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DEDUCTIVE COMPLETENESS

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Abstract. This is an exposition of Lambek's strengthening and generalization of the deduction theorem in categories related to intuitionistic propositional logic. Essential notions of category theory are introduced so as to yield a simple reformulation of Lambek's Functional Completeness Theorem, from which its main consequences can be readily drawn. The connections of the theorem with combinatory logic, and with modal and substructural logics, are briefly considered at the end.

Introduction. On its own, a difficult proof is not a sufficient condition for a good theorem. Neither is it a necessary condition. As a matter of fact, it is normally better to have a simple proof, if such is available. Many a theorem started by having a difficult proof, until the notions involved were clarified enough, and the theorem perhaps reformulated, so as to obtain a simple proof. However, some important theorems always had a simple proof. Such is the theorem that $\sqrt{2}$ is not rational, or Cantor's theorem that the real interval [0, 1] is not denumerable, or again Cantor's theorem on the cardinality of the power set.

The deduction theorem, too, is of that kind: it always had a very simple proof. It is one of the first things one has to prove in an elementary logic course, and both the teacher and the pupil do it easily. This is nevertheless an important theorem.

Its importance is perhaps the greatest for a field of logic where, except at the very beginning, it is hardly ever mentioned: classical model theory. Once we have established strong completeness and compactness, we usually need not bother any more about the consequence relation, and can work exclusively with ordinary Hilbert systems. But this is because the deduction theorem holds.

However, the deduction theorem belongs rather to proof theory than to model theory. And there, it is a completeness theorem of some sort. It tells us that the system is strong enough to mirror its extensions in the same language. The completeness in question is of a syntactical, deductive, sort. For the deductions we shall be able to make in the extension we already have corresponding implications in the system. However, this completeness

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is not such that subsystems or extensions of our system cannot have it. In classical logic we have the deduction theorem, but in intuitionistic logic we have it, too. So the completeness of the deduction theorem does not characterize the system uniquely—at least not without additional requirements, like the requirement that our system be the minimal one possessing this completeness.

It is well known that intuitionistic implication is the minimal one for which the ordinary deduction theorem holds. This fact was discovered by Gentzen, who in [1935] characterized intuitionistic implication by the deduction theorem and modus ponens (the same characterization of intuitionistic implication was also given in [Jaśkowski 1934, §3]). This is a fundamental discovery (like so many of this masterpiece of Gentzen's, which is the *Organon* and *Begriffsschrift* of the theory of deduction). The present paper may be understood as a kind of commentary upon it.

Our concerns will be closer to the spirit of Gentzen's discovery than to the spirit of the first discovery of the deduction theorem, which was made somewhat earlier (see [Tarski 1956, p. 32, footnote]). The next milestone on the way to the subject of this paper is the work of Prawitz in [1965, 1971], which continues Gentzen's investigations of natural deduction. And the subject of the paper is Lambek's result of [1974], which he called the Functional Completeness Theorem. (A more restricted version of the theorem is in [Lambek 1972].)

Lambek showed something interesting for logic, but he showed it in greater generality, as a result about categories related to intuitionistic logic. This was an achievement of his enterprise to introduce categorial methods in proof theory, and proof-theoretical methods in category theory. It explained also the mechanism of the application of the lambda calculus in proof theory.

Lambek's Functional Completeness Theorem shows that the deduction theorem is a superficial aspect of an adjointness involving a system and its extension. This is a strengthening of the deduction theorem, and it leads to a strengthening of Gentzen's characterization of intuitionistic implication through the deduction theorem and modus ponens. It is also a development of Schönfinkel's and Curry's combinatory completeness theorem for combinatory logic. And, like the deduction theorem and the combinatory completeness theorem, it doesn't have a difficult proof (Lambek notes the fact at the end of [1974, §5]). A greater achievement is to formulate the theorem.

As a founding theorem of the field of categorial proof theory, Lambek's theorem deserves a commentary. Lambek's own presentation in [1974] and in the book [Lambek & Scott 1986, Part I] is heuristic, and eminently readable. So, many things need not be rehashed in a commentary such as will be given here. And, especially, not *all* things need, or indeed should, be repeated. Our exposition will only compete with Lambek's by trying to be, from the point of view of category theory, even more pedestrian than his. This paper is not meant for categorists, but for logicians, and we shall reconsider in some detail very elementary concepts of category theory to adapt their definitions to our needs. Eventually, there might be something interesting in that for category theory, too.

Our commentary will be centred around the formulation of Lambek's theorem. There will be no new proof, or indeed any complete proof, of the theorem, here. As we said above, this proof is simple enough, and we shall not repeat every detail of it. Our goal is to reformulate the theorem and draw some consequences of this reformulation. To distinguish this new version of the theorem from Lambek's own, we call it the Deductive Completeness Theorem. It differs from Lambek's Functional Completeness Theorem by referring explicitly to the adjunction implicit in Lambek's formulation. Of that, however, Lambek was fully conscious, and it is presumably because he was addressing logicians that he preferred not to state his theorem as we shall do.

We shall, however, take the risk: after a preparation in the first five parts of the paper, any logician should be able to understand the simple categorial formulation of the Deductive Completeness Theorem in terms of adjointness, given in the sixth part. To compensate for the loss to logic in the formulation, the new name of the theorem should stress better the ties with the perennial concerns of logicians.

The first part of the paper is an introduction to Lambek's brand of sequent systems, which he calls deductive systems. In the second part, one finds, besides Lambek's equational formulation of cartesian categories, an alternative formulation of that kind, in which the ties with cut elimination are stressed. In the third part there is a definition of adjunction, slightly more economical than the most usual one, while the fourth part gives three alternative definitions of comonads. The definitions of these two parts are meant for formulating the Deductive Completeness Theorem, and drawing consequences from it.

The fifth part starts from the ordinary deduction theorem and strengthens it by adding gradually the ingredients of the Deductive Completeness Theorem. The sixth part presents Lambek's formulation of the theorem and the new formulation, and draws some consequences of the latter. It is explained in what sense the extension of a system with a new axiom is contained in the original system when deductive completeness holds. To state that this containment exists is an alternative way to formulate deductive completeness.

Finally, the seventh part deals with three additional topics. It is explained first where implication has been lost on the way from the deduction theorem to the Deductive Completeness Theorem (conjunction is more important for the latter). This is followed by an abstract of some work done in [D. & Petrić 1993] on deductive completeness in a modal and substructural setting. At the end, there is a brief review, without detail, of the ties between deductive completeness and combinatory completeness—a topic that would deserve a separate commentary of its own.

The reader is supposed to have an elementary background and interest in general proof theory, not going beyond [Gentzen 1935] and the main ideas of [Prawitz 1965, 1971]. The background in category theory can be scarcer, and a few basic notions from [Mac Lane 1971] or from the preliminaries of [Lambek & Scott 1986, Part 0] should suffice. But even that is not essential, and a certain taste for category theory could be enough (most, if not all, notions that we need will be defined). The central reference for the theorem that will concern us is the book [Lambek & Scott 1986], through which it was popularized (the text in the book is nearly the same as in [Lambek 1974]). Ours is a commentary on a few chapters from the first part of this book, and so the book should be a constant reference, both where it is and where it is not expressly mentioned.

§1. Deductive systems.

1.1. Sequents singular on both sides. What is usually called a Hilbert system consists in specifying which among all formulae of a language will be the theorems of the system. Gentzen systems, whose particular brands are natural-deduction systems and sequent systems, specify rather a consequence relation among formulae. (We consider again the difference between Hilbert and Gentzen systems in the conclusion of the paper.) A simple kind of sequent system has sequents of the type $A \vdash B$, where the premise A and the conclusion B are formulae. We shall say of such sequents that they are *singular on both sides.*

It is more common to allow a plurality of formulae at least among the premises, on the left-hand side of the turnstile $\vdash$, which parallels the fact that in deduction we may start from more than one premise to reach a conclusion. Premises are collected into sets, or sequences, or multisets, or possibly something else. However, restricting sequents to those singular on both sides need not be an essential limitation if we have connectives enabling us to replace a collection of premises by a synonymous single premise. At least, it is not a limitation for the main things we want to say here (it may be a handicap for cut elimination), while it has the advantage that it permits us to speak about logic in terms of category theory. (If the theory of *multicategories*—in which arrows have sources made of a collection of objects—were more developed, we could rely on it to speak about ordinary sequents, singular just on the right-hand side; cf. Section 2.5.)

A sequent $A \vdash B$ may be read as asserting that there is a natural-deduction proof of B with uncanceled hypotheses collected into A, while we may take that in a Hilbert system we prove sequents of the form $\mathtt{I} \vdash B$, where the propositional constant $\mathtt{I}$ replaces the empty collection of premises. (Of course, usually there is no need to prefix theorems with $\mathtt{I} \vdash$ if every theorem is supposed to have this prefix, but some authors do make a point of prefixing theorems in Hilbert systems at least with $\vdash$.) So, our simple systems with sequents singular on both sides cover natural-deduction systems and Hilbert systems. They may also correspond to ordinary sequent systems (see Section 2.3).

(The turnstile $\vdash$ is not a symbol of the language to which A and B in $A \vdash B$ belong. However, if we envisage an extended language in which $A \vdash B$ is a formula, then a sequent system is like a Hilbert system in this extended language. With sequents singular on both sides, we may even replace the turnstile by the implication sign, and make the system look exactly like an ordinary Hilbert system. This is, however, a Hilbert system of a higher level than those considered here.)

1.2. Codes of proofs. Usually, in sequent systems we are interested only in whether there is a proof of a sequent, and we do not introduce a special notation to distinguish different proofs of the same sequent. It is as if all proofs of the same sequent were equal. However, if inspired by a normalization or cut-elimination procedure we wish to equate some proofs and distinguish others, we shall find handy to write next to a sequent a code recording the history of its proof. Such a coding has become rather common in natural deduction, where typed lambda terms serve as codes (and the coding is called "Curry-Howard isomorphism" or "formulae-as-types interpretation"). If the rules of the sequent system match the rules of a natural-deduction system, we may work with typed lambda coding, but we shall see below that there are, even for natural deduction, other kinds of coding, inspired by category theory.

Codes are to be terms such that axiomatic sequents are coded by atomic terms and rules of inference of the sequent system correspond to operations on terms. This is a slight departure from common practice, where rules of inference are relations not necessarily of a functional kind. For example, the rule of permutation

$$\frac{\Gamma, A, B, \Delta \vdash C}{\Gamma, B, A, \Delta \vdash C}$$

in an ordinary sequent system, with Γ and Δ arbitrary finite, possibly empty, sequences of formulae, does not correspond to a function that applied to the upper sequent yields a single lower sequent: we may permute the first and second formula in the sequence on the left-hand side of $\vdash$, or the second and third etc. However, we can replace such nonfunctional rules by a collection of functional rules, performing separately the many tasks assigned to the original rule. (Permutation may be replaced either by an infinity of functional rules permuting the first and second formula, the second and third formula etc., or by just two functional rules: one to permute the first and second formula and another to put the first formula at the end of the sequence on the left-hand side.) Anyway, most rules we encounter in practice are functional. In general, the operations on terms corresponding to rules of inference are partial operations, because rules may be applicable to sequents of one type, but not of another.

When the proof of a sequent can be converted into another proof of the same sequent by a procedure such as we find in the normalization of natural deduction or in cut elimination, we equate the codes of the two proofs. To eschew triviality, we shouldn't have that codes corresponding to the same sequent are always equated.

1.3. Lambek's deductive systems. We shall now introduce a notion of system of sequents with codes called *deductive systems.* Both the name and the notion stem from Lambek.

We have first a set of things we call, neutrally, *objects*, but which, when we have proof-theoretical matters in mind, we imagine to make a language—then the objects are formulae. We use A, B, …, with indices if needed, as schematic letters for objects. For much of what we do in this paper, we need not specify more precisely what objects are. However, if our language has connectives, these will correspond to *operations on objects.* These operations are taken to be total. By "operation" we understand as usual any function from a domain D^n to the codomain D, not necessarily a one-one function. So "operation on objects" is a more general notion than "connective," which covers some operations that are one-one functions. The algebra of formulae is completely free, while we do not exclude that objects construed in different ways may turn out to be equal. We may also have *special objects*, which are nullary operations on objects.

The main application of what we are exposing is proof-theoretical. In this application we can usually restrict our attention to operations on objects that are one-one functions, which, as we shall see, simplifies matters. However, we don't want to exclude other applications, in particular those of a model-theoretical kind, where these operations cease to be one-one functions. It is an important aspect of the relationship between syntax and semantics that one-one functions of the syntax—not only connectives, but also expressions of other types—are modeled by functions that are not one-one.

Next we have a set of things corresponding to sequents with codes, singular on both sides, which we call *arrows.* And we have two functions assigning to an arrow an object that is its *source*, i.e., its premise, and an object that is its *target*, i.e., its conclusion. There may be more than one arrow with the same source and the same target. The expression $f : A \vdash B$ means that f is an arrow with source A and target B; we say that $A \vdash B$ is the *type* of f. For naming arrows we also use simply $f, g, \ldots$, without specifying the type, if we are not interested in doing so. These arrow-terms correspond to the codes of the preceding section.

A set of objects and a set of arrows with the source and target functions makes a *graph* (in graph theory, this would be called more usually, but less simply, a directed multigraph with loops). In principle, we could have allowed collections of objects and arrows that are not sets but proper classes, but questions pertaining to the size of these collections are unimportant for us here. Having sets, and only small categories later, makes our exposition more determinate.

A *deductive system* is a graph that must have the following *operations on arrows*: for every object A a nullary operation, i.e., special arrow, the *identity arrow* $\mathbf{1}_A : A \vdash A$, and a binary partial operation, *composition*, which applied to the arrows $f : A \vdash B$ and $g : B \vdash C$ yields the arrow $gf : A \vdash C$. Besides these operations on arrows there may be others. Special arrows correspond to axiomatic sequents and n-ary operations on arrows, with $n \geq 1$, correspond to rules of inference. Composition is a singular form of the cut rule:

$$\frac{f : A \vdash B \qquad g : B \vdash C}{gf : A \vdash C}.$$

We shall sometimes display operations on arrows in this manner, as logical rules.

Instead of saying, as would be usual in logic, that an arrow is a *theorem* of a deductive system or is *provable* in it, we now say simply that the arrow is *in* the deductive system, as it is sometimes done in logic, too. When we talk about the arrows *of* a deductive system, we think about the arrows in the deductive system, which should not be confused with the logicians' way of talking about the formulae of a system, meaning formulae of the language of a system, but not necessarily provable in it.

We omit the subscript A from $\mathbf{1}_A$ when we assume it can be recovered from the context. In general, we shall follow such a policy of omitting unessential indices, so as not to encumber notation.

Consequence relations are a generalization of preorders to relations between a collection of objects and an object (or between two collections of objects, when the consequence relations are plural on both sides, as in Gentzen's sequent system for classical logic). With deductive systems we have a different generalization of preordering relations. A preordering relation is a particular kind of deductive system, where for every pair of objects (A, B) there is *at most one* arrow with source A and target B. In general, relations correspond to graphs with this property. Graphs of preorders satisfy the following equivalence, which also holds for every deductive system:

$$\forall A \forall B (\exists f (f : A \vdash B) \Leftrightarrow \forall C (\exists g (g : B \vdash C) \Rightarrow \exists h (h : A \vdash C))).$$

From right to left this is equivalent to reflexivity:

$$\forall A \exists f (f : A \vdash A),$$

while from left to right this is transitivity:

$$\forall A \forall B \forall C \forall f \forall g \exists h ((f : A \vdash B \mathbin{\&} g : B \vdash C) \Rightarrow h : A \vdash C).$$

By "skolemizing" reflexivity and transitivity we obtain the identity arrows and composition of arrows.

