{(x, y) \in X^2 : x = y\}$ on X. By $n (n \in \mathbb{N})$ we denote any set of cardinality n. Then it is clear that

$$1 <_B 2 <_B 3 <_B \cdots <_B \mathbb{N}$$

forms an initial segment of $\leq_B$, and $\mathbb{N} \leq_B E$ for any Borel equivalence relation E with infinitely many equivalence classes. The first non-trivial result concerning $\leq_B$ is the following *Silver Dichotomy*:

First Dichotomy Theorem (Silver [1980]). *Let E be a Borel equivalence relation. Then*

$$E \leq_B \mathbb{N} \text{ or } \mathbb{R} \leq_B E.$$

Equivalently, any Borel equivalence relation E either has countably many equivalence classes or else there is a perfect set consisting of pairwise-inequivalent elements. (Silver proved that this result is also true for co-analytic E.)

Thus we have

$$1 <_B 2 <_B 3 <_B \cdots <_B \mathbb{N} <_B \mathbb{R}$$

as an initial segment of $\leq_B$ and $\mathbb{R} \leq_B E$ for any Borel equivalence relation with uncountably many equivalence classes.

Next denote by E_0 the following equivalence relation on $2^{\mathbb{N}}$:

$$xE_0y \Leftrightarrow \exists n \forall m \geq n(x(m) = y(m)).$$

Up to $\sim_B$ this is the same as the *Vitali equivalence E_V* relation on $\mathbb{R}$:

$$xE_Vy \Leftrightarrow x - y \in \mathbb{Q}.$$

An equivalence relation F is *hyperfinite* if $F = \bigcup_n F_n$, with F_n Borel equivalence relations such that $F_n \subseteq F_{n+1}$ and each F_n-equivalence class is finite. It turns out that $E \leq_B E_0$ iff there is hyperfinite F with $E \leq_B F$.

We now have the following *General Glimm-Effros Dichotomy*:

Second Dichotomy Theorem (Harrington-Kechris-Louveau [1990]). *Let E be a Borel equivalence relation. Then*

$$E \leq_B \mathbb{R} \text{ or } E_0 \leq_B E.$$

Notice that $E \leq_B \mathbb{R}$ is equivalent to saying that there is a Borel map $f : X \to Y$, Y some Polish space, with $xEy \Leftrightarrow f(x) = f(y)$, i.e., E can be completely classified (in a Borel way) by invariants which are members of some Polish space. We express this by saying that E is *concretely classifiable* (or *smooth*). So the Second Dichotomy Theorem asserts that either E is concretely classifiable or else contains a copy of E_0. If FIN denotes the *Frechet ideal* on $\mathbb{N}$, i.e., the ideal of all finite sets, then under the natural identification of $2^{\mathbb{N}}$ with $p(\mathbb{N})$, $2^{\mathbb{N}}/E_0 = p(\mathbb{N})/\mathrm{FIN}$, so another interpretation is that the "Borel cardinality" of X/E, E Borel, is either $1, 2, \ldots, \aleph_0$ (= the "Borel cardinality" of $\mathbb{N}$), $2^{\aleph_0}$ (= the "Borel cardinality" of $\mathbb{R}$), or else $\geq$ the "Borel cardinality" of $p(\mathbb{N})/\mathrm{FIN}$.

So

$$1 <_B 2 <_B 3 <_B \cdots <_B \mathbb{N} <_B \mathbb{R} <_B E_0$$

is an initial segment of $\leq_B$ and $E_0 \leq_B E$ for any E which is not concretely classifiable.

§3. Incomparable equivalence relations. Beyond E_0 linearity of $\leq_B$ breaks down, and it is folklore that there exist Borel E, F with $E \not\leq_B F \,\&\, F \not\leq_B E$. Although $\leq_B$ is not linear, one might hope that it is a well-quasi-ordering (i.e., well-founded with each antichain finite). However Woodin disproved this:

THEOREM 3.1 (Woodin). *There exists a continuum of pairwise $\leq_B$-incomparable Borel equivalence relations.*

Later Louveau-Velickovic improved this result. Below $\subseteq^*$ denotes the following partial ordering on $p(\mathbb{N})$:

$$x \subseteq^* y \Leftrightarrow x \setminus y \in \mathrm{FIN}.$$

THEOREM 3.2 (Louveau-Velickovic [1994]). *The partial ordering $\subseteq^*$ on $p(\mathbb{N}) \setminus \mathrm{FIN}$ can be embedded in $\leq_B$, i.e., there is a map $S \mapsto E_S$ from $p(\mathbb{N}) \setminus \mathrm{FIN}$ into Borel equivalence relations such that*

$$S \subseteq^* T \Leftrightarrow E_S \leq_B E_T.$$

The Louveau-Velickovic equivalence relations E_S are $\mathbf{\Pi}^0_3$, but later Mazur [1996] showed that a similar result holds with $\mathbf{\Sigma}^0_2$ equivalence relations. This is optimal, since all $\mathbf{\Pi}^0_2$ equivalence relations are concretely classifiable, i.e., $\leq_B \mathbb{R}$ (see Harrington-Kechris-Louveau [1990]).

The examples of Louveau-Velickovic and Mazur are all generated by ideals on $\mathbb{N}$. By an *ideal* on $\mathbb{N}$ we mean a subset of $p(\mathbb{N})$ which is closed under finite unions and subsets. For any ideal I on $\mathbb{N}$, let E_I be the corresponding equivalence relation on $p(\mathbb{N})$:

$$xE_I y \Leftrightarrow x \Delta y \in I,$$

where Δ denotes symmetric difference. Louveau-Velickovic assign to each $S \in p(\mathbb{N}) \setminus \mathrm{FIN}$, an ideal I_S with $E_S = E_{I_S}$ as above.

We have thus seen that $\leq_B$ is immensely complicated even at low levels of the Borel hierarchy. This prompts us to restrict attention to important subclasses, which include most natural examples.

§**4. Countable equivalence relations.** A Borel equivalence relation is *countable* if every equivalence class is countable. By a result of Feldman-Moore [1977], these can be equivalently described as induced by Borel actions of countable groups. Up to $\sim_B$ they also include all equivalence relations induced by Borel actions of Polish locally compact groups (see Kechris [1992]), and thus they have particular relevance to areas of mathematics such as ergodic theory, operator algebras, and the study of topological transformation groups. Examples from logic include $\equiv_T$ (Turing equivalence) and $\equiv_A$ (arithmetical equivalence) on $p(\mathbb{N})$.

Countable Borel equivalence relations and some of their subclasses, such as hyperfinite, amenable, and treeable, are studied in Kechris [1991], Dougherty-Jackson-Kechris [1994], Jackson-Kechris-Louveau [∞], Kechris [1994]. We recall here only a few facts and questions relevant to this paper.

