THE BULLETIN OF SYMBOLIC LOGIC
Volume 2, Number 2, June 1996

DEFINABILITY, AUTOMORPHISMS, AND DYNAMIC PROPERTIES OF COMPUTABLY ENUMERABLE SETS

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Abstract. We announce and explain recent results on the computably enumerable (c.e.) sets, especially their *definability* properties (as sets in the spirit of Cantor), their *automorphisms* (in the spirit of Felix Klein's *Erlanger Programm*), their *dynamic* properties, expressed in terms of how *quickly* elements enter them relative to elements entering other sets, and the *Martin Invariance Conjecture* on their Turing degrees, i.e., their *information content* with respect to relative computability (Turing reducibility).

§1. Introduction. All functions are on the nonnegative integers, $\omega = \{0, 1, 2, \dots\}$, and all sets will be subsets of ω. Turing and Gödel *informally* called a function *computable* if it can be calculated by a mechanical procedure, and regarded this as being synonymous with being specified by an "algorithm" or a "finite combinatorial procedure." They each formalized it as follows.[1]

A function is *Turing computable* if it is definable by a Turing machine, as defined by Turing *1936*, see Soare [*1987*, p. 11], and f is *recursive* if it is *general recursive* as defined by Gödel *1934*, see [Kleene, *1952*, p. 274]. A set A is *computably enumerable* (*c.e.*) if A is $\emptyset$ or is the range of a Turing computable function; likewise A is *recursively enumerable* (*r.e.*) if A is $\emptyset$ or is the range of a recursive function. By Turing's Thesis (T.T.) *1936* (which we accept) the (informal) class of *computable* functions is the same as the (formal) class of *Turing computable* functions. Hence, we shall use the term "*computable*" for either class of functions, and likewise we shall

Received November 22, 1995 and in revised form, January 24, 1996.

The first author was supported by National Science Foundation Grant DMS 92-14048, and the second author by National Science Foundation Grants DMS 91-06714 and DMS 94-00825. The second author gave some of these results as part of a lecture at the 10^{th} International Congress for Logic, Methodology and Philosophy of Science, Section 3: Recursion Theory and Constructivism, August 19–25, 1995.

[1]We cite references by their number in LaTeX form in our bibliography but also in the usual convention by author and year (in italics), e.g., [Post, *1944*] or Post *1944*. We omit from our bibliography those references which appear in Soare *1987* and we cite them by year method. In case an author has several publications in the same year they are referred to in order of listing, e.g., [Kleene, *1952*] (= *1952a*), [Kleene, *1952b*], etc.

1079-8986/96/0202-0004/$2.50

use the term "*computably enumerable*" for either class of sets.[2] We do not use "computably enumerable" interchangeably with "recursively enumerable" because they are not *intensionally* equivalent, but only *extensionally* equivalent.

Post *1943* introduced the formalism of a (normal) production system which was a *generational* system in contrast to the *computational* systems of Church, Gödel, Turing, and Kleene. This led Post *1944* to explore the properties of what he called *effectively enumerable sets* or *generated sets* rather than computable *functions*. He was particularly interested in the *information content* of a c.e. set A, namely what we also call its *degree*, usually under Turing reducibility ($A \leq_T B$), but also under certain stronger reducibilities studied by Post such as 1:1, m:1 and tt-reducibilities. Post's *problem* was whether there is a noncomputable c.e. set A which is *incomplete*, namely $\emptyset <_T A <_T K$, for K the complete set $\{W_e : e \in W_e\}$, where $\{W_e\}_{e \in \omega}$ is an effective listing of all c.e. sets. Post's problem was the central theme of Post's paper and of much subsequent research. *Convention.* From now on all sets will be c.e. unless otherwise mentioned.

To this end, Post considered certain "set-theoretic" properties of c.e. sets, i.e., those definable as "sets," e.g., using set inclusion, $\subset$. For example, Post defined a *simple set A* to be a coinfinite c.e. set which intersects every infinite c.e. set, a property which is definable in the language $L(\subset)$, because the predicate $Fin(A)$:"A is finite" is easily seen [12, Ch. X] to be $\subset$-definable (every subset of A is complemented). Let $\mathscr{E}$ denote the partial ordering of the c.e. sets under inclusion, $\mathscr{E} = (\{W_e\}_{e \in \omega}, \subset)$, and let $\mathscr{E}^*$ denote $\mathscr{E}$ modulo the class $\mathscr{F}$ of finite sets.[3] Lachlan has shown that a property of c.e. sets which is closed under finite differences is $\mathscr{E}$-definable iff it is $\mathscr{E}^*$-definable, so we can choose whichever structure is more convenient.