1.4. Implicational deductive systems and Hilbert systems. A *Hilbert system* is a graph with a special object $\mathtt{I}$ whose arrows are all of type $\mathtt{I} \vdash A$, for some objects A. Except in trivial cases, a Hilbert system is not a deductive system. However, interesting Hilbert systems may be obtained by separating out of deductive systems arrows of type $\mathtt{I} \vdash A$.

Suppose we have in a deductive system the object $\mathtt{I}$ and an implication connective $\rightarrow$, such that an arrow of the type $A \vdash B$ is in the deductive system iff an arrow of the type $\mathtt{I} \vdash A \rightarrow B$ is. This corresponds to what we have in usual sequent systems and in natural-deduction systems with the implication-introduction rule and modus ponens. We call such deductive systems *implicational.* In implicational deductive systems, identity arrows are matched by arrows of type $\mathtt{I} \vdash A \rightarrow A$, and composition is matched by the operation that applied to two arrows of the types $\mathtt{I} \vdash A \rightarrow B$ and $\mathtt{I} \vdash B \rightarrow C$ yields an arrow of the type $\mathtt{I} \vdash A \rightarrow C$. When A is $\mathtt{I}$, this amounts to the operation that applied to arrows of the types $\mathtt{I} \vdash B$ and $\mathtt{I} \vdash B \rightarrow C$ yields an arrow of the type $\mathtt{I} \vdash C$, i.e., modus ponens. A Hilbert system separated out of an implicational deductive system will be called *implicational.* An implicational Hilbert system imitates the behaviour of a deductive system.

§2. Cartesian categories.

2.1. Categories. A deductive system is a *category* iff the following equalities hold between its arrows:

(cat 1) $$f\mathbf{1} = f, \quad \mathbf{1}f = f$$

(cat 2) $$h(gf) = (hg)f.$$

In such equalities the arrow-terms on both sides always belong to arrows of the same type, i.e., with the same source and target, and all subterms of these terms must be defined, i.e., partial operations on arrows must match the types of arrows.

These equalities make perfect sense in proof theory. The equalities (cat 1) say that compositions, i.e., cuts, with identity arrows are eliminable, while (cat 2) says that compositions, i.e., cuts, can be permuted with each other. (Gentzen would have required a similar permutation in his cut-elimination procedure had he decided to start by eliminating an arbitrary cut, rather than a cut above which there are no other cuts; but such a more complicated procedure is not needed if the goal is to eliminate *all* cuts.) In the free category generated by an arbitrary arrowless graph, i.e., set of objects, composition is eliminable: in this category we have only identity arrows.

The existence of such free categories is guaranteed by the equational form of the category axioms. (This is analogous to the existence of free algebras in varieties, i.e., equationally presented classes of algebras—the fact that we have here algebraic structures with partial operations doesn't change this sort of result; cf. [Lambek & Scott 1986, I.4].) A free category of some sort is like a purely logical system: it satisfies just what is assumed for this sort of category.

2.2. Functors and natural transformations. A *functor* F from a category $\mathcal{C}$ to a category $\mathcal{D}$ is a function assigning to every object A of $\mathcal{C}$ an object $\mathrm{F}(A)$ of $\mathcal{D}$ and another function assigning to every arrow $f : A \vdash B$ of $\mathcal{C}$ an arrow $\mathrm{F}(f) : \mathrm{F}(A) \vdash \mathrm{F}(B)$ of $\mathcal{D}$ such that

$$\mathrm{F}(\mathbf{1}_A) = \mathbf{1}_{\mathrm{F}(A)}, \quad \mathrm{F}(gf) = \mathrm{F}(g)\,\mathrm{F}(f).$$

As usual, we don't distinguish notationally the F of the function F on objects and the F of the function F on arrows, in order to be able to refer to both by the F of the functor F. (So, below, we shall often have to insist when we are speaking only about the F on objects and not about the whole functor F.)

Let F and G be functors from $\mathcal{C}$ to $\mathcal{D}$. A *natural transformation* from F to G is a family $\mathbf{p}$ of arrows $\mathbf{p}_A : \mathrm{F}(A) \vdash \mathrm{G}(A)$ of $\mathcal{D}$, one for every object A of $\mathcal{C}$, such that for every arrow $f : A \vdash B$ of $\mathcal{C}$ the following equality holds:

$$\mathbf{p}_B\,\mathrm{F}(f) = \mathrm{G}(f)\mathbf{p}_A.$$

(In such equalities, we shall often omit indices that can be recovered from the context, such as the indices A and B of $\mathbf{p}$. The arrows $\mathbf{p}_A$ that belong to the family $\mathbf{p}$ are called the *components* of the natural transformation $\mathbf{p}$.

In an alternative definition, exploited in the next section, a natural transformation from F to G is a family of operations $\mathrm{P}^{\mathrm{G}}_A$, one for every object A of $\mathcal{C}$, which applied to an arrow $g : \mathrm{G}(A) \vdash D$ of $\mathcal{D}$ yield the arrow $\mathrm{P}^{\mathrm{G}}_A(g) : \mathrm{F}(A) \vdash D$ of $\mathcal{D}$, such that the following equalities hold:

$$\mathrm{P}^{\mathrm{G}}(h)\,\mathrm{F}(f) = \mathrm{P}^{\mathrm{G}}(h\,\mathrm{G}(f)), \quad g\,\mathrm{P}^{\mathrm{G}}(h) = \mathrm{P}^{\mathrm{G}}(gh).$$

Every natural transformation P^{G} in this new sense, with the definition

$$\text{(i)} \qquad \mathbf{p}_A =_{\mathrm{def}} \mathrm{P}^{\mathrm{G}}_A(\mathbf{1}_{\mathrm{G}(A)}),$$

gives rise to a natural transformation $\mathbf{p}$ in the old sense, while every natural transformation $\mathbf{p}$ in the old sense, with the definition

$$\text{(ii)} \qquad \mathrm{P}^{\mathrm{G}}_A(g) =_{\mathrm{def}} g\mathbf{p}_A,$$

gives rise to a natural transformation P^{G} in the new sense. Moreover, if we start with P^{G} and obtain $\mathbf{p}$ with (i), and then define P^{G} by (ii), we obtain again the original P^{G}. Similarly, if we start with $\mathbf{p}$ and obtain P^{G} with (ii), and then define $\mathbf{p}$ by (i), we obtain again the original $\mathbf{p}$. For that we have to check that with P^{G} primitive the equality of (ii) holds when we replace $\mathbf{p}_A$ on the right-hand side according to (i), and that with $\mathbf{p}$ primitive the equality of (i) holds when we replace $\mathrm{P}^{\mathrm{G}}_A$ on the right-hand side according to (ii). So the two notions of natural transformation have exactly the same extension.

We shall encounter such a situation several times in this paper—that two notions differently introduced have exactly the same extension. It would be tedious to repeat every time what has to be done to verify that (we have gone into so many details in the preceding paragraph just for the sake of example). To characterize the present situation, we can say that the two notions of natural transformation are *extensionally equivalent by the definitions* (i) *and* (ii). When in the future we say something in the same manner, it is to be understood that this involves the same kind of verification as that made in the preceding paragraph.

Another extensionally equivalent alternative is to define a natural transformation as a family of operations $\mathrm{P}^{\mathrm{F}}_A$ that applied to an arrow $g : D \vdash \mathrm{F}(A)$ of $\mathcal{D}$ yield the arrow $\mathrm{P}^{\mathrm{F}}_A(g) : D \vdash G(A)$ of $\mathcal{D}$ such that the following equalities hold:

$$G(f)\,\mathrm{P}^{\mathrm{F}}(h) = \mathrm{P}^{\mathrm{F}}(\mathrm{F}(f)h), \quad \mathrm{P}^{\mathrm{F}}(h)g = \mathrm{P}^{\mathrm{F}}(hg).$$

Now, the definitions corresponding to (i) and (ii) are

$$\mathbf{p}_A =_{\mathrm{def}} \mathrm{P}^{\mathrm{F}}_A(\mathbf{1}_{F(A)}), \quad \mathrm{P}^{\mathrm{F}}_A(g) =_{\mathrm{def}} \mathbf{p}_A g.$$

If G is a one-one function on objects, we can replace the family of operations $\mathrm{P}^{\mathrm{G}}_A$ by a single operation, while if F is a one-one function on objects, we can do the same for P^{F}.

2.3. Cut-free cartesian categories. Cartesian categories are categories with finite products, including the empty product, i.e., terminal object. Rather than simply copy Lambek's equational formulation of these categories, we will first present another, extensionally equivalent, equational formulation, with a somewhat different proof-theoretical motivation. Lambek's equalities are inspired by normalization in natural deduction—ours will correspond rather to steps in Gentzen's cut-elimination procedure.

A cartesian category in this new axiomatization is a category $\mathcal{C}$ that has a binary operation $\times$ on objects and a binary operation $\times$ on arrows:

$$\frac{f_1 : A_1 \vdash B_1 \qquad f_2 : A_2 \vdash B_2}{f_1 \times f_2 : A_1 \times A_2 \vdash B_1 \times B_2}.$$

All this makes a bifunctor, i.e., a functor from the product category $\mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$, which means that the following equalities hold:

$(\times 1)$ $\qquad \mathbf{1}_{A \times B} = \mathbf{1}_A \times \mathbf{1}_B$

$(\times 2)$ $\qquad (g_1 \times g_2)(f_1 \times f_2) = (g_1 f_1) \times (g_2 f_2).$

The operation $\times$ on objects will play the role not so much of conjunction as of the commas on the left-hand side of ordinary sequents, which have a plurality of formulae on that side of the turnstile. Proof-theoretically, $(\times 1)$ says that axiomatic identity sequents of type $A \times B \vdash A \times B$ can be atomized, i.e., replaced by two identity sequents of simpler type, to which we apply the rule $\times$. The equality $(\times 2)$ permits to permute composition, i.e., cut, with the rule $\times$: the term on the left-hand side has $\times$ inside composition, whereas the term on the right-hand side has composition inside $\times$. From left to right, we have pushed composition inside, i.e., upward.

Moreover, we have in every cartesian category for every object C the operations K^1_C and K^2_C on arrows, and for every object A an operation W_A on arrows:

$$\frac{f : A \vdash B}{\mathrm{K}^1_C(f) : A \times C \vdash B} \qquad \frac{f : A \vdash B}{\mathrm{K}^2_C(f) : C \times A \vdash B} \qquad \frac{f : A \times A \vdash B}{\mathrm{W}_A(f) : A \vdash B}.$$

(If the square function $X \times X$ on objects is one-one, as it is in the category of sets, and, a fortiori, if $\times$ is a connective, a single W operation would do.) These operations satisfy the following equalities:

(K) $$\begin{aligned} &\mathrm{K}^1(h)(f \times g) = \mathrm{K}^1(hf), \quad f\,\mathrm{K}^1(h) = \mathrm{K}^1(fh) \\ &\mathrm{K}^2(h)(f \times g) = \mathrm{K}^2(hg), \quad f\,\mathrm{K}^2(h) = \mathrm{K}^2(fh) \end{aligned}$$

(W) $$\mathrm{W}(h)f = \mathrm{W}(h(f \times f)), \quad f\,\mathrm{W}(h) = \mathrm{W}(fh)$$

(K W) $$\begin{aligned} &\mathrm{W}(\mathrm{K}^1(f)) = f, \quad \mathrm{W}(\mathrm{K}^2(f)) = f \\ &\mathrm{W}(\mathrm{K}^1(f) \times \mathrm{K}^2(g)) = f \times g. \end{aligned}$$

The equalities (K) and (W) say that K^1, K^2 and W are natural transformations in the sense of P^{G} from the preceding section. Proof-theoretically, the K^1 and K^2 operations correspond to the structural rule of thinning and W to the structural rule of contraction. The equalities (K) and (W) enable us to push composition inside K^1, K^2 and W, which corresponds to permuting cut with thinning and contraction. The equalities (K W) correspond to eliminating thinnings followed by contractions of the formulae introduced by thinning—a useless detour that we usually don't envisage in sequent systems.

In general, the permuting of cut with other rules in sequent systems is matched in categories either by the functoriality of an operation on arrows, as with $\times$, or by the operation being a natural transformation in the sense of P^{G} or P^{F} from the preceding section. The permuting of cut with another cut is matched, as we already remarked, by the category axiom (cat 2).

Finally, we have in every cartesian category a special object T, called the terminal object, and for every object A an operation K_A on arrows:

$$\frac{f : \mathsf{T} \vdash B}{\mathrm{K}_A(f) : A \vdash B}$$

which satisfies

$$\text{for } f : A \vdash B, \quad \mathrm{K}_B(h)f = \mathrm{K}_A(h), \quad f\,\mathrm{K}_C(h) = \mathrm{K}_C(fh)$$
$$\text{for } f : \mathsf{T} \vdash B, \quad \mathrm{K}_{\mathsf{T}}(f) = f.$$

The first two equalities say that K is a natural transformation in the sense of P^{G} from the preceding section. The object T replaces the empty collection of premises in sequents (a role that was assigned to $\mathtt{I}$ in the first part of the paper). The K operations correspond also to thinning. The first equality for them corresponds to eliminating a cut preceded by a thinning that has introduced the cut formula, while the second equality corresponds to permuting cut with thinning. The last equality eliminates vacuous thinning, a thing we don't have to envisage in ordinary sequent systems.

This concludes our axiomatization of cartesian categories. Before showing in the next section that it amounts to Lambek's, we prove for this axiomatization that all these comparisons with cut elimination were not empty talk.

CUT ELIMINATION. *For every arrow f in the free cartesian category generated by an arbitrary arrowless graph there is an arrow-term f' without composition such that $f = f'$.*

PROOF. First we define a complexity measure, the *degree*, of an arrow-term f, which we write $\mathrm{d}(f)$. This is (n, m) where n is the number of occurrences of W in f and m the number of occurrences of $\times$, K^1, K^2 and K in f. Then give to the degrees the ordering of type ω^2 where $(n_1, m_1) < (n_2, m_2)$ iff $n_1 < n_2$ or ($n_1 = n_2$ and $m_1 < m_2$).

Take now in a term a subterm gf such that there are no compositions in f and no compositions in g. We call a term such as gf a *single-cut term.* The arrow f in gf can be $\mathbf{1}$, in which case $gf = g$. Next, f can be of the form

$\mathrm{K}^1(h)$, in which case $gf = \mathrm{K}^1(gh)$, with $\mathrm{d}(gh) < \mathrm{d}(gf)$ and gh a single-cut term. The same thing happens if f is $\mathrm{K}^2(h)$, $\mathrm{W}(h)$ or $\mathrm{K}(h)$.