First, there exists a largest, in the sense of $\leq_B$, countable Borel equivalence relation, denoted by

$$E_\infty$$

and called the *universal* countable Borel equivalence relation. It has many, equivalent up to $\sim_B$, manifestations. For example, if $E(F_2, 2)$ denotes the equivalence relation induced by the shift action of F_2, the free group of 2 generators, on 2^{F_2}, then $E(F_2, 2) \sim_B E_\infty$. Slaman-Steel showed that $\equiv_A \sim_B E_\infty$, but it is a basic open problem whether $\equiv_T \sim_B E_\infty$.

By the Second Dichotomy Theorem, if E is countable but not concretely classifiable, then

$$E_0 \leq_B E \leq_B E_\infty.$$

It is known that

$$E_0 <_B E_\infty,$$

and in fact there is a countable Borel E with

$$E_0 <_B E <_B E_\infty.$$

However, it is a basic open problem whether there are $\leq_B$-incomparable countable Borel equivalence relations. It is also open whether $\leq_B$ on this class is a well-quasi-ordering.

§5. $\mathbf{E_1}$. There is a canonical example of a Borel equivalence relation which is not $\leq_B E_\infty$, i.e., not Borel reducible to a countable Borel equivalence relation. It is defined on $(2^{\mathbb{N}})^{\mathbb{N}}$ by

$$xE_1y \Leftrightarrow \exists n \forall m \geq n(x(m) = y(m)),$$

and may be thought of as a "continuous" analog of E_0. The Borel equivalence relations $E \leq_B E_1$ can be characterized as those that can be written in the form $E = \bigcup_{n=0}^{\infty} F_n$, with F_n concretely classifiable, or smooth, so they are called *hypersmooth*. These are studied in Kechris-Louveau [1997] and include many interesting examples, like the "tail" equivalence relations associated to Borel maps, equivalence relations induced by Borel actions of $\mathbb{R}^{<\mathbb{N}}$, the equivalence relations induced by the components of an indecomposable continuum, etc.

It turns out that $E_0 <_B E_1 \not\leq_B E_\infty, E_\infty \not\leq_B E_1$, and that E_1 admits a "local" dichotomy theorem:

Third Dichotomy Theorem (Kechris-Louveau [1997]). *Let $E \leq_B E_1$. Then*

$$E \leq_B E_0 \textit{ or } E \sim_B E_1.$$

Thus E_1 is $\leq_B$-minimal above E_0 and the hypersmooth Borel equivalence relations are all known.

We should point out that E_1 is F_σ and can be also represented as E_{I_1}, where I_1 is the following F_σ ideal on $\mathbb{N}^2$ (which we identify here with $\mathbb{N}$ via some fixed bijection $\langle , \rangle : \mathbb{N}^2 \to \mathbb{N}$):

$$I_2 = \{A \subseteq \mathbb{N}^2 : \exists n(A \subseteq n \times \mathbb{N})\}.$$

Using the usual definition of products of ideals and denoting the ideal $\{\emptyset\}$ by $\emptyset$, $I_2 = \mathrm{FIN} \times \emptyset$.)

If G is a Polish group with a Borel action $(g \cdot x) \mapsto g \cdot x$ on a Polish space X, we say that X is a *Borel G-space*. (See, for example, Becker-Kechris [1996] for the theory of Borel G-spaces.) We denote by E_G^X the corresponding equivalence relation: $xE_G^Xy \Leftrightarrow \exists g \in G(g \cdot x = y)$. In general E_G^X is $\mathbf{\Sigma}_1^1$ but may not be Borel.

It is a very interesting question to understand when a given Borel equivalence E can be, up to $\sim_B$, of the form E_G^X. The following result establishes a basic obstruction.

Theorem 5.1 (Kechris-Louveau [1997]). *For any Polish group G and Borel G-space X,*

$$E_1 \not\leq_B E_G^X.$$

Strengthening a conjecture in Kechris-Louveau [1997], we here propose:

CONJECTURE 1. *Let E be a Borel equivalence relation. Then*

$$E_1 \leq_B E \text{ or } E \sim_B E_G^X$$

for some Polish group G and Borel G-space X.

§6. Polishable ideals. Any ideal I on $p(\mathbb{N})$ is closed under symmetric difference Δ, so it is a subgroup of the Cantor group $(p(\mathbb{N}), \Delta)$. As in, e.g., Becker-Kechris [1996], we call I *Polishable* if it is Borel isomorphic to a Polish group G. Equivalently, I is Polishable if there is a Polish topology on I with the same Borel sets (i.e., the Borel sets of this topology are the Borel subsets of I), which makes (I, Δ) into a topological group. This topology must then be unique. For example, FIN is Polishable, but I_1 is not. If I is Polishable, the equivalence relation E_I is induced by a Borel action of a Polish group, so $E_1 \not\leq_B E_I$. It was conjectured in Kechris-Louveau [1997] that the converse holds: if $E_1 \not\leq_B E_I$, then I is Polishable. This was proved by Solecki.

FIRST DICHOTOMY THEOREM FOR BOREL IDEALS (Solecki [1996]). *Let I be a Borel ideal on $\mathbb{N}$. Then $E_1 \leq_B E_I$ or I is Polishable.*

Solecki [1996] further analyzed the structure of Polishable ideals, showing in particular:

(i) The above theorem holds even for $I \in \mathbf{\Sigma}_1^1$.
(ii) The Polishable ideals are exactly the p-ideals.
(iii) They are characterized, in an appropriate sense, by submeasures on $p(\mathbb{N})$.
(iv) They are all $\mathbf{\Pi}_3^0$.

Some important Polishable ideals arise naturally in this study. The first one is I_3, defined by:

$$\begin{aligned} I_3 &= \{A \subseteq \mathbb{N}^2 : \forall m (A_m \text{ is finite})\} \\ &\ \ (= \emptyset \times \text{FIN}), \end{aligned}$$

where $A_m = \{n : (m, n) \in A\}$. Put

$$E_3 = E_{I_3}.$$

Up to some trivial identifications, $E_3 = E_0^{\mathbb{N}}$. We now have:

SECOND DICHOTOMY THEOREM FOR BOREL IDEALS (Solecki [1996]). *Let I be a Polishable ideal on $\mathbb{N}$. Then*

$$I \in F_\sigma \text{ or } E_3 \leq_B E_I.$$

Using Solecki's work and results of Hjorth on turbulence (mentioned below) Kechris [1996] characterized all Borel ideals I for which $E_I \leq_B E_3$, from which it followed as a corollary that for $\mathrm{FIN} \subseteq I \subsetneqq p(\mathbb{N})$,

$$E_I \leq_B E_3 \Leftrightarrow E_I \sim_B E_0 \text{ or } E_I \sim_B E_3.$$

This suggested the possibility of a local dichotomy analogous to that for E_1.