Among the properties originally considered by Post, simplicity and hh-simplicity are $\mathscr{E}$-definable, but h-simplicity is not (see [12]). (A set A is hh-simple if every superset is complemented modulo A by Lachlan *1968*.) Post was searching for a c.e. noncomputable A whose complement while infinite was so thin that it forced A to be incomplete. Post's Problem was solved by Friedberg and independently by Muchnik with the introduction

[2]This terminology is the same as that introduced in the 1930's and used since then, except for the term "computably enumerable," recently introduced, because Turing and Gödel did not explicitly introduce a term for these corresponding *sets*, but just for the computable *functions*.

[3]Post *1943* and Myhill *1956* noted that $\mathscr{E}$ trivially forms a lattice because c.e. sets are obviously closed under union and intersection. Hence $\mathscr{E}$ is often called "the *lattice* of c.e. sets," but the lattice properties play very little role in the present results and in most of those cited. Thus, we prefer to refer to $\mathscr{E}$ here just as the *partial ordering of the c.e. sets* in order to stress that these results relate the *set theoretic structure* of a c.e. A to its *information content*.

of the priority method. Post's program remained open, however, and stimulated much research on the more general and fundamental question of studying the relationship between the algebraic structure of a c.e. set A and $\deg(A)$.

The set with the thinnest complement of all, a maximal set, was constructed by Friedberg *1958*. Yates *1965* proved that maximal sets could be complete (thereby ruining the *thinness* idea of Post's program); Sacks *1964* proved they could be incomplete; and Martin *1966b* proved that their degrees were exactly the high degrees, $\mathbf{H}_1$, those with jump $0''$. This led to a great deal of activity beginning in the 1960's in classifying the $\mathscr{E}$-definable properties and also led to results connecting the $\mathscr{E}$-definable properties of A to the information content of A.

Marchenkov proved that η-maximal semi-recursive c.e. sets are incomplete, and D. Miller later showed they are all low_2. Soare [12, p. 73] gave a property characterizing low c.e. sets, and Ambos-Spies and Nies gave a property characterizing c.e. sets whose degrees are *cappable* (i.e., halves of a minimal pair). However, these three properties are all non $\mathscr{E}$-definable by Theorem 3.1 and Theorem 3.2 below, respectively.

We now report on several recent results on these broad themes; the full proofs will appear in [8], and five forthcoming papers by Harrington and Soare, [7, 4, 9, 5, 6]. Background information on $\mathscr{E}$ can be found in Soare *1987* and in the *Handbook* Chapter, [Soare, ta]. In §2 we describe a definable solution to Post's program, namely a nontrivial predicate $Q(A)$ definable in $L(\subset)$ over the c.e. sets which insures that A is incomplete. This property is shown to be closely related to a *dynamic* property of being so-called "2-*tardy,*" namely a property like "promptly simple" which is measured with respect to how quickly elements appear in A under some enumeration. However, none of these dynamic properties is $L(\subset)$-definable. The $Q(A)$ property tends to slow down the enumeration of elements into A, forcing A to be incomplete. In contrast we describe an $L(\subset)$-definable property $T(A)$ which allows such a rapid flow of elements into A that A must be *complete* even though A may possess many other properties such as being promptly simple. We also present a related property $NL(A)$ which has a slower flow but fast enough to guarantee that A is not low, even though A may possess virtually all other related lowness properties (low_2 and others) and A may simultaneously be promptly simple. In §3 we describe a new method generating automorphisms of $\mathscr{E}$, which yields results that may be considered in opposition to the definability results and we study the possibility of coding a set into every nontrivial orbit. In §4 we consider an old conjecture of Martin about which classes of c.e. degrees are *invariant*, and we supply new evidence for the conjecture. Throughout all the sections runs the idea of dynamic properties as a kind of *leitmotiv*.

§2. $\mathscr{E}$-definability and information content. We begin by producing an $\mathscr{E}$-definable property $Q(A)$ which guarantees that A is *incomplete* and thereby represents an $\mathscr{E}$-definable solution to Post's problem. Next we exhibit a dual property $T(A)$ which guarantees that A is *complete* but possibly simple, even promptly simple, and then a third property $NL(A)$ which holds of *no low* set A but holds of certain *low*$_2$ sets and thus gives a kind of discrimination between sets of low information content and some others, and also gives a barrier to certain conjectured automorphisms.

2.1. An $\mathscr{E}$-definable solution to Post's problem. During a given point in the construction of a c.e. set X we let X denote the finite set of elements enumerated in X so far, and let X_s denote the approximation to X by the end of stage s. We begin with a definition of terminology and notation.