The remaining case is that f is of the form $f_1 \times f_2$. Then g can be $\mathbf{1}$, in which case $gf = f$. Next, g can be $\mathrm{K}^1(h)$, in which case $gf = \mathrm{K}^1(hf_1)$, with $\mathrm{d}(hf_1) < \mathrm{d}(gf)$ and hf_1 a single-cut term. The case when g is $\mathrm{K}^2(h)$ is quite analogous. If g is $\mathrm{W}(h)$, then $gf = \mathrm{W}(h(f \times f))$, with, again, $\mathrm{d}(h(f \times f)) < \mathrm{d}(gf)$ and $h(f \times f)$ a single-cut term (the definition of degree is specially designed to fit this case). If g is $\mathrm{K}(h)$, then $gf = \mathrm{K}(h)$, and there are no compositions in $\mathrm{K}(h)$. Finally, if g is $g_1 \times g_2$, then $gf = (g_1 f_1) \times (g_2 f_2)$, with $\mathrm{d}(g_1 f_1) < \mathrm{d}(gf)$ and $\mathrm{d}(g_2 f_2) < \mathrm{d}(gf)$, where both $g_1 f_1$ and $g_2 f_2$ are single-cut terms. An induction on degrees then proves that the composition of gf can be eliminated. $\dashv$

Note first that the cut-elimination procedure of this proof is quite analogous to Gentzen's (to deal with contraction we have induction up to ω^2, while Gentzen introduces a multiple cut rule, called "mix"). Note also that in the proof we used all our equalities except the equalities (cat 2), ($\times 1$), (K W) and $\mathrm{K}_{\mathsf{T}}(f) = f$. (This is as it should be, because all the unused equalities except (cat 2) don't involve composition, whereas (cat 2) is not needed when we concentrate on single-cut terms, as we already indicated in Section 2.1.)

Only operations corresponding to the structural rules of thinning and contraction are mentioned above, but we have operations corresponding to other structural rules as defined operations. In particular, we can define permutation:

$$\frac{f : A \times B \vdash C}{\mathrm{W}(f(\mathrm{K}^2(\mathbf{1}) \times \mathrm{K}^1(\mathbf{1}))) : B \times A \vdash C}.$$

There are also operations corresponding to the structural rules of association, which move parentheses at will around $\times$.

In an alternative formulation of cartesian categories where the natural transformations K^1, K^2, W and K would be of the P^{F} kind from the preceding section, instead of P^{G} as here, we would not be able to prove cut elimination (cf. the definition of permutation above). And, of course, there would be no cut elimination with natural transformations of the $\mathbf{p}$ kind. (We might also envisage that some of K^1, K^2, W and K are of the P^{G} kind and others of the P^{F} kind.)

2.4. Lambek's cartesian categories. In Lambek's axiomatization of cartesian categories we have instead of the operations on arrows $\times$, K^1, K^2, W and K: special arrows, the projections, $\mathbf{k}^1_{A,B} : A \times B \vdash A$ and $\mathbf{k}^2_{A,B} : A \times B \vdash B$, for all objects A and B ([Lambek and Scott 1986] write π and π' where we write $\mathbf{k}^1$ and $\mathbf{k}^2$), and the pairing operation that applied to arrows $f : C \vdash A$ and $g : C \vdash B$ yields the arrow $\langle f, g \rangle : C \vdash A \times B$; we also have the special arrows $\mathbf{k}_A : A \vdash \top$ ([Lambek and Scott 1986] write $\bigcirc_A$ for $\mathbf{k}_A$). These satisfy

$$(\times\beta) \qquad \mathbf{k}^1\langle f, g\rangle = f, \quad \mathbf{k}^2\langle f, g\rangle = g$$

$$(\times\eta) \qquad h = \langle \mathbf{k}^1 h, \mathbf{k}^2 h\rangle$$

$$(\top\eta) \qquad \text{for } h : A \vdash \top, \quad h = \mathbf{k}_A.$$

In this formulation, which is more economical than what we had in the preceding section, there is a parallel with introduction-elimination rules for conjunction in natural deduction. The equalities $(\times\beta)$, $(\times\eta)$ and $(\top\eta)$ correspond to steps in the normalization of natural-deduction proofs in conjunctive logic. These steps cover not only conversions of the β kind, i.e., banishing the introduction of a connective followed by its elimination, but also conversions of the η kind, i.e., atomizing the middle part of proofs coming after the eliminations and preceding the introductions. Conversions of the β kind correspond to atomizing cuts, while those of the η kind correspond to atomizing the axiomatic identity sequents $A \vdash A$. However, we don't have cut elimination in Lambek's formulation of cartesian categories (as we don't have it in natural-deduction systems).

The notion of cartesian category of the preceding section and Lambek's notion are extensionally equivalent by the definitions

$$\text{(i)} \qquad \begin{aligned} &\mathbf{k}^1_{A,B} =_{\text{def}} \mathrm{K}^1_B(\mathbf{1}_A), \quad \mathbf{k}^2_{A,B} =_{\text{def}} \mathrm{K}^2_A(\mathbf{1}_B) \\ &\langle f, g\rangle =_{\text{def}} \mathrm{W}(f \times g) \\ &\mathbf{k}_A =_{\text{def}} \mathrm{K}_A(\mathbf{1}_\top) \end{aligned}$$

$$\text{(ii)} \qquad \begin{aligned} &f \times g =_{\text{def}} \langle f\mathbf{k}^1, g\mathbf{k}^2\rangle \\ &\mathrm{K}^1(f) =_{\text{def}} f\mathbf{k}^1, \quad \mathrm{K}^2(f) =_{\text{def}} f\mathbf{k}^2 \\ &\mathrm{W}(f) =_{\text{def}} f\langle \mathbf{1}, \mathbf{1}\rangle \\ &\mathrm{K}_A(f) =_{\text{def}} f\mathbf{k}_A \end{aligned}$$

(this extensional equivalence is to be interpreted as in Section 2.2).

2.5. Cartesian closed and bicartesian closed categories. We shall not try to give a formulation of cartesian closed and bicartesian closed categories along the lines of Section 2.3. In general, the format of categories doesn't lend itself very well to cut elimination. We managed to eliminate cut with cartesian categories, where we have a single sort of connective, finite conjunction, or, indeed, no connectives, but only structural rules. With additional connectives like implication and disjunction we would have more difficulties. It would presumably be better to switch then to something more practical for cut elimination: namely, Lambek's *multicategories*, where arrows have sources made of a collection of objects (see [Lambek 1969, 1989, 1993]). However, this theory is far less developed than the theory of categories.

A *cartesian closed* category is a cartesian category that has an operation on objects $\rightarrow$, for all objects A and B a special arrow $\varepsilon_{A,B} : (A \rightarrow B) \times A \vdash B$, and for all objects A and C an operation that applied to an arrow $f : C \times A \vdash B$ yields the arrow $f^* : C \vdash A \rightarrow B$; these satisfy the equalities

$$(\rightarrow \beta) \qquad \varepsilon(f^* \times \mathbf{1}) = f$$

$$(\rightarrow \eta) \qquad (\varepsilon(g \times \mathbf{1}))^* = g$$

which are related to normalization in natural deduction. The arrows ε correspond to modus ponens, while * corresponds to the implication-introduction rule. These two rules characterize intuitionistic implication.

We shall not define in detail bicartesian closed categories, which in addition to all that we had for cartesian closed categories have finite coproducts, corresponding to disjunction; this includes also the empty coproduct, i.e., initial object, corresponding to the constant absurd proposition (see [Lambek and Scott 1986, I.8]; an object is initial in a category iff there is a unique arrow from it to every other object). These categories correspond to full intuitionistic propositional logic. However, bicartesian closed categories don't play a major role in our exposition.

Cartesian closed categories are also less important for our exposition than cartesian categories in general, because in the approach we follow conjunction is a more basic connective than implication. In nontrivial Hilbert systems we cannot do without implication (it should at least be definable), but in Gentzen systems we can, and there, conjunction, which serves to join premises into a single formula, may be taken as a "structural" symbol, like the comma on the left-hand side of ordinary sequents. This is evident in the axiomatization of cartesian categories of Section 2.3, where we have rules closer to the structural rules of thinning and contraction than to rules for a conjunction connective. Investigating such deductive systems means investigating the structural foundations of logic, rather than any particular connective.

Cartesian categories don't cover only intuitionistic logic but classical logic, too. Intuitionistic conjunction and classical conjunction don't differ; or rather, from the point of view of the preceding paragraph, intuitionistic and classical structural principles concerning premises are the same. What differs is the turnstile, and structural principles concerning conclusions. That is why when we introduce implication, we leave the common ground of the two logics.

Other important sorts of categories correspond to fragments of various logics. In particular, symmetric monoidal closed categories (see [Mac Lane

1971, VII.1,7]) correspond to the product-implication fragment of intuitionistic linear logic. Of course, to match exactly purely logical deductive systems all these categories have to be freely generated from a set of atomic formulae.

§3. Adjunctions.

3.1. Definition of adjunction. Let $\mathcal{A}$ and $\mathcal{B}$ be two categories. The objects of $\mathcal{A}$ will be designated by A, A_1, A_2, ..., and the arrows of $\mathcal{A}$ by f, f_1, f_2, ..., while the objects of $\mathcal{B}$ will be designated by B, B_1, B_2, ..., and the arrows of $\mathcal{B}$ by $g, g_1, g_2, \ldots$.

Let F be a function from the objects of $\mathcal{B}$ into the objects of $\mathcal{A}$ and G a function from the objects of $\mathcal{A}$ into the objects of $\mathcal{B}$. (We don't yet assume the functions on objects F and G are tied to functors: that will follow from other assumptions.)

Finally, let there be for every pair of objects (A, B) a function $\Phi_{B,A}$ assigning to an arrow $g : B \vdash \mathrm{G}(A)$ of $\mathcal{B}$ an arrow $\Phi_{B,A}(g) : \mathrm{F}(B) \vdash A$ of $\mathcal{A}$, and a function $\Gamma_{B,A}$ assigning to an arrow $f : \mathrm{F}(B) \vdash A$ of $\mathcal{A}$ an arrow $\Gamma_{B,A}(f) : B \vdash \mathrm{G}(A)$ of $\mathcal{B}$. We denote by Φ the family of all the functions $\Phi_{B,A}$ and by Γ the family of all the functions $\Gamma_{B,A}$. In general, we omit the subscripts of $\Phi_{B,A}$ and $\Gamma_{B,A}$ in equalities.

Then $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ is an *adjunction* between $\mathcal{A}$ and $\mathcal{B}$ iff the following equalities hold:

$$(\Gamma\Phi) \qquad \Gamma(\Phi(g)) = g$$

$$(\Phi\Gamma) \qquad \Phi(\Gamma(f)) = f$$

$$(\Gamma\Gamma) \qquad \Gamma(f_2 f_1) = \Gamma(f_2 \Phi(\mathbf{1}))\Gamma(f_1).$$

Of course, we assume that in these equalities Γ, Φ, $\mathbf{1}$, g, f, f_1 and f_2 are of such types that all the terms of the equalities are defined. For example, Φ and Γ must be $\Phi_{B,A}$ and $\Gamma_{B,A}$ in $(\Gamma\Phi)$ and $(\Phi\Gamma)$, while in $(\Gamma\Gamma)$ the identity arrow $\mathbf{1}$ must be $\mathbf{1}_{\mathrm{G}(A)}$ for some A.

We can replace the equality $(\Gamma\Gamma)$ by the equality

$$(\Phi\Phi) \qquad \Phi(g_2 g_1) = \Phi(g_2)\Phi(\Gamma(\mathbf{1})g_1).$$

Here is a derivation of $(\Phi\Phi)$:

$$\begin{aligned} \Phi(g_2)\Phi(\Gamma(\mathbf{1})g_1) &= \Phi(\Gamma(\Phi(g_2)\Phi(\Gamma(\mathbf{1})g_1))), \text{ by } (\Phi\Gamma) \\ &= \Phi(\Gamma(\Phi(g_2)\Phi(\mathbf{1}))\Gamma(\mathbf{1})g_1), \text{ by } (\Gamma\Gamma) \text{ and } (\Gamma\Phi) \\ &= \Phi(g_2 g_1), \text{ by } (\Gamma\Gamma) \text{ and } (\Gamma\Phi) \end{aligned}$$

and we can derive $(\Gamma\Gamma)$ analogously from $(\Gamma\Phi)$, $(\Phi\Gamma)$ and $(\Phi\Phi)$.

The arrows $\varphi_A : \mathrm{F}(\mathrm{G}(A)) \vdash A$ and $\gamma_B : B \vdash \mathrm{G}(\mathrm{F}(B))$ (often written ε_A and η_B) are defined by

$$\varphi_A =_{\mathrm{def}} \Phi_{\mathrm{G}(A),A}(\mathbf{1}_{\mathrm{G}(A)}), \quad \gamma_B =_{\mathrm{def}} \Gamma_{B,\mathrm{F}(B)}(\mathbf{1}_{\mathrm{F}(B)}).$$

Usually, we omit the subscripts of φ_A and γ_B. Once the functors involved are defined, it can be verified that these arrows make two natural transformations, which are called respectively the *counit* and *unit* of the adjunction.

For every arrow g of $\mathcal{B}$ and every arrow f of $\mathcal{A}$ we define $\mathrm{F}(g)$ and $\mathrm{G}(f)$ by

$$\mathrm{F}(g) =_{\mathrm{def}} \Phi(\gamma g), \quad \mathrm{G}(f) =_{\mathrm{def}} \Gamma(f\varphi).$$

Then it can be verified that F and G are functors. These are the functors that are said to be *adjoint*, F being left-adjoint and G right-adjoint. With these last definitions $(\Phi\Phi)$ and $(\Gamma\Gamma)$ read

$$\Phi(g_2 g_1) = \Phi(g_2)\,\mathrm{F}(g_1), \quad \Gamma(f_2 f_1) = \mathrm{G}(f_2)\Gamma(f_1).$$

In the standard definition of adjunction of the kind we have given, with Φ and Γ, it is presupposed from the outset that F and G are functors, while in our, extensionally equivalent, definition it is supposed for F and G that they are only functions on objects: the functions F and G on arrows are defined subsequently, and their functorial character is something derived. Our definition is more practical for the application we have in mind, and is also more economical. There is an additional equality presupposed in the standard definition (see [Mac Lane 1971, IV.1, p.79]), which we can derive as follows:

$$\begin{aligned}\Gamma(f\,\mathrm{F}(g)) &= \Gamma(f\Phi(\mathbf{1}))\Gamma(\Phi(\Gamma(\mathbf{1})g)), \text{ by } (\Gamma\Gamma) \text{ and the definition of F}\\ &= \Gamma(f)g, \text{ by } (\Gamma\Phi) \text{ and } (\Gamma\Gamma).\end{aligned}$$

Adjunction is one of the main notions of mathematics in general, and of logic in particular. We shall see below how this notion is involved in the deduction theorem, but this certainly does not exhaust its importance for logic. Another application it has in categorial proof theory is the analysis of logical constants—the connectives, quantifiers and equality—through adjointness. Roughly speaking, these constants are tied to functors left or right-adjoint to functors of a structural sort—structural in the sense of Gentzen. Still another application is the explanation of normalization and cut elimination: among the equalities between arrows induced by these procedures one finds equalities of adjunctions. However, all these applications are connected to each other.

3.2. Local and global bijections of adjunctions. In an adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ between the categories $\mathcal{A}$ and $\mathcal{B}$, for every pair of objects (A, B), where A is from $\mathcal{A}$ and B is from $\mathcal{B}$, the functions $\Gamma_{B,A}$ and $\Phi_{B,A}$ are components of two transformations natural in A and B. These two functions give a bijection and its inverse between the set $\{f \mid f : \mathrm{F}(B) \vdash A\}$ of arrows of $\mathcal{A}$ and the set $\{g \mid g : B \vdash \mathrm{G}(A)\}$ of arrows of $\mathcal{B}$ (such sets are called *hom-sets*). These bijections are *local.* We first fix A and B, and then find the bijection between two sets of arrows restricted by A and B. Since this restriction is on two sides of the arrows, we shall say that the bijections are *two-sided* local.