Conjecture 2. *Let E be a Borel equivalence relation. If $E \leq_B E_3$, then*

$$E \leq_B E_0 \text{ or } E \sim_B E_3.$$

What can be said for Polishable $\mathbf{\Sigma}^0_2$ ideals? Beyond FIN, the simplest example is the following so-called *summable* ideal:

$$I_2 = \left\{ A \subseteq \mathbb{N} : \sum_{n \in A} 1/(n+1) < \infty \right\}.$$

(The use of the exact sequence $1/(n+1)$ is not important here. Any sequence $\{a_n\}$ of positive numbers for which $\sum_{n=0}^{\infty} a_n = \infty$ would give an ideal I with $E_I \sim_B E_{I_2}$, so equivalent to I_2 for our purposes.)

The following has been conjectured by Kechris and Mazur.

Conjecture 3. *Let I be an F_σ Polishable ideal,* $\mathrm{FIN} \subseteq I \subsetneqq p(\mathbb{N})$. *Then*

$$E_I \sim_B E_0 \text{ or } E_2 \leq_B E_I.$$

In particular, this implies that if $E_I \leq_B E_2$, then $E_I \leq_B E_0$ or $E_I \sim_B E_2$, and again leads to a further conjecture.

Conjecture 4. *Let E be a Borel equivalence relation. If $E \leq_B E_2$, then $E \leq_B E_0$ or $E \sim_B E_2$.*

Recently Hjorth very nearly verified this conjecture:

Fourth Dichotomy Theorem (Hjorth [1996]). *Let E be a Borel equivalence relation. If $E \leq_B E_2$, then*

$$E \leq_B E_\infty \text{ or } E \sim_B E_2.$$

By results of Solecki [1996] this implies that for Borel ideals $\mathrm{FIN} \subseteq I \subsetneqq p(\mathbb{N})$, if $E_I \leq_B E_2$, then $E_I \sim_B E_0$ or $E_I \sim_B E_2$.

§7. Actions of S_∞. An important class of equivalence relations are those induced by Borel actions of the infinite symmetric group S_∞, the Polish group of permutations of $\mathbb{N}$, and its closed subgroups. By the results of Becker-Kechris [1996] these are represented exactly by the isomorphism relation on the countable models of an $L_{\omega_1\omega}$ sentence. The analysis of Borel equivalence relations induced by Borel actions of closed subgroups of S_∞ has been undertaken in Hjorth-Kechris [1996], Hjorth-Kechris-Louveau [1996] using the descriptive measure of complexity of a Borel equivalence relation E in terms of its potential Wadge class. We say that E is *potentially of a given Wadge class* Γ if for some F in Γ we have $E \leq_B F$, and we define the *potential Wadge class* of E to be the smallest such class Γ. We have already mentioned that for arbitrary Borel E:

(i) E is potentially $\mathbf{\Pi}^0_2 \Leftrightarrow E \leq_B \mathbb{R}$.

We now have:

(ii) (Hjorth-Kechris [1996]). If E is induced by a Borel action of a closed subgroup of S_∞, then

$$E \text{ is potentially } \mathbf{\Sigma}^0_2 \Leftrightarrow E \leq_B E_\infty,$$

(iii) (Hjorth-Kechris-Louveau [1996]). If E is as in (ii), the next possible potential Wadge class for E is $\mathbf{\Pi}^0_3$, and moreover

$$E \text{ is potentially } \mathbf{\Pi}^0_3 \Leftrightarrow E \leq_B E_3^*,$$

where E_3^* is the following Borel equivalence relation on $(2^{\mathbb{N}})^{\mathbb{N}}$

$$(x_n)E_3^*(y_n) \Leftrightarrow \{x_n : n \in \mathbb{N}\} = \{y_n : n \in \mathbb{N}\},$$

so that $(2^{\mathbb{N}})^{\mathbb{N}}/E_3^*$ is essentially $p_{\aleph_0}(2^{\mathbb{N}}) = \{A \subseteq 2^{\mathbb{N}}\colon A \text{ is countable}\}$.

The simplest known example of an E whose potential Wadge class is $\mathbf{\Pi}^0_3$ is again E_3. So this leads to:

Conjecture 5. *For every Borel equivalence relation E induced by a Borel action of a closed subgroup of S_∞, we have*

$$E \leq_B E_\infty \text{ or } E_3 \leq_B E.$$

Note that the failure of the first alternative says exactly that the potential Wadge class of E is at least $\mathbf{\Pi}^0_3$.

§8. Actions of general Polish groups. As discussed in Becker-Kechris [1996], every equivalence relation of the form E^X_G, whose G is a closed subgroup of S_∞ is $\sim_B$ to an equivalence relation of the form $E^X_{S_\infty}$. We will use the notation $E \leq_B E(S_\infty)$ (or $E \sim_B E(S_\infty)$) to denote the fact that $E \leq_B E^X_{S_\infty}$ (or $E \sim_B E^X_{S_\infty}$) for some Borel S_∞-space X.

Having $E \leq_B E(S_\infty)$ essentially means that E can be classified by countable structures up to isomorphism (for instance linear orderings, groups, rings, fields, and so on), so it is quite interesting to understand for a given E whether $E \leq_B E(S_\infty)$ is possible or not. For the case $E = E^X_G$ the situation has been analyzed in Hjorth [1995], and given a more complete exposition in Kechris [1996a], using the notions of *local orbit* and *turbulence*.

Suppose a Polish group G acts continuously on a Polish space X, a situation we summarize by saying that X is a *Polish G-space*. Fix an open set $U \subseteq X$ and an open symmetric neighborhood V of $1 \in G$, and define the following reflexive, symmetric relation $R_{U,V}$ on U:

$$xR_{U,V}y \Leftrightarrow x, y \in U \,\&\, \exists g \in V(g \cdot x = y).$$

For $x \in U$, let

$$\begin{aligned}\mathcal{O}(x, U, V) &= \text{ the connected component of } x \text{ in } R_{U,V} \\ &= \{y : \exists x_0, \dots, x_n(x_0 = x, x_n = y, \forall i < n(x_i R_{U,V} x_{i+1}))\}.\end{aligned}$$

This is called the (U, V)-*local orbit* of x, since if $U = X, V = G$ this is the usual orbit of x. We say that the action, or more loosely the G-space X, is *turbulent* if every orbit is dense and meager and every local orbit is somewhere dense, in the sense that every $\overline{\mathcal{O}(x, U, V)}$ has nonempty interior.

For example, no continuous action of a closed subgroup of S_∞ or a Polish locally compact group is turbulent (Hjorth). Under the necessarily unique topology witnessing that (I_2, Δ) is Polishable, we have that $p(\mathbb{N})$ is a Polish I_2-space under translation and it is in fact turbulent. Thus, in particular, E_2 arises as the orbit equivalence relation of a turbulent action.