DEFINITION 2.1. (i) A subset $A \subset C$ is a *major subset* of C $(A \subset_{\mathrm{m}} C)$ if $C - A$ is infinite and $(\forall e)[\,\overline{C} \subseteq W_e \Longrightarrow \overline{A} \subseteq^* W_e\,]$. (Note that if $A \subset_{\mathrm{m}} C$ then both A and C are noncomputable.)

(ii) $A \sqsubset B$ if there exists C such that $A \sqcup C = B$ (i.e., $A \cup C = B$ and $A \cap C = \emptyset$).

(iii) If $\{X_s\}_{s\in\omega}$ and $\{Y_s\}_{s\in\omega}$ are computable enumerations of c.e. sets X and Y define $X \backslash Y = \{z : (\exists s)[z \in X_s - Y_s]\}$, the elements enumerated in X before Y and $X \searrow Y = (X \backslash Y) \cap Y$, the elements enumerated first in X and later in Y.

Properties of c.e. sets stated in terms of X_s, $X \backslash Y$, or $X \searrow Y$ are called *dynamic* properties because they refer to the *order* of enumeration of elements in X and Y, but $\mathscr{E}$-definable properties are *static* because they refer only to the elements eventually in X or Y.

THEOREM 2.2 (Harrington-Soare [8]). *There is a property $Q(A)$ which guarantees that $A <_T K$ and which holds of some noncomputable set.*

Define the property $Q(A)$ by:

$$Q(A): \quad (\exists C)_{A \subset_{\mathrm{m}} C}\ (\forall B \subseteq C)(\exists D \subseteq C)(\forall S)_{S \sqsubset C}\ [$$

$$[B \cap (S - A) = D \cap (S - A)] \tag{1}$$

$$\Longrightarrow (\exists T)[\overline{C} \subset T\ \&\ A \cap (S \cap T) = B \cap (S \cap T)]]. \tag{2}$$

We may visualize the property $Q(A)$ as a two person game in the sense of Lachlan *1970* between the $\exists$-player (called RED) who plays the c.e. sets A, C, D and T and the $\forall$-player (called BLUE) who plays the c.e. sets B and S. For simplicity ignore all the sets but C, D, B, and A, since the others are necessary only to give us a suitable domain on which to play the following strategy. Visualize $C \supseteq D \supseteq B \supseteq A$, and let $v_1, v_2, \ldots, v_5$ denote the differences of c.e. sets (called *d.c.e. sets*): $\omega - C$, $C - D$, $D - B$, $B - A$,

A respectively, but viewed dynamically like e-states, so an element can pass from v_i to v_j, $i < j$. The oversimplified $Q(A)$ property now asserts that if BLUE plays: $(1)'$ $D = B$ on $\overline{A}$, then RED will play: $(2)'$ $B = A$. In particular, if both players are following their best strategies, then for an element x to enter A, it must pass through the v-states in the order v_1, v_2, $\dots$, v_5 as proved in [8]. However, the set B acts like a wall of restraint, like the minimal pair restraint of Lachlan and Yates in [12, p. 153]. When presented with an $x \in D - B$, BLUE may hold x as long as he likes, but must eventually put x into B at which point RED is free to put x into A but not before. This implies that A is tardy (i.e., not of promptly simple degree, see Ambos-Spies, Jockusch, Shore, and Soare in [12, p. 284]), but it has just been discovered [4] that $Q(A)$ imposes a much stronger tardiness property which helps us classify those sets which can be coded into any nontrivial orbit, see §3.

By analogy with the standard definition of A being tardy, define a c.e. set A to have the much stronger property [4] called 2-*tardy* if for every nondecreasing computable function $p(s)$,

$$(3) \quad (\exists W_i \supseteq \overline{A})(\exists W_e = A)(\forall y)(\forall s)[y \in W_{i,s} - W_{e,s} \Longrightarrow y \notin A_{p(s)}].$$

For the intuition, suppose that RED plays sets A, W_i and W_e, and BLUE plays the function $p(s)$. Then all elements of $\overline{A}$ must eventually enter W_i. However, if A is noncomputable then RED will want to put infinitely many elements x into A *after* $x \in W_i$. To accomplish this, RED must first *announce* that intention by putting x into W_e, say at some stage s, and then RED must delay until some stage $t \geq p(s)$ before putting x into A. Since $p(s)$ is total, this must eventually occur. (Hence, the delay function $p(s)$ is similar to, but much stronger than, the minimal pair restraint $r(e, s)$ of [12, p. 154] which has $\liminf_s r(e, s) < 0$.)