The existence of such two-sided local bijections does not guarantee that there is a global bijection between $\{f \mid \exists A \exists B (f : \mathrm{F}(B) \vdash A)\}$ and $\{g \mid \exists A \exists B (g : B \vdash \mathrm{G}(A))\}$. Such a bijection exists if in the adjunction both F and G are one-one functions on objects. We can then paste together the members of the families of bijections Γ and Φ to obtain a single bijection and its inverse satisfying $(\Gamma\Phi)$, $(\Phi\Gamma)$ and $(\Gamma\Gamma)$. Such global bijections can replace the two-sided local bijections, which are recovered by restricting the domains of the global bijections.

If we assume just for G that it is a one-one function on objects, then for every given B there is a bijection between $\{f \mid \exists A (f : \mathrm{F}(B) \vdash A)\}$ and $\{g \mid \exists A (g : B \vdash \mathrm{G}(A))\}$. This bijection and its inverse, Γ_B and Φ_B, will be called *one-sided* local bijections. We can then replace the families of bijections Γ and Φ indexed on two variables by two families of functions Γ_B and Φ_B indexed only on the first variable, provided functions in the latter families satisfy $(\Gamma\Phi)$, $(\Phi\Gamma)$ and $(\Gamma\Gamma)$. The two-sided local bijections are obtained by restricting the domains of these one-sided local bijections. We have analogous one-sided local bijections when it is assumed just for F that it is a one-one function on objects.

3.3. Inclusive adjunctions. Suppose the objects of the category $\mathcal{A}$ are included among the objects of the category $\mathcal{B}$. Then we call an adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ between $\mathcal{A}$ and $\mathcal{B}$ a *right-inclusive* adjunction of $\mathcal{A}$ into $\mathcal{B}$ iff the function G is the inclusion function on objects; i.e., for every object A of $\mathcal{A}$ we have that $\mathrm{G}(A)$ is A.

The category $\mathcal{A}$ is a *subcategory* of $\mathcal{B}$ iff the objects of $\mathcal{A}$ are included among the objects of $\mathcal{B}$, the arrows of $\mathcal{A}$ are included among the arrows of $\mathcal{B}$ and have the same sources and targets as in $\mathcal{B}$, they are closed under composition of arrows in $\mathcal{B}$ and the identity arrows of $\mathcal{A}$ are those of $\mathcal{B}$. If $\mathcal{A}$ is a subcategory of $\mathcal{B}$, then an adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ is a *reflection* of $\mathcal{B}$ in $\mathcal{A}$ iff the functor G is the inclusion functor; i.e., G is the inclusion function both on objects and on arrows. Reflections of $\mathcal{B}$ in $\mathcal{A}$ are particular cases of right-inclusive adjunctions of $\mathcal{A}$ into $\mathcal{B}$. However, not every right-inclusive adjunction is a reflection (as we shall find in Section 6.1).

In the opposite direction, when the objects of $\mathcal{B}$ are included among the objects of $\mathcal{A}$ and in the adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ between $\mathcal{A}$ and $\mathcal{B}$ the function F is the inclusion function on objects, we have a *left-inclusive* adjunction of $\mathcal{B}$ into $\mathcal{A}$. A *coreflection* of $\mathcal{A}$ in its subcategory $\mathcal{B}$, where the inclusion functor has a right adjoint, is a particular case of such a situation.

3.4. Modalities in adjunctions. In an adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ between the categories $\mathcal{A}$ and $\mathcal{B}$ let $\mathrm{F\,G}$ and $\mathrm{G\,F}$ be the operations on, respectively, the objects of $\mathcal{A}$ and the objects of $\mathcal{B}$ defined by

$$\mathrm{F\,G}(A) =_{\mathrm{def}} \mathrm{F}(\mathrm{G}(A)), \quad \mathrm{G\,F}(B) =_{\mathrm{def}} \mathrm{G}(\mathrm{F}(B)).$$

The operation $\mathrm{F\,G}$ behaves like an S4 necessity operator since in $\mathcal{A}$ we have the operations on arrows

$$\frac{f : A_1 \vdash A_2}{f\varphi : \mathrm{F\,G}(A_1) \vdash A_2} \qquad \frac{f : \mathrm{F}(B) \vdash A}{\mathrm{F}(\Gamma(f)) : \mathrm{F}(B) \vdash \mathrm{F\,G}(a)}$$

analogous to sequent rules for the necessity operator of S4. The objects $\mathrm{F}(B)$ of $\mathcal{A}$ are like formulae modalized in the sense of necessity.

On the other hand, the operation $\mathrm{G\,F}$ behaves like an S4 possibility operator since in $\mathcal{B}$ we have the operations on arrows

$$\frac{g : B \vdash \mathrm{G}(A)}{\mathrm{G}(\Phi(g)) : \mathrm{G\,F}(B) \vdash \mathrm{G}(A)} \qquad \frac{g : B_1 \vdash B_2}{\gamma g : B_1 \vdash \mathrm{G\,F}(B_2)}.$$

Now the objects $\mathrm{G}(A)$ of $\mathcal{B}$ are like formulae modalized in the sense of possibility.

If the adjunction is right-inclusive, in the sense of the preceding section, then F behaves within $\mathcal{B}$ like an S4 possibility operator and the objects of $\mathcal{A}$ are like formulae modalized in the sense of possibility. Among these objects we find all the objects of the form $\mathrm{F}(B)$, but maybe others, too.

These matters are related to the fact that $\mathrm{F\,G}$ is tied to the functor of a comonad in $\mathcal{A}$ and $\mathrm{G\,F}$ to the functor of a monad in $\mathcal{B}$. We define these notions in the next part of the paper, and concentrate on the first.

§4. Comonads.

4.1. Definitions of comonad. A *comonad* (sometimes called a *cotriple*) in a category $\mathcal{A}$ is usually defined as a triple $\langle L, \mathbb{I}, \mathbf{d} \rangle$ where L is a functor from $\mathcal{A}$ to $\mathcal{A}$ and $\mathbb{I}$ and $\mathbf{d}$ are two natural transformations with the components $\mathbb{I}_A : \mathrm{L}(A) \vdash A$ and $\mathbf{d}_A : \mathrm{L}(A) \vdash \mathrm{L}(\mathrm{L}(A))$ such that the following equalities hold

$$\mathbb{I}\mathbf{d} = \mathbf{1}, \quad \mathrm{L}(\mathbb{I})\mathbf{d} = \mathbf{1}, \quad \mathbf{dd} = \mathrm{L}(\mathbf{d})\mathbf{d}.$$

(Instead of $\mathbb{I}$ and $\mathbf{d}$ one often finds ε and δ, but the latter symbols are used for other things in this paper or those we refer to.)

Alternatively, we can define a comonad via its Kleisli category. We do that as follows. Let L be a unary operation on the objects of $\mathcal{A}$ (we don't assume yet that L is tied to a functor) and let there be in $\mathcal{A}$ for every object A an arrow $\mathbb{I}_A : \mathrm{L}(A) \vdash A$ (we don't assume yet that these are components of a natural transformation); moreover, let there be in $\mathcal{A}$ an operation on arrows

$$\frac{f_1 : \mathrm{L}(A_1) \vdash A_2 \qquad f_2 : \mathrm{L}(A_2) \vdash A_3}{f_2 \circ f_1 : \mathrm{L}(A_1) \vdash A_3}.$$

The *Kleisli Deductive System* $\mathcal{A}_\mathrm{L}$ of $\langle \mathrm{L}, \mathbb{I}, \circ \rangle$ has as objects the objects of $\mathcal{A}$, while its arrows are defined as follows: for every arrow $f : \mathrm{L}(A_1) \vdash A_2$ of $\mathcal{A}$ and for every A such that $\mathrm{L}(A)$ is the same object as $\mathrm{L}(A_1)$ take the arrow f indexed by A, which we write f^A and call the *A-copy* of f, to be an arrow of $\mathcal{A}_\mathrm{L}$ of the type $A \vdash A_2$. Since L is not necessarily one-one, a single arrow of $\mathcal{A}$ may split into several different copies in $\mathcal{A}_\mathrm{L}$; such a splitting is excluded if L is one-one. The identity arrow $\mathbf{1}_A$ in $\mathcal{A}_\mathrm{L}$ is $(\mathbb{I}_A)^A$ and the composition $f_2^{A_2} f_1^{A_1}$ in $\mathcal{A}_\mathrm{L}$ of $f_1^{A_1}$ and $f_2^{A_2}$ is $(f_2 \circ f_1)^{A_1}$.

The deductive system $\mathcal{A}_\mathrm{L}$ is a category iff the following equalities hold in $\mathcal{A}$:

$$f \circ \mathbb{I} = f, \quad \mathbb{I} \circ f = f$$
$$f_3 \circ (f_2 \circ f_1) = (f_3 \circ f_2) \circ f_1.$$

Our new definition of comonad says that $\langle \mathrm{L}, \mathbb{I}, \circ \rangle$ is a comonad in $\mathcal{A}$ iff the Kleisli deductive system $\mathcal{A}_\mathrm{L}$ is a category and, moreover, the following equality holds in $\mathcal{A}$:

(shift) $$f_3(f_2 \circ f_1) = (f_3 f_2) \circ f_1.$$

This equality can be replaced by either of the following two equalities:

$$f_3(\mathbf{1} \circ f_1) = f_3 \circ f_1, \quad (f_3 \mathbb{I}) \circ f_1 = f_3 f_1.$$

The new notion of comonad $\langle \mathrm{L}, \mathbb{I}, \circ \rangle$ and the old notion of comonad $\langle \mathrm{L}, \mathbb{I}, \mathbf{d} \rangle$ are extensionally equivalent by the definitions

(i) $$\mathrm{L}(f) =_{\mathrm{def}} \mathbf{1} \circ (f \mathbb{I})$$
$$\mathbf{d}_A =_{\mathrm{def}} \mathbf{1}_{\mathrm{L}(\mathrm{L}(A))} \circ \mathbf{1}_{\mathrm{L}(A)}$$

(ii) $$f_2 \circ f_1 =_{\mathrm{def}} f_2 \, \mathrm{L}(f_1) \mathbf{d}$$

(this extensional equivalence is to be interpreted as in Section 2.2).

The category $\mathcal{A}_\mathrm{L}$ of a comonad $\langle \mathrm{L}, \mathbb{I}, \circ \rangle$ in $\mathcal{A}$ is called the *Kleisli category* of this comonad. Between the category $\mathcal{A}$ and the Kleisli category $\mathcal{A}_\mathrm{L}$ (the latter being the category $\mathcal{B}$ of the definition of adjunction) there is the adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ where $\mathrm{F}(B)$ is $\mathrm{L}(B)$ and $\mathrm{G}(A)$ is A, while for an arrow $f : \mathrm{L}(B) \vdash A$ of $\mathcal{A}$, the arrow $\Phi_{B,A}(f^B)$ is f and $\Gamma_{B,A}(f)$ is f^B. For an arrow $f^{B_1} : B_1 \vdash B_2$ of $\mathcal{A}_{\mathrm{L}}$, the arrow $\mathrm{F}(f^{B_1})$ of $\mathcal{A}$, which is by definition $\Phi(\Gamma(\mathbf{1})f^{B_1})$, is $\mathbf{1} \circ f : \mathrm{L}(B_1) \vdash \mathrm{L}(B_2)$, while for an arrow $f : A_1 \vdash A_2$ of $\mathcal{A}$, the arrow $\mathrm{G}(f)$ of $\mathcal{A}_{\mathrm{L}}$, which is by definition $\Gamma(f\,\Phi(\mathbf{1}))$, is $(f\,\mathbb{I})^{A_1} : A_1 \vdash A_2$, i.e., the A_1-copy of $f\,\mathbb{I} : \mathrm{L}(A_1) \vdash A_2$. The two equalities above that can replace the (shift) equality correspond respectively to $(\Phi\Phi)$ and $(\Gamma\Gamma)$. Since $(\Gamma\Phi)$ and $(\Phi\Gamma)$ are trivially satisfied here, the adjunction between $\mathcal{A}$ and the Kleisli category $\mathcal{A}_{\mathrm{L}}$ is based on (shift).

Still another extensionally equivalent definition of comonad can be given as follows. Consider $\langle \mathrm{L}, \mathbb{I}, \mathrm{N}\rangle$ where, as before, L is a unary operation on the objects of the category $\mathcal{A}$ and for every object A of $\mathcal{A}$ there is an arrow $\mathbb{I}_A : \mathrm{L}(A) \vdash A$ (we don't assume yet that L is tied to a functor and that $\mathbb{I}$ is a natural transformation); N is an operation on the arrows of $\mathcal{A}$:

$$\frac{f : \mathrm{L}(A_1) \vdash A_2}{\mathrm{N}(f) : \mathrm{L}(A_1) \vdash \mathrm{L}(A_2)}.$$

We say that $\langle \mathrm{L}, \mathbb{I}, \mathrm{N}\rangle$ is a comonad in $\mathcal{A}$ iff the following equalities hold:

$$\mathbb{I}\,\mathrm{N}(f) = f, \quad \mathrm{N}(f_2\,\mathrm{N}(f_1)) = \mathrm{N}(f_2)\,\mathrm{N}(f_1), \quad \mathrm{N}(\mathbb{I}) = \mathbf{1}.$$

The notion of comonad $\langle \mathrm{L}, \mathbb{I}, \mathrm{N}\rangle$ and the notion of comonad $\langle \mathrm{L}, \mathbb{I}, \circ\rangle$ are extensionally equivalent by the definitions

(i-N) $$f_2 \circ f_1 =_{\mathrm{def}} f_2\,\mathrm{N}(f_1)$$

(ii-N) $$\mathrm{N}(f) =_{\mathrm{def}} \mathbf{1}_{\mathrm{L}(A_2)} \circ f.$$

So all the three notions of comonad are extensionally equivalent.

If we don't economize on primitives (and don't insist that there be only three things in a "triple" or "cotriple"), we can give a more perspicuous, extensionally equivalent, version of the last definition. Now a comonad in $\mathcal{A}$ would be a quadruple $\langle \mathrm{L}, \mathbb{I}, \circ, \mathrm{N}\rangle$ where L is a unary operation on objects, for every object A there is an arrow $\mathbb{I}_A : \mathrm{L}(A) \vdash A$, the operations on arrows $\circ$ and N are as above, and the following equalities hold:

$$\mathbb{I}\,\mathrm{N}(f) = f, \quad \mathrm{N}(f_2 \circ f_1) = \mathrm{N}(f_2)\,\mathrm{N}(f_1), \quad \mathrm{N}(\mathbb{I}) = \mathbf{1}.$$

These are the equalities of the preceding definition, except for the change in the second equality. But this change enables us to derive the equalities of (i-N) and (ii-N).

With $\langle \mathrm{L}, \mathbb{I}, \circ\rangle$ we have defined a comonad by its Kleisli deductive system being a category and by (shift), on which is based the adjunction with this category. With $\langle \mathrm{L}, \mathbb{I}, \circ, \mathrm{N}\rangle$ we have defined a comonad by there being a morphism of deductive systems from the Kleisli deductive system $\mathcal{A}_{\mathrm{L}}$ into $\mathcal{A}$ that would be one-one, both on objects and on arrows, if L were one-one (see the next section). This morphism corresponds to the left adjoint of the adjunction between $\mathcal{A}$ and the Kleisli category $\mathcal{A}_{\mathrm{L}}$.