Hjorth [1995] shows:

THEOREM 8.1 (Hjorth [1995]). *If the Polish group G acts continuously on the Polish space X and the action is turbulent, then $E_G^X \not\leq_B E(S_\infty)$.*

And goes on to conjecture:

CONJECTURE 6. *Let G be a Polish group and X a Borel G-space. Then either*

(i) *There is a turbulent G-space Y with $E_G^Y \leq_B E_G^X$,*

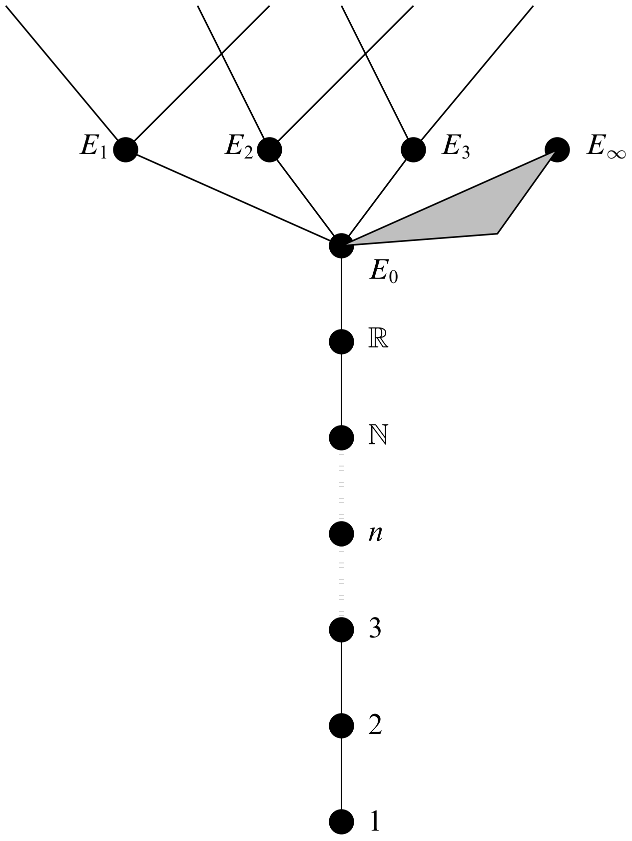
or

(ii) $E_G^X \sim_B E(S_\infty)$.

Note that this conjecture is extremely strong as it implies that the Topological Vaught Conjecture is equivalent to the Vaught Conjecture for $L_{\omega_1\omega}$ theories (see Becker-Kechris [1996] for a discussion of the Vaught Conjecture). It has been, however, partially verified.

FIFTH DICHOTOMY THEOREM (Hjorth [1995]). *Let G be a sufficiently "nice" Polish group, for instance one admitting an invariant metric. Then either there is a turbulent G-space Y with $E_G^Y \leq_B E_G^X$ or else $E_G^X \sim_B E(S_\infty)$. This holds for any Polish group if $\sim_B$ is replaced by $\sim_\Gamma$, when Γ is the class of provably $\mathbf{\Delta}^1_2$ functions.*

The structure of E_G^X, when X is a turbulent G-space is as yet poorly understood. The simplest example of such an E_G^X seems to be E_2, which suggests the possibility that $E_2 \leq_B E_G^X$, for any turbulent G-space X. This was however disproved by Hjorth [1996], who showed that if $G = c_0, X = \mathbb{R}^{\mathbb{N}}$, and the action is left-translation, then this action is turbulent but $E_2 \not\leq_B E_{c_0}^{\mathbb{R}^{\mathbb{N}}}$. It is still however possible that there is a small, even finite, collection of Borel equivalence relations E induced by turbulent actions such that every E_G^X, for a turbulent G-space X, is $\geq_B$ to one of them. The simplest such collection not known to fail is $\{E_2, E_{c_0}^{\mathbb{R}^{\mathbb{N}}}\}$, so that leads to our last, perhaps overly optimistic, conjecture:



CONJECTURE 7. *Let G be a Polish group acting continuously on a Polish space X. If the action is turbulent, then*

$$E_2 \leq_B E_G^X \text{ or } E_{c_0}^{\mathbb{R}^{\mathbb{N}}} \leq_B E_G^X.$$

Since it is known that $E_3 \leq_B E_{c_0}^{\mathbb{R}^{\mathbb{N}}}$ it would follow from this conjecture that for any turbulent Polish G-space X,

$$E_2 \leq_B E_G^X \text{ or } E_3 \leq E_G^X.$$

§9. Towards a global picture. The preceding results and conjectures together imply the following picture concerning the structure of $\leq_B$:

In other words, the equivalence relations E_1, E_2, E_3 are minimal above E_0 and incomparable with each other and E_∞, and every Borel equivalence relation is either $\leq_B E_\infty$ or else $\geq_B$ one of E_1, E_2, E_3. The shaded area denotes the Borel equivalence relations E with $E_0 \leq_B E \leq_B E_\infty$.

One arrives at this picture as follows: First, E_1 is minimal above E_0 by the Second Dichotomy Theorem. Next E_2 is minimal above E_0 by Conjecture 4 and E_3 is minimal above E_0 by Conjecture 2. Moreover $E_1 \nsim_B E_2$ by Theorem 5.1, $E_2 \nsim_B E_3$ by Theorem 8.1, and $E_1 \nsim_B E_3$ by Theorem 5.1. Finally E_1, E_2, E_3 are each incomparable with E_∞ by §5 (for E_1), Theorem 8.1 and Conjecture 4 (for E_2), Louveau [1994] and Conjecture 2 (for E_3).

Now let E be an arbitrary Borel equivalence relation. If $E_1 \not\leq_B E$, then, by Conjecture 1, $E \sim_B E_G^X$ for a Polish group G and Borel G-space X. By Conjecture 6, either $E_G^Y \leq E_G^X$ for a turbulent G-space Y, in which case $E_2 \leq_B E$ or $E_3 \leq_B E$ by Conjecture 7 and the remark following it, or else $E_G^X \sim_B E(S_\infty)$. But then, by Conjecture 5, $E \leq_B E_\infty$ or $E_3 \leq_B E$.

Remark. Solecki [1996] has shown that for a Borel ideal I, if $E_I \leq_B E_\infty$, then $E_I \leq_B E_0$, so the above picture implies that for the equivalence relations of the form E_I, I a Borel ideal, we have $E_I \leq_B E_0$ or else $E_1 \leq_B E_I$ or $E_2 \leq_B E_I$ or $E_3 \leq_B E_I$.

We readily admit that this final global picture is rather optimistic and arises as the conjunction of a sequence of already bold conjectures, any of which may fail. However, it serves the purpose of crystallizing the boundary of our present knowledge and suggesting some concrete test problems on which further progress rests.

§10. New dichotomies. Here we prove the following two new dichotomies, of which the first proves Conjecture 2 and the second partially verifies Conjecture 5.

Sixth Dichotomy Theorem. *Let E be a Borel equivalence relation. If $E \leq_B E_3$, then*

$$E \leq_B E_0 \text{ or } E \sim_B E_3.$$

Seventh Dichotomy Theorem. *Let $G \subseteq S_\infty$ be a closed subgroup of S_∞ admitting an invariant metric. If X is a Borel G-space and E_G^X is Borel, then* *for any* $E \leq_B E_G^X$

$$E \leq_B E_\infty \text{ or } E_3 \leq_B E.$$

In terms of the picture from §9, we have already seen that E_1, E_2, E_3 are incomparable with each other and that E_1 is minimal above E_0. By the Sixth Dichotomy Theorem we now have that E_3 is minimal above E_0, and so the only question of this nature still to be decided is the truth of Conjecture 4, to the effect that E_2 is minimal above E_0. Indeed, since E_1 and E_3 are known to be incomparable with E_∞, we might describe the current situation by saying that Conjecture 4 is the only issue from the lower part of the global picture still unresolved.

The Seventh Dichotomy Theorem has as a consequence that no equivalence relation of the form E_G^X with G a closed subgroup of S_∞ admitting an invariant metric, can provide a counterexample to the global conjecture proposed in §9.

§11. Sketches of proofs, I. We will first deal with the Sixth Dichotomy Theorem. We argue that it is enough to prove the following two results.

THEOREM 11.1. *Let $G_i, i = 1, 2, \dots,$ be closed subgroups of S_∞, put $G = \prod_{i=1}^\infty G_i, G^n = \prod_{i=1}^n G_i$ and let X be a Borel G-space. If $E \leq_B E_G^X$ and $E \leq_B E_\infty$, then there are Borel G^n-spaces $Y_{n,m}$, so that $E \leq_B \bigoplus_{n,m} E_{G_n}^{Y_{n,m}}$, where $\bigoplus$ denotes direct sum.*

THEOREM 11.2. *Let E be a Borel equivalence relation. If $E \leq_B E_\infty^{\mathbb{N}}$, then*

$$E \leq_B E_\infty \text{ or } E_3 \sqsubseteq_c E,$$

where $\sqsubseteq_c$ means that there is an injective continuous reduction.

To see that these suffice, assume that $E \leq_B E_3$. Since $E_3 \sim_B E_0^{\mathbb{N}}$, we have $E \leq_B E_0^{\mathbb{N}} \leq_B E_\infty^{\mathbb{N}}$, so, by Theorem 11.2, either $E_3 \sqsubseteq_c E$, thus, in particular, $E_3 \leq_B E$, or else $E \leq_B E_\infty$. If the last alternative holds, we have that $E \leq_B E_0^{\mathbb{N}}$ and $E \leq_B E_\infty$.

As discussed in Dougherty-Jackson-Kechris [1994] $E_0 \sim E_{\mathbb{Z}}^{2^{\mathbb{Z}}}$, where the action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ is simply the shift-action. So $E \leq_B E_0^{\mathbb{N}} \sim_B (E_{\mathbb{Z}}^{2^{\mathbb{Z}}})^{\mathbb{N}} \sim_B E_{\mathbb{Z}^{\mathbb{N}}}^X$, where $X = (2^{\mathbb{Z}})^{\mathbb{N}}$ and $\mathbb{Z}^{\mathbb{N}}$ acts on $(2^{\mathbb{Z}})^{\mathbb{N}}$ coordinatewise. Since $\mathbb{Z}$ is a closed subgroup of S_∞, Theorem 11.1 implies that E can be Borel reduced to the direct sum of a sequence of equivalence relations of the form $E_{\mathbb{Z}^n}^Y$. As discussed in Jackson-Kechris-Louveau [∞], it is a theorem of Weiss that any orbit equivalence relation associated to a Borel $\mathbb{Z}^n$-space is hyperfinite, i.e., $\leq_B E_0$. It then follows that $E \leq_B E_0$.

We now sketch the proof of Theorem 11.1. Theorem 11.2 is a special case of the Seventh Dichotomy Theorem, which we discuss in §12, but it can be also given an independent simpler proof.

PROOF OF THEOREM 11.1 (sketch). Fix a basis $\{U_k^n\}_{k\in\mathbb{N}}$ for each G^n, invariant under right multiplication. This uses the fact that each G_i is a closed subgroup of S_∞; see Becker-Kechris [1996]. Identifying U_k^n with $U_k^n \times G_{n+1} \times G_{n+2} \times \cdots$, $\{U_k^n\}_{k,n\in\mathbb{N}}$ is a basis for G, and for each n, $\{U_k^n\}_{k\in\mathbb{N}}$ is invariant under right multiplication by G. If E lives on the Polish space Z, fix a basis $\{W_m\}$ for Z, and if E_∞ lives on Y, fix a basis $\{N_n\}$ for Y. We can assume that there are *continuous* $f : Z \to X$, $h : Z \to Y$ witnessing that $E \leq_B E_G^X$, $E \leq_B E_\infty$.

Put

$$P(x, g, z, y) \Leftrightarrow f(z) = g \cdot x \,\&\, h(z) = y;$$

so that $P \subseteq X \times G \times Z \times Y$ is closed. Clearly for $x \in [f(z)]_{E_G^X}$, $\mathrm{proj}_Y(P(x))$ is a $\mathbf{\Sigma}_1^1$ non-$\emptyset$ countable subset of Y, so there are n, k, m, p with $\mathrm{proj}_Y(P(x)\cap (U_k^n \times W_m \times N_p)) = \{y\}$ for some y. Then for all $N_q \subseteq N_p$,

$$y \notin N_q \Leftrightarrow \ \mathrm{proj}_Y\ (P(x) \cap (U_k^n \times W_m \times N_q)) = \emptyset.$$

Put

$$R(x, n, k, m, p) \Leftrightarrow \ \mathrm{proj}_Y\ (P(x) \cap (U_k^n \times W_m \times N_p)) = \emptyset.$$

This is $\mathbf{\Pi}_1^1$ and $\sim$-invariant, where $\sim$ is the following $\mathbf{\Sigma}_1^1$-equivalence relation:

$$\begin{aligned}(x, n, k, m, p) \sim (x', n', k', m', p') \Leftrightarrow & m = m' \,\&\, p = p' \,\&\, n' = n \,\&\\ & \exists g_0 (g_0^{-1} \cdot x = x' \,\&\, U_k^n g_0 = U_{k'}^n).\end{aligned}$$

Notice that we are using here the invariance of $\{U_k^n\}_{k\in\mathbb{N}}$ under right multiplication. It then follows by a theorem of Solovay, discussed in Kechris [1995], that there is an $\sim$-invariant $\mathbf{\Pi}_1^1$-rank $\varphi : R \to \omega_1$. It follows that

$$\begin{aligned}\forall z \forall y \in [h(z)]_{E_\infty} \exists \alpha < \omega_1^{z,a} \exists n, k, m, p[y \in N_p \,\&\, \forall q (N_q \subseteq N_p \Rightarrow \\ (y \in N_q \Leftrightarrow \varphi(f(z), n, k, m, q) \geq \alpha)],\end{aligned}$$

where $a \in \mathbb{N}^\mathbb{N}$ is an appropriate fixed parameter, independent of z. Using boundedness, it follows that there is a fixed $\alpha_0 < \omega_1$ so that

$$\begin{aligned}\forall z \exists n \exists y \in [h(z)]_{E_\infty} \exists \alpha < \alpha_0 \exists k, m, p[y \in N_p \,\&\, \forall q (N_q \subseteq N_p \Rightarrow \\ (y \in N_q \Leftrightarrow \varphi(f(z), n, k, m, q) \geq \alpha)].\end{aligned}$$