A *monad* (also called a *triple*) is a comonad in the category with arrows reversed—sources become targets and targets sources. We may, of course, define monads by definitions analogous to any of the definitions of comonads of this section. The operation L of a comonad resembles an S4 necessity operator, while the corresponding operation of a monad resembles an S4 possibility operator.

4.2. Injective comonads. Let us say that a comonad, in any of the senses of the preceding section, is *injective* iff L is a one-one function on objects.

The Kleisli category $\mathcal{A}_{\mathrm{L}}$ of an injective comonad in $\mathcal{A}$ is isomorphic to the subcategory $\mathrm{N}(\mathcal{A}_{\mathrm{L}})$ of $\mathcal{A}$ whose objects are of the form $\mathrm{L}(A)$ for A an object of $\mathcal{A}$ and whose arrows are of the form $\mathrm{N}(f) : \mathrm{L}(A_1) \vdash \mathrm{L}(A_2)$ for $f : \mathrm{L}(A_1) \vdash A_2$ an arrow of $\mathcal{A}$. That the category $\mathrm{N}(\mathcal{A}_{\mathrm{L}})$ is well defined is guaranteed by the second and third equalities of the last two definitions of comonads, while the isomorphism follows from the remaining, first, equality. (Two categories are isomorphic iff there is a functor from one to the other that is a bijection both on objects and on arrows.)

It can be shown without supposing that the comonad is injective that the category $\mathrm{N}(\mathcal{A}_{\mathrm{L}})$ is isomorphic to the category $\mathcal{A}^{\mathrm{L}}_{\mathrm{free}}$ of "free coalgebras of the comonad", namely, the category whose objects are pairs $(\mathrm{L}(A), \mathbf{d}_A)$ and whose arrows are arrows $g : \mathrm{L}(A_1) \vdash \mathrm{L}(A_2)$ of $\mathcal{A}$, with source $(\mathrm{L}(A_1), \mathbf{d}_{A_1})$ and target $(\mathrm{L}(A_2), \mathbf{d}_{A_2})$, such that

$$\text{(homo)} \qquad \mathrm{L}(g)\mathbf{d} = \mathbf{d}g$$

(by definition, this is $\mathrm{N}(g) = \mathrm{N}(\mathbf{1})g$, which can be shown equivalent to $g = \mathrm{N}(\mathbb{I}g)$). The (homo) equality (which corresponds to the "homomorphism" equality for algebras of a monad) is equivalent to assuming that for some $f : \mathrm{L}(A_1) \vdash A_2$ we have $g = \mathrm{N}(f)$.

In general, when it is not supposed that the comonad is injective, we can assert only that $\mathcal{A}_{\mathrm{L}}$ and $\mathrm{N}(\mathcal{A}_{\mathrm{L}})$ are equivalent categories, not necessarily isomorphic, and the same with $\mathcal{A}^{\mathrm{L}}_{\mathrm{free}}$ instead of $\mathrm{N}(\mathcal{A}_{\mathrm{L}})$. (That two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent means that there are two functors, F_1 from $\mathcal{C}$ to $\mathcal{D}$ and F_2 from $\mathcal{D}$ to $\mathcal{C}$, such that from the composite functors $\mathrm{F}_2\,\mathrm{F}_1$ and $\mathrm{F}_1\,\mathrm{F}_2$ to the identity functors on $\mathcal{C}$ and $\mathcal{D}$ there are natural transformations whose components are isomorphisms; i.e., these components have inverses with which they compose to give identity arrows.)

4.3. Comonads of adjunctions. Every adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ between the categories $\mathcal{A}$ and $\mathcal{B}$ gives rise to a comonad $\langle \mathrm{L}, \mathbb{I}, \circ \rangle$ in $\mathcal{A}$ where

$$\begin{aligned}
&\mathrm{L}(A) =_{\mathrm{def}} \mathrm{F}\,\mathrm{G}(A) = \mathrm{F}(\mathrm{G}(A))\\
&\mathbb{I} =_{\mathrm{def}} \varphi = \Phi(\mathbf{1})\\
&f_2 \circ f_1 =_{\mathrm{def}} \Phi(\Gamma(f_2)\Gamma(f_1)).
\end{aligned}$$

We call this comonad the *comonad of the adjunction.* Its Kleisli category is denoted by $\mathcal{A}_{\mathrm{F\,G}}$. To show that the Kleisli deductive system $\mathcal{A}_{\mathrm{F\,G}}$ is a category we don't need $(\Gamma\Gamma)$ and $(\Phi\Phi)$: the equalities $(\Gamma\Phi)$ and $(\Phi\Gamma)$ suffice. We need $(\Gamma\Gamma)$ or $(\Phi\Phi)$ for (shift).

A subcategory of a category $\mathcal{C}$ obtained by specifying a subset of the objects of $\mathcal{C}$ and taking all the arrows of $\mathcal{C}$ between the objects in the subset is called a *full* subcategory of $\mathcal{C}$. Let $\mathrm{G}(\mathcal{A})$ be the full subcategory of $\mathcal{B}$ whose objects are all the objects $\mathrm{G}(A)$ for A an object of $\mathcal{A}$ and whose arrows are all the arrows of $\mathcal{B}$ between such objects. Then we can state and prove the following proposition.

PROPOSITION. *Let* $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ *be an adjunction between* $\mathcal{A}$ *and* $\mathcal{B}$ *where* G *is a one-one function on objects. Then the categories* $\mathcal{A}_{\mathrm{F\,G}}$ *and* $\mathrm{G}(\mathcal{A})$ *are isomorphic.*

PROOF. We define a functor H from $\mathcal{A}_{\mathrm{F\,G}}$ to $\mathrm{G}(\mathcal{A})$ in the following way. For every object A of $\mathcal{A}_{\mathrm{F\,G}}$, which is by definition an object of $\mathcal{A}$, let $\mathrm{H}(A)$ be $\mathrm{G}(A)$. For every arrow $f^{A_1} : A_1 \vdash A_2$ of $\mathcal{A}_{\mathrm{F\,G}}$, which is by definition the A_1-copy of an arrow $f : \mathrm{F}(\mathrm{G}(A_1)) \vdash A_2$ of $\mathcal{A}$, let $\mathrm{H}(f^{A_1})$ be $\Gamma_{\mathrm{G}(A_1),A_2}(f) : \mathrm{G}(A_1) \vdash \mathrm{G}(A_2)$. To check that H is a functor we have

$$\Gamma(\mathbb{I}) = \Gamma(\Phi(\mathbf{1})) = \mathbf{1}, \text{ by definition and } (\Gamma\Phi)$$

$$\Gamma(f_2 \circ f_1) = \Gamma(\Phi(\Gamma(f_2)\Gamma(f_1))) = \Gamma(f_2)\Gamma(f_1), \text{ by definition and } (\Gamma\Phi).$$

Relying on the fact that G is one-one, we define the functor H^{-1} from $\mathrm{G}(\mathcal{A})$ to $\mathcal{A}_{\mathrm{F\,G}}$ by taking that $\mathrm{H}^{-1}(\mathrm{G}(A))$ is A and that for $g : \mathrm{G}(A_1) \vdash \mathrm{G}(A_2)$ the arrow $\mathrm{H}^{-1}(g)$ is $(\Phi_{\mathrm{G}(A_1),A_2}(g))^{A_1} : A_1 \vdash A_2$, i.e., the A_1-copy of $\Phi_{G(A_1),A_2}(g) : \mathrm{F}(\mathrm{G}(A_1)) \vdash A_2$. It remains to use the equalities $(\Phi\Gamma)$ and $(\Gamma\Phi)$ to verify that $\mathrm{H}^{-1}(\mathrm{H}(f^{A_1})) = f^{A_1}$ and $\mathrm{H}(\mathrm{H}^{-1}(g)) = g$. $\dashv$

Note that, since we don't need $(\Gamma\Gamma)$ and $(\Phi\Phi)$ for this proof, we could establish a stronger proposition where for $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ we don't suppose that it is an adjunction, but only that $(\Gamma\Phi)$ and $(\Phi\Gamma)$ are satisfied.

Note also that since F is not necessarily a one-one function on objects there need not be a global bijection between the set $\{f \mid \exists A_1 \exists A_2 (f : \mathrm{F}(\mathrm{G}(A_1)) \vdash A_2)\}$ of arrows of $\mathcal{A}$ and the set $\{g \mid \exists A_1 \exists A_2 (g : \mathrm{G}(A_1) \vdash \mathrm{G}(A_2))\}$ of arrows of $\mathrm{G}(\mathcal{A})$, i.e., $\mathcal{B}$. However, a single arrow from the first set can split in $\mathcal{A}_{\mathrm{F\,G}}$ into different copies of types $A_1' \vdash A_2$, $A_1'' \vdash A_2$, etc., with which we obtain a global bijection between the arrows of $\mathcal{A}_{\mathrm{F\,G}}$ and the arrows of $\mathrm{G}(\mathcal{A})$. If F is a one-one function on objects as well as G, then Φ and Γ determine a global bijection between the sets above, and this *same* bijection then makes $\mathcal{A}_{\mathrm{F\,G}}$ isomorphic to $\mathrm{G}(\mathcal{A})$. The Proposition then has a sharper form.

This is an immediate consequence of the Proposition:

COROLLARY. *Let* $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ *be an adjunction between* $\mathcal{A}$ *and* $\mathcal{B}$ *where* G *is a bijection between the objects of* $\mathcal{A}$ *and the objects of* $\mathcal{B}$. *Then the categories* $\mathcal{A}_{\mathrm{F\,G}}$ *and* $\mathcal{B}$ *are isomorphic.*

It is rather trivial to establish a kind of converse to the Corollary. It says that if the Kleisli category $\mathcal{A}_{\mathrm{L}}$ of a comonad in $\mathcal{A}$ is isomorphic to a category $\mathcal{B}$, then there is an adjunction $\langle \mathrm{L\,G}^{-1}, \mathrm{G}, \Phi, \Gamma \rangle$ between $\mathcal{A}$ and $\mathcal{B}$ where G is a bijection from the objects of $\mathcal{A}$ to the objects of $\mathcal{B}$. This is a consequence of the fact that there is an adjunction between $\mathcal{A}$ and $\mathcal{A}_{\mathrm{L}}$, as we saw in Section 4.1. The bijection G on objects of the adjunction $\langle \mathrm{L\,G}^{-1}, \mathrm{G}, \Phi, \Gamma \rangle$ is obtained from the isomorphism between $\mathcal{A}_{\mathrm{L}}$ and $\mathcal{B}$, since $\mathcal{A}$ has the same objects as $\mathcal{A}_{\mathrm{L}}$.

§5. Beyond the deduction theorem.

5.1. The ordinary deduction theorem. The ordinary deduction theorem is formulated for Hilbert systems. We are supposed to have in such a system an implication connective, and to qualify as an implication, this connective must presumably satisfy some minimal properties. We take that these are the properties we find in the implicational Hilbert systems of Section 1.4. We have there at least arrows of type $\mathtt{I} \vdash A \to A$ and closure under transitivity of implication and modus ponens.

So let $\mathcal{S}$ be an implicational Hilbert system and let $\mathcal{S}[x]$ be the extension of $\mathcal{S}$ with a new axiomatic arrow $x : \mathtt{I} \vdash C$, called an *indeterminate* by Lambek. The arrow x is not in $\mathcal{S}$, but C, and $\mathtt{I}$ too, are objects of $\mathcal{S}$. Usually, in logic, we are interested in cases where the new arrow $x : \mathtt{I} \vdash C$ is such that there are no arrows of the type $\mathtt{I} \vdash C$ in $\mathcal{S}$, but we need not exclude the case where $\mathcal{S}$ already has such an arrow. This would happen if we introduce a new proof for a theorem we already have. Anyway, this case is covered by the ordinary deduction theorem.

The objects of $\mathcal{S}[x]$ coincide with the objects of $\mathcal{S}$, while the arrows of $\mathcal{S}[x]$ are obtained by taking all the arrows of $\mathcal{S}$ plus x and closing under the operations on arrows assumed for implicational Hilbert systems, and perhaps other operations on arrows assumed for $\mathcal{S}$. (Sometimes, we may take for $\mathcal{S}[x]$ only a proper subset of the rules of inference assumed for $\mathcal{S}$, because these rules were meant for $\mathcal{S}$ and not for all of its extensions; such is the rule of universal generalization in the predicate calculus, which we may wish not to keep for extensions with axioms having free variables, or the rule of necessitation in modal logic. However, the operations on arrows of implicational Hilbert systems are always meant for extensions of $\mathcal{S}$, too.)

The ordinary deduction theorem states that

> For every arrow of the type $\mathtt{I} \vdash A$ in $\mathcal{S}[x]$ there is an arrow of the type $\mathtt{I} \vdash C \to A$ in $\mathcal{S}$.

Note that the following converse of the deduction theorem is an immediate consequence of the closure of $\mathcal{S}[x]$ under modus ponens:

> For every arrow of the type $\mathtt{I} \vdash C \rightarrow A$ in $\mathcal{S}$ there is an arrow of the type $\mathtt{I} \vdash A$ in $\mathcal{S}[x]$.

5.2. The deduction theorem in deductive systems. Suppose now $\mathcal{S}$ is a deductive system and $\mathcal{S}[x]$ is obtained from $\mathcal{S}$ as follows. The objects of $\mathcal{S}[x]$ coincide with the objects of $\mathcal{S}$, while the arrows of $\mathcal{S}[x]$ are obtained by taking the graph of $\mathcal{S}$ plus a new indeterminate arrow $x : \mathtt{I} \vdash C$ and closing under the operations on arrows of deductive systems; i.e., we must have in $\mathcal{S}[x]$ the identity arrows—which are already there, because they are in $\mathcal{S}$, with the same objects—and we must close under composition. This if we want $\mathcal{S}[x]$ to be just a deductive system. We may, however, want $\mathcal{S}[x]$ to be a deductive system of a particular sort, to which $\mathcal{S}$ belongs, too. In that case we may require that the arrows of $\mathcal{S}[x]$ be closed under additional operations.

If $\mathcal{S}$ is a category, we may impose on the arrows of $\mathcal{S}[x]$ equalities that hold in all categories; if $\mathcal{S}$ is a category of a particular sort, we may impose on the arrows of $\mathcal{S}[x]$ additional equalities, pertaining to this particular sort of category. This is how we obtain Lambek's *polynomial* categories, which we shall consider in the next section. It is essential for all these constructions of $\mathcal{S}[x]$ out of $\mathcal{S}$ that $\mathcal{S}[x]$ is a *free* extension of $\mathcal{S}$ of some sort: the graph of $\mathcal{S}$ with the arrow x added generates freely an implicational Hilbert system, or a deductive system, or a particular sort of deductive system, or a category, or a particular sort of category. We shall formulate this freedom with more precision in categorial terms in the next section.