Put

$$\begin{aligned}Z_{n,\alpha,m,p} = \{z : \exists y \in [h(z)]_{E_\infty} \exists k[y \in N_p \,\&\, \forall q (N_q \subseteq N_p \Rightarrow \\ (y \in N_q \Leftrightarrow \varphi(f(z), n, k, m, q) \geq \alpha)]\}.\end{aligned}$$

Then, $Z_{n,\alpha,m,p}$ is Borel E-invariant and $\bigcup_{n,\alpha<\alpha_0,m,p} Z_{n,\alpha,m,p} = Z$, so it is enough to find a Borel G^n-space Y_n with $E|Z_{n,\alpha,m,p} \leq_B E_{G_n}^{Y_n}$. We simply take $Y_n =$

$(Y \oplus \{\infty\})^{\{U_k^n\}_{k\in\mathbb{N}}}$, where ∞ is a symbol not in Y and we let G^n act on Y_n by the shift action

$$g \cdot H(U_k^n) = H(U_k^n g).$$

Clearly this is a Borel G^n-space. It is now easy to check that the function $Q = Q_{n,\alpha,m,p}$ defined below witnesses that $E|Z_{n,\alpha,m,p} \leq_B E_{G^n}^{Y_n}$: $Q(z)(U_k^n)$ is the unique $y \in [h(z)]_{E_\infty}$ such that $y \in N_p$ and $\forall q(N_q \subseteq N_p \Rightarrow [y \in N_q \Leftrightarrow \varphi(f(z), n, k, m, q) \geq \alpha])$, if such exists; otherwise $Q(z)(U_k^n) = \infty$.

§12. Sketches of proofs, II. We will sketch here the proof of the Seventh Dichotomy Theorem.

First we note the following fact:

LEMMA 1. *If $G \subseteq S_\infty$ is a closed subgroup of S_∞ admitting an invariant metric, then there is a sequence G_n of countable (discrete) groups such that G is isomorphic (as a topological group) to a closed subgroup of $\prod_n G_n$.*

PROOF. The hypothesis allows us to find a nbhd basis $\{U_n\}$ of $1 \in G$ consisting of open normal subgroups. Let $\Omega_n = G/U_n$. Then the canonical action of G on Ω_n gives a homomorphism π_n of G onto a countable subgroup G_n of the symmetric group on Ω_n and $\pi = (\pi_n)$ gives an isomorphism of G with a closed subgroup of $\prod_n G_n$. ⊣

By a result of Mackey (see Becker-Kechris [1996, 2.3.5]), if G is a closed subgroup of H then for any Borel G-space X there is a Borel H-space Y with $E_G^X \sim_B E_H^Y$, so it is enough to prove the result for G a countable product of countable groups and thus, without loss of generality, for the group $G = H^{\mathbb{N}}$, where H is the direct sum of countably many copies of the free group on $\aleph_0$ generators. (This group H has the technical advantage that $H^{n+1} \cong H$ for any $n \in \mathbb{N}$.)

So fix this $G = H^{\mathbb{N}}$ and a Borel G-space X with E_G^X Borel and let $E \leq_B E_G^X$. It is not hard to see that we can assume that the Borel reduction of E into E_G^X is 1-1. If $\mathcal{A}_0$, a structure in some language L_0 with universe $\mathbb{N}$, has automorphism group, $\mathrm{Aut}(\mathcal{A}_0)$, isomorphic to G, then by Becker-Kechris [1996, pp. 31–32] the G-space X is Borel embeddable in the relativized logic action $J_{L_0\cup L}^{\mathcal{A}_0}$ (L a countable language disjoint from L_0) of $\mathrm{Aut}(\mathcal{A}_0)$ on $Y_{L_0\cup L}^{\mathcal{A}_0} = \{\mathcal{M} \in X_{L_0\cup L} : \mathcal{M}|L_0 = \mathcal{A}_0\}$, with $X_{L_0\cup L}$ denoting the Polish space of $L_0 \cup L$-structures with universe $\mathbb{N}$. If Y is the range of the embedding, then E_G^X is Borel isomorphic to $\cong |Y$. If Z is the closure of Y under isomorphism in the space $X_{L_0\cup L}$, then it is easy to check that Z is Borel and $\cong |Z$ is Borel. In particular, there is a sentence $\sigma \in (L_0 \cup L)_{\omega_1\omega}$ with $Z = \mathrm{Mod}(\sigma) = \{\mathcal{M} \in X_{L_0\cup L} : \mathcal{M} \models \sigma\}$ (Lopez-Escobar; see Kechris [1995]).

We now choose a particular such $\mathcal{A}_0$, which is technically convenient for our purposes. Let

$$\mathcal{A}_0 = \left\langle \bigcup_{n=0}^{\infty} H^{n+1}, \{Q_n^{\mathcal{A}_0}\}_{n\in\mathbb{N}}, \{F_h^{\mathcal{A}_0}\}_{h\in H}, \{p_{ij}^{\mathcal{A}_0}\}_{0\leq i<j} \right\rangle$$

with $Q_n^{\mathcal{A}_0}(a) \Leftrightarrow a \in H^{n+1}, F_h^{\mathcal{A}_0}((g_0,\dots,g_n)) = (g_0(h_0^n)^{-1},\dots,g_n(h_n^n)^{-1})$, with $h \mapsto (h_0^n,\dots,h_n^n)$ an isomorphism of H with H^{n+1}, $p_{ij}^{\mathcal{A}_0}((g_0,\dots,g_n)) = 1$ if $j \neq n; = (g_0,\dots,g_i)$, if $j = n$. By a simple coding we can assume that the universe of $\mathcal{A}_0$ is $\mathbb{N}$. It is not hard to check that the map $g \mapsto \rho_g$, where if $g = (h_0, h_1, \dots)$ then $\rho_g((g_0,\dots,g_n)) = (h_0g_0,\dots,h_ng_n)$, is an isomorphism of G with $\mathrm{Aut}(\mathcal{A}_0)$. A key technical property of $\mathcal{A}_0$ is that for every $n, k \geq n$ every element of $(Q_n)^{\mathcal{A}_0}$ is definable by a term from any element of $(Q_k)^{\mathcal{A}_0}$.

Also fix for each n an element $p_n \in (Q_n)^{\mathcal{A}_0}$. Using these we can define a homomorphism $\pi : G \to S_\infty$ so that $\pi(G) \supseteq \mathrm{Aut}(\mathcal{A}_0)$ and for $g = (h_0, h_1, \dots)$, $\pi(g)$ acts on $(Q_n)^{\mathcal{A}_0}$ by $h_n \cdot F_h(p_n) = F_{hh_n^{-1}}(p_n)$, so that this depends only on h_n. Put $\pi_g = \pi(g)$.