Suppose now $\mathcal{S}$ and $\mathcal{S}[x]$ are implicational deductive systems in the sense of Section 1.4; i.e., they are such that an arrow of the type $A \vdash B$ is in them iff an arrow of the type $\mathtt{I} \vdash A \rightarrow B$ is. This is the sort of deductive system out of which implicational Hilbert systems are separated. The ordinary deduction theorem applied to such $\mathcal{S}$ and $\mathcal{S}[x]$ is equivalent to the statement

> For every arrow of the type $\mathtt{I} \vdash A$ in $\mathcal{S}[x]$ there is an arrow of the type $C \vdash A$ in $\mathcal{S}$.

Since $\rightarrow$ does not occur in the formulation of this statement, we may inquire whether the statement holds also for deductive systems that are not necessarily implicational—they may lack $\rightarrow$ (but they must, however, have $\mathtt{I}$).

Note that this version of the deduction theorem is a different matter from the deduction theorem codified by the implication-introduction rule we may have in the deductive systems $\mathcal{S}$ and $\mathcal{S}[x]$ when they correspond to natural-deduction systems or ordinary sequent systems. The latter is tied to the presence of $\rightarrow$ and the * operations on arrows of Section 2.5, whereas the former does not depend on that. The deductive systems $\mathcal{S}$ and $\mathcal{S}[x]$ may have * without satisfying the new version of the deduction theorem, and they may lack * and satisfy it. But, of course, there is an analogy between these two things.

The converse of the new deduction theorem is

> For every arrow of the type $C \vdash A$ in $\mathcal{S}$ there is an arrow of the type $\mathtt{I} \vdash A$ in $\mathcal{S}[x]$.

This is an immediate consequence of the fact that $\mathcal{S}[x]$ is a deductive system; so its arrows are closed under composition.

Assume now that in the implicational deductive systems $\mathcal{S}$ and $\mathcal{S}[x]$ we have a connective $\bullet$ and closure under the operation that to an arrow of the type $D \vdash A \rightarrow B$ assigns an arrow of the type $D \bullet A \vdash B$; we also assume that we have arrows of type $A \vdash A \bullet \mathtt{I}$. Then the preceding version of the deduction theorem is equivalent to the statement

> For every arrow of the type $B \vdash A$ in $\mathcal{S}[x]$ there is an arrow of the type $C \bullet B \vdash A$ in $\mathcal{S}$.

This last version of the deduction theorem has the right form for a proof by induction on the length of proof of $B \vdash A$ in $\mathcal{S}[x]$. With the preceding version we would run into trouble in such an inductive proof because the arrow of the type $\mathtt{I} \vdash A$ could be obtained by composition from arrows of types $\mathtt{I} \vdash D$ and $D \vdash A$. Of course, we presumably have to make more assumptions than above concerning $\bullet$ for this inductive proof to go through. But we can make these assumptions independently of $\rightarrow$, which is again not mentioned in the statement of the deduction theorem.

We generalize the last version of the deduction theorem to the following statement about an unidentified function assigning to every object B an object $\mathrm{F}(B)$:

> For every arrow of the type $B \vdash A$ in $\mathcal{S}[x]$ there is an arrow of the type $\mathrm{F}(B) \vdash A$ in $\mathcal{S}$.

The function $C \bullet \_$ is just a particular case of F, and we may envisage other functions in various sorts of deductive system for which the last statement holds. These functions may be tied to connectives, but not as simply as $C \bullet \_$ is (cf. Section 7.2). We may also envisage functions F that are not tied to connectives, and are even not one-one. That happens, in particular, when our deductive system is not syntax, but a model of syntax. For example, we take a concrete cartesian category, like the category of sets, as a model of a free cartesian category, i.e., conjunctive logic; then $\bullet$ corresponds to the cartesian product, and $\emptyset \times \_$ is not a one-one function.

But we might perhaps envisage a similar situation in syntax too, where F would consist in some sort of coding, which would presumably involve C, but in a more complicated way than $C \bullet \_$ does. If, contrary to what we had up to now, there are in $\mathcal{S}[x]$ objects that are not in $\mathcal{S}$ (perhaps C itself is not an object of $\mathcal{S}$), then F could be involved in the translation of the extended language in the language of $\mathcal{S}$, a translation that need not be one-one—though such translations are often one-one.

We might also envisage a weakening of the deduction theorem where the function F would not be uniformly given for all the arrows of type $B \vdash A$ in $\mathcal{S}[x]$. We would then have only that for every arrow of the type $B \vdash A$ in $\mathcal{S}[x]$ there is a function F on objects and an arrow of the type $\mathrm{F}(B) \vdash A$ in $\mathcal{S}$ (cf. the conclusion of [D. & Petrić 1993]).

If, however, we keep the assumption that $\mathcal{S}$ and $\mathcal{S}[x]$ have the same objects, and wish to have something not too distant from the usual deduction theorem, then in syntax F will be tied to a connective of $\mathcal{S}$, and will be one-one.

The converse of the last version of the deduction theorem is

> For every arrow of the type $\mathrm{F}(B) \vdash A$ in $\mathcal{S}$ there is an arrow of the type $B \vdash A$ in $\mathcal{S}[x]$.

This statement is the obvious generalization of statements converse to previous versions of the deduction theorem, statements that were tied to having modus ponens and composition in $\mathcal{S}[x]$, and were hence quite unproblematic.

5.3. Polynomial categories and the heritage functor. Let the schematic expression "cart-category" range over "cartesian category," "cartesian closed category" and "bicartesian closed category." A cart-functor from a cart-category to a cart-category is one that preserves the cart-category structure.

In the future we shall take that the deductive system $\mathcal{S}$ from the preceding section is a cart-category $\mathcal{A}$, while $\mathcal{S}[x]$ will be the polynomial cart-category $\mathcal{A}[x]$ obtained by extending $\mathcal{A}$ with the indeterminate arrow $x : \top \vdash C$ for $\top$ the terminal object and C an object of $\mathcal{A}$. As we indicated in the two preceding sections, the construction of $\mathcal{A}[x]$ consists first in adding the arrow x to the graph of $\mathcal{A}$ and closing the resulting graph under the operations on arrows of cart-categories. This yields a deductive system that may be called the cart-deductive system $\mathcal{A}$ plus x. On the arrows of this deductive system we then impose the equalities that must hold between arrows of all cart-categories. To impose these equalities means defining an appropriate smallest equivalence relation between the arrows of the cart-deductive system $\mathcal{A}$ plus x and taking the equivalence classes as the arrows of $\mathcal{A}[x]$. This construction is exposed in [Lambek & Scott 1986, Part I, Chapters 4 and 5] and in previous papers of Lambek, and we shall not repeat it here.

From $\mathcal{A}$ to $\mathcal{A}[x]$ there is a functor G that is identity on objects and that to every arrow f of $\mathcal{A}$ assigns the arrow of $\mathcal{A}[x]$ resulting from f in the graph of $\mathcal{A}$ plus x after arrows have been equated in $\mathcal{A}[x]$. Formally, the arrows of $\mathcal{A}[x]$ are equivalence classes of arrows in the cart-deductive system $\mathcal{A}$ plus x, and $\mathrm{G}(f)$ is the equivalence class of f. We call this functor the *heritage* functor, because it maps the arrows of $\mathcal{A}$ onto the indeterminate-free polynomial arrows of $\mathcal{A}[x]$ that originate in $\mathcal{A}$, and for which we may therefore say that they are inherited from $\mathcal{A}$. (Note that in $\mathcal{A}[x]$ an arrow-term not containing x is always equal to one containing x since $f\mathbf{k}^2\langle x\mathbf{k}, \mathbf{1}\rangle = f$; and for $x : \top \vdash \top$ we have $x = \mathbf{k}_\top$.) From the proof-theoretical point of view, the heritage functor G gives an image of the old system in the extension. It is clear that this functor is a cart-functor.

In general, the heritage functor is not the inclusion functor, because in $\mathcal{A}[x]$ the arrow f may be equated to arrows distinct from it in $\mathcal{A}$. This can be expressed by saying that the functor G need not be faithful (which means that the induced functions, which assign $\mathrm{G}(f)$ to f, from the hom-sets $\{f \mid f : A_1 \vdash A_2\}$ of arrows of $\mathcal{A}$ to the hom-sets $\{g \mid g : \mathrm{G}(A_1) \vdash \mathrm{G}(A_2)\}$ of arrows of $\mathcal{A}[x]$ are not one-one). Faithfulness gives in general only "local" embeddings, but for the heritage functor, which is identity on objects and assigns to every arrow in its domain an arrow designated by the same arrow-term, faithfulness amounts to being the inclusion functor. The main example where G is not faithful is when $\mathcal{A}$ is the cart-category of sets with functions and x is of the type $\{\emptyset\} \vdash \emptyset$. Then any two arrows of $\mathcal{A}$ with the same source and target are equated in the polynomial cart-category $\mathcal{A}[x]$ (since $\emptyset \times A$ is $\emptyset$ and $f_1\mathbf{k}^2_{\emptyset,A} = f_2\mathbf{k}^2_{\emptyset,A}$ in $\mathcal{A}$, by composing with $\langle x\mathbf{k}_A, \mathbf{1}_A\rangle$ in $\mathcal{A}[x]$ we obtain $\mathrm{G}(f_1) = \mathrm{G}(f_2)$).

However, the heritage functor G is faithful in special cases. First, when there is already in $\mathcal{A}$ an arrow of the same type as x (see [Lambek 1974] and [Lambek and Scott 1986, I.5]), which is a case we don't usually find interesting for logic (though it may be). The second case is quite interesting for logic: it is the case when "cart-category" means "cartesian closed category" and $\mathcal{A}$ is a free cartesian closed category (see [Čubrić 1994]; it is shown in this paper that for cartesian closed categories G is faithful iff the unique arrow $\mathbf{k}_C$ from the C of $x : \top \vdash C$ to the terminal object $\top$ is epi, i.e., right-cancelable in compositions). The faithfulness of G can probably be also shown when "cart-category" means "cartesian category" and $\mathcal{A}$ is a free cartesian category.

The freedom of the extension of $\mathcal{A}$ to $\mathcal{A}[x]$ is expressed by the following property involving the heritage functor G:

> For every cart-functor M from $\mathcal{A}$ to a cart-category $\mathcal{D}$ and every arrow $h : \top \vdash \mathrm{M}(C)$ in $\mathcal{D}$ there is a unique cart-functor M' from $\mathcal{A}[x]$ to $\mathcal{D}$ such that $\mathrm{M}'(x) = h$ and G composed with M' equals M.

This property characterizes $\mathcal{A}[x]$ up to isomorphism.

5.4. Strengthening the deduction theorem in categories. Instead of the deductive systems $\mathcal{S}$ and $\mathcal{S}[x]$ of Section 5.2 we now take a cart-category $\mathcal{A}$ and a polynomial cart-category $\mathcal{A}[x]$ obtained by extending $\mathcal{A}$ with $x : \top \vdash C$. We shall phrase the numbered statements below concerning these categories so that they can be compared with what we had in the previous parts of the paper, and in particular Section 3.1. The category $\mathcal{A}[x]$ will correspond to $\mathcal{B}$. The conventions of Section 3.1 will apply under the following assumptions:

the objects of $\mathcal{A}$ are included among the objects of $\mathcal{A}[x]$;
F is a function from the objects of $\mathcal{A}[x]$ into the objects of $\mathcal{A}$;
G is the inclusion function from the objects of $\mathcal{A}$ into the objects of $\mathcal{A}[x]$.

These assumptions impose somewhat more care than strictly needed, since the objects of $\mathcal{A}$ and $\mathcal{A}[x]$ actually coincide, but we prefer to have statements that could cover also a more general situation (such as some of those envisaged at the end of Section 5.2). The object $\mathrm{F}(B)$ of $\mathcal{A}$ is $C \times B$, which corresponds to the $C \bullet B$ we had before, while $\mathrm{G}(A)$ is A. The function F is actually a unary operation on the objects of $\mathcal{A}$, and G is the identity function on the objects of $\mathcal{A}$.

Then the last version of the deduction theorem from Section 5.2 reads in the new context:

For every arrow $g : B \vdash A$ in $\mathcal{A}[x]$ there is an arrow $f : \mathrm{F}(B) \vdash A$ in $\mathcal{A}$.

As a first strengthening of this deduction theorem we take the following statement, obtained by "skolemizing":

(1) There is a function $\Phi_{B,A}$ such that for every arrow $g : B \vdash A$ in $\mathcal{A}[x]$ the arrow $\Phi_{B,A}(g) : \mathrm{F}(B) \vdash A$ is in $\mathcal{A}$.

This strengthening is quite natural if we consider how the deduction theorem is usually proven by induction on the length of proof. We should define $\Phi_{B,A}(g)$ by induction on the complexity of g, i.e., by induction on the length of proof of the arrow g in $\mathcal{A}[x]$. With cart-categories, $\Phi_{B,A}$ is Lambek's κ_x (an avatar of λx), defined inductively as follows on the complexity of $g : B \vdash A$ to which it is applied in $\kappa_x(g)$; in the last clause of this definition, $\mathbf{b} : (C \times B) \times A_1 \vdash C \times (B \times A_1)$, where A is $A_1 \rightarrow A_2$, is an arrow definable in every cartesian category:

$$\kappa_x(\mathrm{G}(f)) = f\mathbf{k}^2_{C,B}, \text{ for G the heritage functor}$$
$$\kappa_x(x) = \mathbf{k}^1_{C,\top}$$
$$\kappa_x(g_2 g_1) = \kappa_x(g_2)\langle \mathbf{k}^1_{C,B}, \kappa_x(g_1)\rangle$$
$$\kappa_x(\langle g_1, g_2\rangle) = \langle \kappa_x(g_1), \kappa_x(g_2)\rangle$$
$$\kappa_x(h^*) = (\kappa_x(h)\mathbf{b})^*.$$

Consider now the following version of the statement last displayed in Section 5.2, which is motivated by the presence of modus ponens and composition in $\mathcal{A}[x]$:

> For every arrow $\Phi_{B,A}(g) : \mathrm{F}(B) \vdash A$ in $\mathcal{A}$ there is an arrow $h : B \vdash A$ in $\mathcal{A}[x]$.

By "skolemizing" this statement we obtain

> There is a function $\Gamma_{B,A}$ such that for every arrow $\Phi_{B,A}(g) : \mathrm{F}(B) \vdash A$ in $\mathcal{A}$ the arrow $\Gamma_{B,A}(\Phi_{B,A}(g))$: $B \vdash A$ is in $\mathcal{A}[x]$.

Then we may envisage that in $\mathcal{A}[x]$ we have

$$\Gamma_{B,A}(\Phi_{B,A}(g)) = g$$

which is the equality $(\Gamma\Phi)$ of Section 3.1. This amounts to strengthening further the deduction theorem by stating the following in addition to (1):

(2) The function $\Phi_{B,A}$ is one-one.

Of course, $(\Gamma\Phi)$ implies also that $\Gamma_{B,A}$ is onto the arrows of $\mathcal{A}[x]$ of the type $B \vdash A$.

It is only now that we make an essential use of our categorial framework. The statement (1) makes sense also for arbitrary deductive systems, whereas (2) is worth considering only if we have a significant equality relation between arrows. This is the first real strengthening of the deduction theorem, because, as we said above, (1) is implicit in the ordinary way of proving this theorem.

The main reason for considering (2) is that we have it for cart-categories. There, we have a binary *application* operation—a kind of cut—that to $x : \mathsf{T} \vdash C$ and an arrow $f : C \times B \vdash A$ assigns the arrow $f'x : B \vdash A$ (when B is T and f is of the type $C \vdash A$, then instead of application we may consider fx, i.e., x composed with f). The function $\Gamma_{B,A}$ is composition with $'x$, which is defined as $\langle x\mathbf{k}_B, \mathbf{1}_B\rangle$; so $f'x$ is $f\langle x\mathbf{k}_B, \mathbf{1}_B\rangle$. One can then prove by induction on the complexity of g the $(\Gamma\Phi)$ equality $\kappa_x(g)'x = g$, an equality analogous to β conversion in the lambda calculus (see [Lambek 1974], [Lambek & Scott 1986, I.6]).

A further strengthening of (1) is obtained by stating the following, in addition to (2):

(3) The function $\Phi_{B,A}$ is onto the arrows of $\mathcal{A}$ of the type $\mathrm{F}(B) \vdash A$.

This amounts to asserting that for every arrow $f : \mathrm{F}(B) \vdash A$ of $\mathcal{A}$ we have in $\mathcal{A}$

$$\Phi_{B,A}(\Gamma_{B,A}(f)) = f$$

which is the equality $(\Phi\Gamma)$ of Section 3.1. This strengthening can be proven for κ_x and $'x$, for which we have $\kappa_x(f'x) = f$, an equality analogous to η conversion in the lambda calculus. So this is an extensionality principle.

This extensionality is perhaps easier to understand if we approach it through the $\Gamma_{B,A}$ functions, for which it tells us that they are one-one; namely, if $f_1'x = f_2'x$, then $f_1 = f_2$.

A final strengthening of (1) is

$$(\Gamma\Gamma) \qquad \Gamma(f_2 f_1) = \Gamma(f_2\Phi(\mathbf{1}))\Gamma(f_1)$$

(with the indices of Γ, Φ and $\mathbf{1}$ omitted). This can also readily be checked for κ_x and $'x$ (we have $(f_2\kappa_x(\mathbf{1}))'x = f_2$ and $f_2(f_1'x) = (f_2 f_1)'x$). Equivalently, we could take the equality $(\Phi\Phi)$ of Section 3.1, and verify it for κ_x and $'x$ (for that the clause for $\kappa_x(g_2 g_1)$ in the definition of κ_x is essential).

The statements (1), (2), (3) and $(\Gamma\Gamma)$ are all summed up in the statement

$\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma\rangle$ is an adjunction between $\mathcal{A}$ and $\mathcal{A}[x]$.

§6. The Deductive Completeness Theorem.

6.1. Functional completeness and deductive completeness. In Lambek's formulation of the Functional Completeness Theorem, κ_x is not explicitly mentioned (it appears in the proof). It is said instead that composition with $'x$ is onto and one-one. This amounts to stating (1), (2) and (3) from the preceding section. The equalities $(\Gamma\Gamma)$ and $(\Phi\Phi)$ are also not explicitly mentioned. But Lambek envisages the adjunction these equalities yield in [1974, §5] and [Lambek & Scott 1986, I.7].

It is easily verified that in this adjunction $\mathrm{F}(g)$, which is by definition $\Phi(\Gamma(\mathbf{1})g)$, i.e., $\kappa_x((\mathbf{1}'x)g)$, is equal to $\langle \mathbf{k}^1, \kappa_x(g)\rangle$ in $\mathcal{A}$ (so the right-hand side of the clause for $\kappa_x(g_2 g_1)$ in the definition of κ_x in the preceding section amounts to $\kappa_x(g_2)\,\mathrm{F}(g_1)$), while $\mathrm{G}(f)$, which is by definition $\Gamma(f\Phi(\mathbf{1}))$, i.e., $(f\kappa_x(\mathbf{1}))'x$, is equal to f in $\mathcal{A}[x]$. So the right adjoint of this adjunction is the heritage functor from $\mathcal{A}$ to $\mathcal{A}[x]$. We can then express concisely the last statement of the preceding section in the following manner.

DEDUCTIVE COMPLETENESS THEOREM. *The heritage functor from $\mathcal{A}$ to $\mathcal{A}[x]$ has a left adjoint.*

Of course, much information is hidden with such a formulation. We are not told anything about the particular form of the elements of the adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma\rangle$. This we learn in a preparation such as we had in the preceding two sections, which amounts to a proof of the theorem. The less informative statement has the advantage that it can be asserted as it is in other situations. When we envisage that, even the information that the functor adjoint to the heritage functor is left-adjoint could be suppressed. The essence of deductive completeness would be that the heritage functor has an adjoint functor—left or right, never mind (cf. Section 7.1). However, for the time being we don't want to lose sight of Lambek's Functional Completeness Theorem.

As indicated in Section 3.2, if in an adjunction $\langle \mathrm{F}, \mathrm{G}, \Phi, \Gamma \rangle$ we have that G is a one-one function on objects, then we can replace the families of two-sided local bijections $\Phi_{B,A}$ and $\Gamma_{B,A}$ by two families of one-sided local bijections Φ_B and Γ_B. From the latter families we can recover the former. In the adjunction of deductive completeness, G is the identity function on objects, and is hence one-one. So we can do with Φ_B and Γ_B.

This does not yet mean that there is a global bijection between the arrows of $\mathcal{A}[x]$ and the set $\{f \mid \exists A \exists B (f : C \times B \vdash A)\}$ of arrows of $\mathcal{A}$, unless the operation $C \times \_$, i.e., F, is a one-one function on objects. And this function is quite often one-one. In particular, it is such in the logically interesting case where $\times$ is the conjunction connective. In the main example of a cart-category, the category of sets with functions, $C \times \_$ is always a one-one function except when C is the empty set.

The adjunction of the Deductive Completeness Theorem fails in general to be a reflection of $\mathcal{A}[x]$ in $\mathcal{A}$. The category $\mathcal{A}$ is not always a subcategory of $\mathcal{A}[x]$ and the heritage functor G is not always the inclusion functor: in the general case, this functor fails to be faithful. However, this adjunction is right-inclusive in the sense of Section 3.3: the function G, being the identity function on objects, is also the inclusion on objects. (This is why $C \times \_$ behaves within $\mathcal{A}[x]$ like an S4 possibility operator.)

It sometimes happens, nevertheless, that the heritage functor G is the inclusion functor and that $\mathcal{A}$ is a subcategory of $\mathcal{A}[x]$. This is, in particular, the case when “cart-category” means “cartesian closed category” and $\mathcal{A}$ is a free cartesian closed category (and probably also when “cart-category” means “cartesian category” and $\mathcal{A}$ is a free cartesian category; cf. Section 5.3). The category $\mathcal{A}$ is such a free category when it is a purely logical system. When “cart-category” means “bicartesian closed category,” it is not enough to have freedom for $\mathcal{A}$. To ensure in this case that the unique arrow $\mathbf{k}_C$ from the C of $x : \mathsf{T} \vdash C$ to the terminal object T is epi (a necessary and sufficient condition for the faithfulness of G as far as cartesian closed categories are concerned; see [Čubrić 1994]) it is necessary, and perhaps also sufficient for a free bicartesian closed category $\mathcal{A}$, that C should not be isomorphic to the initial object; i.e., C should not be a contradiction.

So in some important logical cases the Deductive Completeness Theorem asserts that there is a reflection of $\mathcal{A}[x]$ in $\mathcal{A}$. This phrasing of the theorem is quite near to the spirit of the ordinary deduction theorem. It suggests that $\mathcal{A}[x]$ is somehow contained in $\mathcal{A}$. We shall consider in the next section what in general, with or without reflection, this being contained exactly means.

6.2. The polynomial category contained in the original category. In the comonad $\langle \mathrm{L}, \mathbb{I}, \circ \rangle$ of the adjunction of the Deductive Completeness Theorem, $\mathrm{L}(A)$ is $\mathrm{F}(A)$, i.e., $C \times A$, the arrow $\mathbb{I}_A$ is $\mathbf{k}^2_{C,A} : C \times A \vdash A$ and $f_2 \circ f_1$ is $f_2\langle \mathbf{k}^1, f_1 \rangle$ (note that $\kappa_x(g_2 g_1)$ is defined as $\kappa_x(g_2) \circ \kappa_x(g_1)$). In this comonad, the arrow $\mathrm{L}(f)$ is $\mathbf{1}_C \times f$, the arrow $\mathbf{d}_A$ is $\langle \mathbf{k}^1_{C,A}, \mathbf{1}_{C\times A}\rangle$ and $\mathrm{N}(f)$ is $\langle \mathbf{k}^1, f\rangle$. (The operation $C \times \_$ behaves within $\mathcal{A}$ like an S4 necessity operator. This means that within $\mathcal{A}[x]$, too, $C \times \_$ behaves in this manner, in addition to behaving like a possibility operator; indeed, in $\mathcal{A}[x]$ we have both $\langle x\mathbf{k}_A, \mathbf{1}_A\rangle : A \vdash C \times A$ and $\mathbf{k}^2_{C,A} : C \times A \vdash A$, which correspond respectively to the unit γ and counit φ of the adjunction.)

Since in the adjunction of the Deductive Completeness Theorem the identity function G is a bijection between the objects of $\mathcal{A}$ and the objects of $\mathcal{A}[x]$, we can conclude according to the Corollary of Section 4.3 that $\mathcal{A}[x]$ is isomorphic to the Kleisli category $\mathcal{A}_{\mathrm{F}}$ of the comonad of the adjunction. (To this fact is devoted §5 of [Lambek 1974] and Chapter 7, Part I, of [Lambek & Scott 1986].) The indeterminate $x : \mathsf{T} \vdash C$ corresponds to the T-copy of the arrow $\mathbf{k}^1_{C,\mathsf{T}} : C \times \mathsf{T} \vdash C$ in this isomorphism, as can be seen in the clause for $\kappa_x(x)$ in the definition of κ_x in Section 5.4. How the functor from $\mathcal{A}[x]$ to $\mathcal{A}_{\mathrm{F}}$ involved in the isomorphism preserves the cart-structure of $\mathcal{A}[x]$ can be seen by studying the clauses of this definition.

As we indicated in the last paragraph of Section 4.3, proving independently the isomorphism of $\mathcal{A}[x]$ with $\mathcal{A}_{\mathrm{F}}$ yields an alternative proof of the Deductive Completeness Theorem. This isomorphism can be proved either via Φ and Γ, or by showing that $\mathcal{A}_{\mathrm{F}}$ has the property recorded for $\mathcal{A}[x]$ at the end of Section 5.3. So the Deductive Completeness Theorem and the isomorphism of $\mathcal{A}[x]$ with $\mathcal{A}_{\mathrm{F}}$ entail each other.

For the comonad of the adjunction of the Deductive Completeness Theorem, the objects of the category $\mathrm{N}(\mathcal{A}_{\mathrm{F}})$, defined as in Section 4.2, are of the form $C \times A$, while its arrows are the arrows $g : C \times A \vdash C \times B$ of $\mathcal{A}$ such that for some arrow $f : C \times A \vdash B$ of $\mathcal{A}$ we have $g = \langle \mathbf{k}^1_{C,A}, f\rangle$, which is equivalent to assuming for g the equality $\mathbf{k}^1_{C,B} g = \mathbf{k}^1_{C,A}$. This equality is equivalent to the (homo) equality in the present context. When we make the arrows $\mathbf{k}^1_{C,A}$ play the role of the objects $C \times A$ of $\mathrm{N}(\mathcal{A}_{\mathrm{F}})$, we obtain a full subcategory $\mathcal{A}//C$ of the slice category of $\mathcal{A}$ over C (cf. [Lambek & Scott 1986, p. 64] and [Makkai 1993]; the slice category of $\mathcal{A}$ over C is isomorphic to the Eilenberg-Moore category, i.e., the category of coalgebras, of our comonad). If our comonad is injective (which means that F, i.e., $C \times \_$, is a one-one function on objects), then $\mathrm{N}(\mathcal{A}_{\mathrm{F}})$ is isomorphic to $\mathcal{A}_{\mathrm{F}}$; but, in general, $\mathrm{N}(\mathcal{A}_{\mathrm{F}})$ and $\mathcal{A}_{\mathrm{F}}$ are only equivalent categories.

The isomorphism of $\mathcal{A}[x]$ with $\mathcal{A}_{\mathrm{F}}$ is interesting because it tells in what sense $\mathcal{A}[x]$ is contained in $\mathcal{A}$. We may, however, represent that fact in alternative ways when $C \times \_$ is a one-one function, which is the case when $\times$ is a connective. Then $\mathcal{A}[x]$ is isomorphic to $\mathrm{N}(\mathcal{A}_{\mathrm{F}})$ (or its variant $\mathcal{A}//C$), and also to the category $\mathcal{A}^{\mathrm{F}}_{\mathrm{free}}$ of free coalgebras of the comonad (see Section 4.2). The isomorphism of $\mathcal{A}[x]$ with $\mathrm{N}(\mathcal{A}_{\mathrm{F}})$ is especially worth noting since

$\mathrm{N}(\mathcal{A}_{\mathrm{F}})$ is easy to grasp. The category $\mathrm{N}(\mathcal{A}_{\mathrm{F}})$ is a subcategory of $\mathcal{A}$, which is not the case for $\mathcal{A}_{\mathrm{F}}$ and $\mathcal{A}^{\mathrm{F}}_{\mathrm{free}}$, and when $\mathcal{A}[x]$ is isomorphic to it, which happens usually in the proof-theoretical context, then $\mathcal{A}[x]$ is almost literally contained in $\mathcal{A}$.

§7. Beyond deductive completeness.

7.1. Deductive completeness with implication. In a cartesian closed category there is a bijection between the hom-sets $\{f \mid f : C \times A_1 \vdash A_2\}$ and $\{f \mid f : C \vdash A_1 \to A_2\}$. So we could reformulate the Deductive Completeness Theorem by turning everything upside down.

The category $\mathcal{B}$ would be a given cartesian closed or bicartesian closed category and $\mathcal{B}[x]$ (corresponding to $\mathcal{A}$) would be the polynomial cartesian closed or bicartesian closed category obtained by extending $\mathcal{B}$ with the indeterminate $x : \top \vdash C$. The object $\mathrm{G}(B)$ is $C \to B$, while the identity function F on objects is part of the heritage functor F from $\mathcal{B}$ to $\mathcal{B}[x]$. The families of functions $\Phi_{B,A}$ and $\Gamma_{B,A}$ are given by two families of functions Φ_A and Γ_A, where for $g : B \vdash C \to A$ the arrow $\Phi_A(g) : B \vdash A$ is obtained from g and x with the help of the modus ponens arrow $\varepsilon_{C,A} : (C \to A) \times C \vdash A$, while for $f : B \vdash A$ the arrow $\Gamma_A(f) : B \vdash C \to A$ is obtained from $\kappa_x(f) : C \times B \vdash A$ by composing with $\langle \mathbf{k}^2, \mathbf{k}^1 \rangle : B \times C \vdash C \times B$ and applying the implication-introduction operation * of Section 2.5.

The Deductive Completeness Theorem says now that the heritage functor from $\mathcal{B}$ to $\mathcal{B}[x]$ has a right adjoint. This adjunction gives rise to a monad in $\mathcal{B}$, and the Kleisli category of this monad is isomorphic to $\mathcal{B}[x]$ (see Lambek 1974, Theorem 3.6, Corollaries 3.8, 5.4] and [Lambek & Scott 1986, I.7, p. 64, Exercise 1]). This new version of deductive completeness explains why we said in Section 6.1 that one shouldn't insist that the adjoint of the heritage functor is a left adjoint.