To summarize: We fix $\mathcal{A}_0$ as before (in the language of L_0), a sentence $\sigma \in (L_0 \cup L)_{\omega_1\omega}$ (L some appropriate language disjoint from L_0), so that $\cong | \mathrm{Mod}(\sigma)$ is Borel and every $\mathcal{M} \in \mathrm{Mod}(\sigma)$ is isomorphic to an expansion of $\mathcal{A}_0$, a Borel injection $f : W \to \mathrm{Mod}(\sigma)$, where E lives on W, such that $xEy \Leftrightarrow f(x) \cong f(y)$ and $f[W] = X_0 \subseteq \{\mathcal{M} \in \mathrm{Mod}(\sigma) : \mathcal{M}|L_0 = \mathcal{A}_0\}$.

By relativization we can assume that all these data are effective, i.e., L is recursive (clearly L_0 is too), $\sigma \in L_{\omega_1^{ck}}, \cong | \mathrm{Mod}(\sigma), f, X_0$ are Δ_1^1, and $\{g \in G : \pi_g \in \mathrm{Aut}(\mathcal{A}_0)\}$ admits a countable dense set consisting of recursive elements.

Below we use standard notions from the theory of the logics $\mathcal{L}_{\omega_1\omega}$, like fragments, quantifier rank, etc., as in Barwise [1975]. For us *fragment* will always mean *countable* fragment. For $\mathcal{M} = \langle \mathcal{M}, -\rangle$ an $\mathcal{L}$-structure, $F \subseteq \mathcal{L}_{\omega_1\omega}$ a fragment and $\bar{a} \in M^{<\mathbb{N}}$, $\mathrm{Th}_F(\mathcal{M}, \bar{a}) = \{\varphi \in F : \mathcal{M} \models \varphi(\bar{a})\}$.

For $A \subseteq \mathrm{Mod}(\sigma), \mathcal{M} \in \mathrm{Mod}(\sigma), \bar{a} \in M^{<\mathbb{N}}$ put

$$(\mathcal{M}, \bar{a}) \models A \Leftrightarrow \exists g \in S_\infty(g(a_i) = a_i \,\&\, g \cdot \mathcal{M} \in A)$$

(where $g \cdot \mathcal{M}$ is the structure we obtain from $\mathcal{M}$ by applying g).

The following is the key concept used in the proof:

Let $F \subseteq (L_0 \cup L)_{\omega_1\omega}$ be a fragment, $A \subseteq \mathrm{Mod}(\sigma), \mathcal{M} \in \mathrm{Mod}(\sigma)$ and $a \in (Q_n)^{\mathcal{M}}, b \in (Q_k)^{\mathcal{M}}, k > n$. We say that A *isolates* $\mathrm{Th}_F(\mathcal{M}, a, b)$ *over* $(\mathcal{M}, a)$ if

(i) $(\mathcal{M}, a, b) \models A$;

(ii) $\forall \mathcal{M}_0 \in \mathrm{Mod}(\sigma) \forall g \in S_\infty(g(a) = a \,\&(\mathcal{M}_0, a, b) \models A \,\&(g\cdot\mathcal{M}_0, a, b) \models A \Rightarrow \mathrm{Th}_F(\mathcal{M}_0, a, b) = \mathrm{Th}_F(g \cdot \mathcal{M}_0, a, b))$.

We say that $\psi \in (L_0 \cup L)_{\omega_1\omega}$ *isolates* $\mathrm{Th}_F(\mathcal{M}, a, b)$ *over* $(\mathcal{M}, a)$ if

(i) $(\mathcal{M}, a, b) \models \psi$;

(ii) $\forall \mathcal{M}_0 \in \mathrm{Mod}(\sigma) \forall a_0 \in (Q_n)^{\mathcal{M}_0} \forall b_0, b_0' \in (Q_k)^{\mathcal{M}_0}[(\mathcal{M}_0, a_0, b_0) \models \psi \;\&\; (\mathcal{M}_0, a_0, b_0') \models \psi \Rightarrow \mathrm{Th}_F(\mathcal{M}_0, a_0, b_0) = \mathrm{Th}_F(\mathcal{M}_0, a_0, b_0',)]$.

We now consider two cases:

CASE I. $\forall$ fragments $F \subseteq (L_0 \cup L)_{\omega_1^{ck}}, F \in L_{\omega_1^{ck}} \forall \mathcal{M} \in X_0 \exists n \exists a \in (Q_n)^{\mathcal{M}} \forall k > n \forall b \in (Q_k)^{\mathcal{M}} \exists A \in \Sigma_1^1$ (A isolates $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$).

In this case we show that $E \leq_B E_\infty$. This is done by using the case hypothesis and reflection arguments to find an increasing transfinite sequence $\{F_\alpha\}_{\alpha<\omega_1^{ck}}$ of fragments Δ_1-definable over $L_{\omega_1^{ck}}$ such that for any $\alpha < \omega_1^{ck}$, and any $\mathcal{M} \in X_0$ there is n and $a \in (Q_n)^{\mathcal{M}}$, so that $\forall k > n \forall b \in (Q_k)^{\mathcal{M}} \exists \psi \in F_\alpha$ (ψ isolates $\mathrm{Th}_{F_{<\alpha}}(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$), where $F_{<\alpha} = \bigcup_{\beta<\alpha} F_\beta$.

We then show, by an argument reminiscent of the Scott analysis, that for any $\alpha < \omega_1^{ck}$, if $\mathcal{M}, \mathcal{M}_0 \in X_0$ and $a \in (Q_n)^{\mathcal{M}}, a_0 \in (Q_n)^{\mathcal{M}}$ have the above property, then $\mathrm{Th}_{F_\alpha}(\mathcal{M}, a) = \mathrm{Th}_{F_\alpha}(\mathcal{M}_0, a_0)$ implies that $(\mathcal{M}, a), (\mathcal{M}_0, a_0)$ agree on all formulas of $(L_0 \cup L)_{\omega_1\omega}$ of rank α.

Since $\cong |\,\mathrm{Mod}(\sigma)$ is Δ_1^1, it follows by Becker-Kechris [1996, 7.1,4], that there is $\alpha_0 < \omega_1^{ck}$, so that if $\mathcal{M}, \mathcal{M}_0 \in \mathrm{Mod}(\sigma)$ and $\mathcal{M}, \mathcal{M}_0$ agree on formulas of rank α_0, then $\mathcal{M} \cong \mathcal{M}_0$.

Using the Kreisel Selection Theorem, there is a Δ_1^1 function assigning to each $\mathcal{M} \in X_0$ some $n_{\mathcal{M}} \in \mathbb{N}$ and $a_{\mathcal{M}} \in (Q_{n_{\mathcal{M}}})^{\mathcal{M}}$ so that $a_{\mathcal{M}}$ has the above property with respect to α_0. Then for $\mathcal{M} \in X_0$ put $U(\mathcal{M}) = \mathrm{Th}_{F_{\alpha_0}}(\mathcal{M}, a_{\mathcal{M}})$. This is a Δ_1^1 function from X_0 into $2^{F_{\alpha_0}}$ and $U(\mathcal{M}) = U(\mathcal{M}_0) \Rightarrow \mathcal{M} \cong \mathcal{M}_0$. Moreover $\{U(\mathcal{M}_0) : \mathcal{M}_0 \in X_0 \,\&\, \mathcal{M} \cong \mathcal{M}_0\}$ is countable. This implies, see, e.g., Hjorth [1996], that $\cong |X_0 \leq_B E_\infty$, thus $E \leq_B E_\infty$.

CASE II. $\exists F \in L_{\omega_1^{ck}} \exists \mathcal{M} \in X_0 \forall n \forall a \in (Q_n)^{\mathcal{M}} \exists k > n \exists b \in (Q_k)^{\mathcal{M}} \forall A \in \Sigma_1^1$ (A does not isolate $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$).

We will then show that $E_0^{\mathbb{N}} \leq_B E$. Let

$$\begin{aligned} Y_0 = \{\mathcal{M} \in \mathrm{Mod}(\sigma) : \mathcal{M}|L_0 = \mathcal{A}_0 \,\&\, \exists \mathcal{N} \in X_0(\mathcal{M} \cong \mathcal{N}) \,\&\, \\ \forall n \forall a \in (Q_n)^{\mathcal{M}} \exists k > n \exists b \in (Q_k)^{\mathcal{M}} \forall A \in \Sigma_1^1 \\ (A \text{ does not isolate } \mathrm{Th}_F(\mathcal{M}, a, b) \text{ over } (\mathcal{M}, a))\}. \end{aligned}$$

Then Y_0 is nonempty, Σ_1^1, and invariant under the action of $\mathrm{Aut}(\mathcal{A}_0)$. We show that $E_0^{\mathbb{N}} \leq_c \cong |Y_0$. Since there is a C-measurable reduction of $\cong |Y_0$ into $\cong |X_0 \sqsubseteq_B E$, it follows that there is a C-measurable reduction of $E_0^{\mathbb{N}}$ into E and thus a continuous reduction of $E_0^{\mathbb{N}}|D$ into E, where D is a comeager subset of $(2^{\mathbb{N}})^{\mathbb{N}}$. Now it can be shown that $E_0^{\mathbb{N}} \sqsubseteq_B E_0^{\mathbb{N}}|D$, so $E_0^{\mathbb{N}} \leq_B E$. (One way to do that is to use the Sixth Dichotomy Theorem, which can be given an independent proof, and Louveau [1994].)

Let $V_n = \{g = (h_0, h_1, \dots) \in G : \pi_g \in \mathrm{Aut}(\mathcal{A}_0) \,\&\, h_0 = h_1 = \cdots = h_n = 1\}$.

We will use in the sequel the following notation: $\langle m, j\rangle$ is the usual Cantor bijection of $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$, given by

$$\langle m, j\rangle = \frac{(m+j)(m+j+1)}{2} + j.$$

Let $L(n) = \max\{k : \exists i(\langle k, i\rangle \leq n)\}$, so that $L(\langle m, 0\rangle) = m$, $L(n) \leq L(n+1)$, $L(n) = L(n-1)$, if $n = \langle m, j\rangle$ with $j > 0$, and $L(n) = L(n-1)+1$, if $n = \langle m, 0\rangle > 0$.

By induction on $n \geq 0$ we can define, using a rather complicated construction and the Gandy-Harrington topology, the following:

(i) Nonempty Σ^1_1 sets A_s, for $s \in 2^{n+1}$, so that $A_\emptyset = Y_0, A_{s\hat{\ }i} \subseteq A_s$, $\operatorname{diam}(A_{x|i}) \to 0$ as $i \to \infty$ for any $x \in 2^{\mathbb{N}}$, and $\bigcap_i A_{x|i} = \{\mathcal{M}_x\}$ is a singleton.

(ii) $k_m \in \mathbb{N}$, for $m \leq L(n)$. These will be chosen so that $0 < k_0 < k_1 < \cdots$ and we will also have:

(iii) $\mathcal{M} \in A_{0^{n+1}} \Rightarrow \forall r \leq L(n) \forall a \in (Q_r)^{\mathcal{M}} \exists b \in (Q_{k_r})^{\mathcal{M}} \forall A \in \Sigma^1_1$ (A does not isolate $\mathrm{Th}_F(\mathcal{M}, a, b)$ over $(\mathcal{M}, a)$).

(iv) $A_s, s \in 2^{n+1}$, is invariant under $\pi(V_{k_{L(n)}})$.

(v) $g_s \in H^{\mathbb{N}}$, for $s \in 2^{n+1}$, with $g_{0^{n+1}} = 1, g_s$ recursive, and $\pi_{g_s} \in \mathrm{Aut}(\mathcal{A}_0)$.

(vi) (*links*) We will also have $\pi_{g_s} \cdot A_{0^{n+1}} = A_s$, for $s \in 2^{n+1}$.

(vii) (*positive requirements*) For $s, t \in 2^{n+1}$, put $g_{s,t} = g_t g_s^{-1}$. If $\bar{n} < n$ and $(\bar{s}, \bar{t}) \subseteq (s, t)$ with $\bar{s}, \bar{t} \in 2^{\bar{n}+1}$, then we will have for any $\ell \leq L(n)$:

$$[\forall \bar{\ell} \leq \ell \forall \langle \bar{\ell}, i\rangle \in (n+1) \setminus (\bar{n}+1)(s(\langle \bar{\ell}, i\rangle = t(\langle \bar{\ell}, i\rangle))] \Rightarrow g_{s,t} \equiv_\ell g_{\bar{s},\bar{t}},$$

where for $g, h \in G^{\mathbb{N}}, \ell \in \mathbb{N}$ we let

$$g \equiv_\ell h \Leftrightarrow \forall i \leq \ell(g_i = h_i).$$

(viii) (*negative requirements*) Fix a recursive enumeration $\{\tilde{h}_0, \tilde{h}_1, \dots\}$ of H with $\tilde{h}_0 = 1$. If $s, t \in 2^{n+1}, n = \langle m, j\rangle$, then we must have

$$s(n) \neq t(n) \Rightarrow (\mathcal{M} \in A_s \,\&\, g \in H^{\mathbb{N}} \,\&\, \pi_g \in \mathrm{Aut}(\mathcal{A}_0)\&$$
$$g(m), g(k_m) \in \{\tilde{h}_0, \dots, \tilde{h}_n\} \Rightarrow \pi_g \cdot \mathcal{M} \notin A_t).$$

Once these have been constructed, it follows that $x \mapsto \mathcal{M}_x$ is continuous and

$$x E_0^{\mathbb{N}} y \Leftrightarrow \mathcal{M}_x \cong \mathcal{M}_y,$$

so, as $\mathcal{M}_x \in Y_0$, this shows that $E_0^{\mathbb{N}} \leq_c \cong | Y_0$.

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