The new version of the Deductive Completeness Theorem may have the advantage that implication is now involved in it; so it should be closer to the ordinary deduction theorem. However, since we deal with deductive systems that have the turnstile $\vdash$ instead of the implication $\to$ of Hilbert systems, we don't really need implication. As we said in Section 2.5, conjunction may be taken as a more basic connective than implication.

This primacy of conjunction is reflected in the former version of the Deductive Completeness Theorem having a wider span than this new one, which doesn't cover cartesian categories that are not closed. We may increase the span of the new theorem by considering categories with $\to$, i.e., exponentiation, analogous to cartesian closed categories but lacking $\times$ (in the style of [Eilenberg & Kelly 1966]). Such categories, which seem quite a bit more complicated to describe than cartesian and cartesian closed categories, correspond to Hilbert systems, where everything hinges upon implication. (They also correspond to Schönfinkel's and Curry's way of keeping only functions with a single argument in combinatory logic.)

7.2. Modal deductive completeness. The Modal Functional Completeness Theorem of [D. & Petrić 1993] is proven for a hierarchy of sorts of categories that correspond to modal extensions of substructural logics. The weakest substructural logic envisaged for this hierarchy is Lambek's nonassociative calculus with a unit $\mathtt{I}$ for the product $\bullet$ and lattice connectives. The assumptions for the corresponding categories may be roughly described as follows.

A category $\mathcal{A}$ of this sort has a special object $\mathtt{I}$ and the binary operations on objects $\bullet$, $\rightarrow$ and $\leftarrow$, the operation $\bullet$ being tied to a bifunctor. If $\mathcal{T}$ is the trivial category with the single object $\mathtt{I}$ and the single arrow $\mathbf{1}_{\mathtt{I}}$, then there is an adjunction between the product category $\mathcal{T} \times \mathcal{A}$ and $\mathcal{A}$, with $\mathrm{F}(A)$ being $(\mathtt{I}, A)$ and $\mathrm{G}((\mathtt{I}, A))$ being $\mathtt{I} \bullet A$, and an analogous adjunction between the product category $\mathcal{A} \times \mathcal{T}$ and $\mathcal{A}$. There is also for every object C of $\mathcal{A}$ an adjunction between $\mathcal{A}$ and $\mathcal{A}$ with $\mathrm{F}(A)$ being $C \bullet A$ and $\mathrm{G}(A)$ being $C \rightarrow A$, and another adjunction between $\mathcal{A}$ and $\mathcal{A}$ with $\mathrm{F}(A)$ being $A \bullet C$ and $\mathrm{G}(A)$ being $A \leftarrow C$ (the implication $\rightarrow$ of cartesian closed categories as presented in Section 2.5 corresponds rather to $\leftarrow$ than to $\rightarrow$, but in that context the two implications are naturally isomorphic). Moreover, $\mathcal{A}$ is a bicartesian closed category with another set of operations on objects.

Among the sorts of categories in the hierarchy, obtained by gradually adding assumptions related to structural rules only (assumptions of a combinatory nature; see the next section), one finds in particular symmetric monoidal closed categories with $\bullet$ as tensor product, $\mathtt{I}$ as unit and $\rightarrow$ as internal hom-functors. Moreover, these categories are bicartesian with another set of operations on objects. They correspond to intuitionistic linear logic.

A category of one of these sorts S may be extended to a modal category of the sort S $\Box$ in a manner roughly described as follows. It is assumed for the extended category $\mathcal{A}$ that it has a unary operation $\Box$ on objects and that there is a subcategory $\mathcal{A}_\Box$ of $\mathcal{A}$ that is cartesian and is generated by the objects $\Box A$ of $\mathcal{A}$. In $\mathcal{A}_\Box$ the operation $\bullet$ behaves as cartesian product and $\mathtt{I}$ is the terminal object. Assumptions for $\Box$ and the associated arrows are drawn from an adjunction between $\mathcal{A}$ and $\mathcal{A}_\Box$, where F is the inclusion functor and $\mathrm{G}(A)$ is $\Box A$. This adjunction is a coreflection of $\mathcal{A}$ in $\mathcal{A}_\Box$; i.e., the inclusion functor has a right adjoint. (However, if S is not as strong as the sort of category of linear logic, we may need additional assumptions concerning modalized versions of the structural rules of association and permutation that are not induced by the coreflection.) These assumptions give to $\Box$ the properties of an S4 necessity operator that mimics moreover the missing structural rules. This makes for the same kind of modal extension one finds in linear logic.

From an S$\Box$ category $\mathcal{A}$ we obtain a polynomial S$\Box$ category $\mathcal{A}[x]$ in a manner analogous to what we had with cart-categories (assumptions for S$\Box$ categories are equational, too). Then the Modal Functional Completeness Theorem can be phrased exactly as the Deductive Completeness Theorem of Section 6.1. Now $\mathrm{F}(B)$ is $\Box C \bullet B$. (The formulation of the Modal Functional Completeness Theorem in [D. & Petrić 1993] imitates Lambek's formulation of the Functional Completeness Theorem. The comments about the bijection $'x$ in that paper should be understood in the light of Section 3.2 of the present paper. When the heritage functor is not faithful, the proof of Lemma 1 of [D. & Petrić 1993] should be phrased as the proof of Lemma 2.9 in [Lambek 1974] or the corresponding part of the proof of Proposition 6.1 in [Lambek & Scott 1986, I.6]; however, this is based on the same calculations as those of the proof of Lemma 1.)

Deductive completeness with $\Box C \bullet \_$ may be understood as being more general than deductive completeness with $C \times \_$ of Part 6. The latter is obtained by specializing the former: $\bullet$ becomes $\times$ and $\Box C$ becomes isomorphic to C. The operation $\Box$ may also serve for determining assumptions necessary for the Deductive Completeness Theorem (a task to which Section 5 of [D. & Petrić 1993] is devoted). Roughly speaking, $\Box$ serves to keep track of the C of $\mathcal{A}[x]$ when it is brought down into $\mathcal{A}$, and this helps in determining what exactly we should be able to do with this C in $\mathcal{A}$ in order to get the theorem. It happens that assumptions made for $\Box$ above are not only sufficient, but also necessary, for deductive completeness with $\Box C \bullet \_$; so we may say that the adjointness of this deductive completeness characterizes the operation $\Box$. Some of the conditions necessary for deductive completeness, namely, those for $(\Phi\Gamma)$, i.e., on the η side, are of the same kind as conditions necessary for coherence in various sorts of categories—in particular, symmetric monoidal closed categories.

If $\Box C$ is isomorphic to C, then assumptions for cartesian categories, concerning both the existence of arrows and the equalities between them, are not only sufficient, but also necessary, for the Deductive Completeness Theorem of Part 6. This may be taken as a vindication and strengthening of Gentzen's characterization of intuitionistic implication through the deduction theorem and modus ponens. Only, we don't speak exactly about implication, but rather about an intuitionistic turnstile.

7.3. Combinatory aspects of deductive completeness. One can approach deductive completeness by thinking about the connection between natural deduction and the typed lambda calculus. The gist of that is how the implication-introduction rule, i.e., the deduction theorem, is coded by lambda abstraction. The arrow $\Phi(g)$ of $\mathcal{A}$ assigned to the polynomial arrow g of $\mathcal{A}[x]$ by the Deductive Completeness Theorem is like that function abstracted by applying the lambda operator.

But how do we know we have something like a lambda operator in cart-categories? This is not explicitly postulated there, as it is in the lambda calculus. (The kind of lambda abstraction we have postulated in cartesian closed categories with the implication-introduction operations * is not of the kind we need for deductive completeness. Anyway, we have deductive completeness also for cartesian categories that are not closed.)

The proof of the Deductive Completeness Theorem consists then in the definition of Φ. Once we have that, the proof is pretty simple. A more important thing is to formulate the theorem.

The fact that we can define Φ in cart-categories is analogous to the fact that we can define lambda abstraction in systems of combinators. This property of combinatory logic is often called "combinatory completeness" (Curry called it "combinatorial completeness"), but Lambek, following [Rosenbloom 1950], calls it "functional completeness." (The latter denomination is quite reasonable, provided the notion is not confused with the functional completeness of classical propositional logic in, for example, implication and negation as only primitive connectives, which is a different matter.) Lambek called also "functional completeness" the analogous property of cart-categories. Indeed, "combinatory completeness" may seem misplaced in this context, where, at first glance at least, one doesn't see combinators. But the combinators are there.

Lambek's functional completeness is primarily tied to the cartesian structure of cart-categories, i.e., to the product $\times$. The other operations on objects are secondary. This is because $\mathrm{F}(B)$ is $C \times B$. Axioms concerning $\times$ in cartesian categories as usually presented don't seem to be about combinators. However, these axioms can be given in terms of combinators. These are not Schönfinkel's and Curry's combinators, which are tied to $\rightarrow$, but combinators tied to $\times$. And the combinators of $\times$ are easier to work with than the combinators of $\rightarrow$. (Most of the complications with combinators, and especially with those of $\rightarrow$, arise when extensionality is axiomatized by equalities, rather than by a rule. Lambek's version of the Functional Completeness Theorem of [1972], which involves only cartesian closed categories, is based explicitly on Schönfinkel's and Curry's combinators of $\rightarrow$.)

A combinatory axiomatization of cartesian categories is obtained by extending the usual axiomatization of symmetric monoidal categories. So we start with a bifunctor $\times$ and natural isomorphisms of types

$$(A \times B) \times C \vdash A \times (B \times C), \quad A \times B \vdash B \times A$$

which satisfy Mac Lane's pentagonal and hexagonal equalities of coherence. (An arrow is an isomorphism in a category iff it has an inverse that composed with it on both sides gives identity arrows.) The first kind of arrow corresponds to the combinator **B** and the second to the combinator **C**.

The arrows we have to add to these to obtain the product of cartesian categories are natural transformations with components of types

$$A \times B \vdash A, \quad A \times B \vdash B, \quad A \vdash A \times A$$

which have to satisfy appropriate coherence equalities, where they are mixed with arrows of the **B** and **C** kind and with each other (see [D. & Petrić 1993, 1994]). The first two types are the types of projections, which correspond to the combinator **K**, while the third is the type of diagonal arrows, which correspond to the combinator **W**. (We have used the same letters to name the corresponding operations on objects of the "cut-free" axiomatization of cartesian categories in Section 2.3.)

We didn't mention the arrows, inherited from symmetric monoidal categories, involving the unit object (or empty product), which in the presence of arrows of the **K** and **W** kind will make of this unit object a terminal object. Their role, though it may be indispensable for deductive completeness, is nevertheless auxiliary, and their combinatory aspects are less familiar. But we must mention arrows assumed for every category, the identity arrows of type $A \vdash A$, which correspond to the combinator **I**. And for every category we assume composition of arrows, which corresponds to combinatory application.

Among the combinatory arrows **I**, **B**, **C**, **K** and **W** tied to $\times$ there is a hierarchy easier to perceive than with Schönfinkel's and Curry's combinators. There is a basic layer with **I**, above which there is a layer with **B** and **C**, and a third layer with **K** and **W**. The isomorphisms are all in the first two layers (cf. [D. & Petrić 1994], which shows that no new isomorphism can arise in the third layer). Within the second layer, we may put **B**, which comes from monoidal categories, below **C**, which comes from symmetric monoidal categories. With **C** arrows of type $A \times A \vdash A \times A$ we obtain arrows of type $B \vdash B$ that are not identities—below **C** we don't have them. (There are also interesting sorts of categories without **K** or without **W** in the third layer, but they don't show clearly that among **K** and **W** one is above the other.) The **B** and **C** combinatory arrows of the second layer are definable in terms of the **K** and **W** combinatory arrows of the third layer, which is not the case with the corresponding combinators of Schönfinkel and Curry, where **B**, **C**, **K** and **W** are all independent. (However, independence may be regained in a modal context, or by restricting **K** and **W** arrows to product-free objects A and B, or by not assuming the functoriality of $\times$, or by sticking to arrows of a Hilbert system, as Schönfinkel and Curry do.)

It is well known that having all of **I**, **B**, **C**, **K** and **W**, together with application, is sufficient (and also necessary) for combinatory completeness. So, after all, one might call functional completeness "combinatory completeness" (but "deductive completeness" should stress better the ties with the deduction theorem and with deduction in general).

When clauses in the inductive definition of Lambek's κ_x, i.e., Φ, are expressed explicitly in terms of the combinatory arrows tied to $\times$, it becomes clear what features of cart-categories require particular combinators, and why all these combinatory arrows are needed for deductive completeness. (For example, in the clause for $\kappa_x(h^*)$ in Section 5.4 there is a defined combinatory arrow of the **B** kind.) This also shows how one can modify just small bits of the setting while still retaining a modified form of deductive completeness for categories corresponding to substructural logics, like symmetric monoidal closed categories (see [D. & Petrić 1993, Section 6]).

Conclusion. At the beginning of the first part of this paper we mentioned two styles of axiomatizing logic—Hilbert's style and Gentzen's style. (Tying Hilbert's name to the first style is a convention—Hilbert had predecessors. Tying Gentzen's name to the second style is more than a convention.) In Hilbert systems we have a minimum of rules of inference (we cannot get rid of rules completely, unless we postulate all theorems as axioms), whereas in Gentzen's natural-deduction systems we have a minimum of axioms. For propositional logic in Hilbert's style we usually have just modus ponens as a rule, and the deduction theorem has to be proved, whereas in Gentzen's style we have no axioms at all, but at the cost of postulating the deduction theorem as a primitive rule.

These two styles of axiomatizing logic correspond to the two styles of combinatory logic. Hilbert's style is in systems of combinators, where application, corresponding to modus ponens, is the only operation, and combinatory completeness, i.e., the existence of lambda abstraction, has to be proved. Gentzen's style is in the lambda calculus, where we have no primitive combinators at all, but at the cost of postulating lambda abstraction as primitive.

Proving the Deductive Completeness Theorem for cart-categories means working on Hilbert's side of the matter. (It is the Hilbert side of an underlying Gentzen system.) On Gentzen's side we would postulate deductive completeness and try to see what sort of category this yields—as Gentzen postulated the deduction theorem with modus ponens and obtained intuitionistic implication. At the end of Section 7.2 we mentioned that, working on Gentzen's side, the cartesian structure of cart-categories is induced by deductive completeness. And this should not be taken as a vindication of conjunction, but rather as a vindication of the usual structural rules corresponding to the **I**, **B**, **C**, **K** and **W** combinators, with which the cartesian structure of cart-categories is built.

It is well known that cartesian categories can be characterized by an adjointness involving the product category of such a category with itself and this category (whose counit is a pair made of the natural transformations $\mathbf{k}^1$ and $\mathbf{k}^2$, and whose unit is the natural transformation $\langle \mathbf{1}, \mathbf{1} \rangle$, i.e., $\mathrm{W}(\mathbf{1})$). The adjointness of deductive completeness is another adjointness that characterizes cartesian categories.

The Deductive Completeness Theorem need not seem very surprising—especially after the combinatory structure of cartesian categories is made explicit. One can have the feeling it is one of these things that had to hold. But such are many completeness theorems, starting with Gödel's completeness theorem for predicate logic. (To be really surprised we need something else than completeness, as the case of another theorem of Gödel's shows with such force.) With the deduction theorem we also feel it had to hold, if implication is implication.

This lack of surprise may also be a sign of merit. Results that clarify matters quickly acquire an air of familiarity, so that it is hard to believe there were times when they were not known.

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