Since this delay imposed by $p(s)$ is very similar to that imposed by B in $Q(A)$, one might suspect that the two properties coincide, and indeed this is almost true. If $Q(A)$ then A is 2-tardy [4, Thm 3.3]. If A is 2-tardy then $Q(A)$ holds providing A is in the right *environment*, namely A is a small major subset of some C, written $A \subset_{\mathrm{sm}} C$ [4, Thm 3.8], where small subsets are defined below. (Thus, the dynamic property, 2-tardy, which is much easier to work with (see §3) captures the main spirit of Q in the cases we are interested in.)

In connection with his study of the elementary theory of the c.e. sets, Lachlan *1968d* (see [12, p. 193, Def. 4.10]) defined A to be a *small* subset of C (written $A \subset_{\mathrm{s}} C$) using three quantifiers, $(\forall X)(\forall Y)(\exists Z)$. Harrington and Soare [9] recently noticed that this is equivalent to the *dynamic* property which they call *small tardy*, defined as follows, where f ranges over nondecreasing computable functions.

$$(4) \qquad (\forall f)(\exists T)[\overline{C} \subseteq T \ \&\ (\forall x)[x \in (T \cap C)_{\text{at s}} \Longrightarrow x \notin A_{f(s)}]].$$

This dynamic property 4 in turn led to an equivalent $\mathscr{E}$-definable property [9, §2] simpler than Lachlan's with only *two* quantifiers and provided intuition into the concept of smallness.

$$(5) \qquad (\forall Y)[[(C - A) \subseteq Y] \Longrightarrow (\exists Z)[\overline{C} \subseteq Z \ \&\ Z \cap C \subseteq Y]].$$

2.2. $\mathscr{E}$-definability and complete sets. Harrington has given an $\mathscr{E}$-definable property which held of exactly the creative sets [12, p. 339]. Here we give a more sophisticated example of an $\mathscr{E}$-definable property which guarantees completeness but holds for noncreative sets, in particular for promptly simple sets, and which answers a conjecture on automorphisms.

Theorem 2.3 (Harrington-Soare [5]). *There is an $\mathscr{E}$-definable property T satisfied by a promptly simple set A such that for all W, $T(W)$ implies that $K \leq_T W$.*

This also negatively answers a question raised by Cholak of whether for every promptly simple set A and high c.e. degree $\mathbf{d}$ there exists $B \in \mathbf{d}$ such that A is automorphic to B. Define $T(A)$ by:

$$(6) \quad T(A): \quad (\exists C \supseteq A)(\forall B \subset C)(\exists \text{ a computable set } R)_{(R-C \text{ not c.e.})}$$

$$(7) \qquad [B \subset_{\mathrm{s}} C \quad .\Longrightarrow. \quad A \supseteq (B \cap R) \ \&\ A \not\supseteq^* C \cap R].$$

To comprehend $T(A)$ we restrict attention to the set R and visualize that there we have $C \supset B \supseteq A$. Consider the states $R - C$, $C - B$, $B - A$, and A denoted by v_1, v_2, v_3, and v_4, respectively. To see that $T(A)$ implies $K \leq_T A$, BLUE defines a Turing reduction $K = \Delta^A$. When he defines $\Delta^A(n)$, he simultaneously defines the use function $\delta^A(n)$ to be an element y_n of $(C - B) \cap R$. If later n enters K then BLUE puts y_n into B but $A \supseteq B \cap R$ by the second clause of (7), so y_n eventually enters A allowing $\Delta(n)$ to be redefined. We can construct a promptly simple A satisfying $T(A)$ or sets C and A such that: $A \subset_{\mathrm{m}} C$; A is promptly simple; and C is r-maximal; by a construction like the Lachlan small major subset construction [12, p. 194].

2.3. $\mathscr{E}$-definability and low sets. The previous property $T(A)$ guaranteed such a rapid flow of elements from $C - B$ into A that A was complete. The next property $NL(A)$ is more subtle but guarantees a sufficiently large flow into A so that A is nonlow, but A can still be low$_2$ and hence incomplete. (Recall that a c.e. set A is *low$_n$* if $A^{(n)} \equiv_T \emptyset^{(n)}$ and *high$_n$* if $A^{(n)} \equiv_T \emptyset^{(n+1)}$, and similarly for c.e. *degrees*.) Define $NL(A)$ by:

$$NL(A): \ (\exists C \supseteq A)(\forall B_0, B_1)[\tag{8}$$

$$[B_0 \sqcup B_1 = C] \,\&\, [B_1 \subseteq A] \quad . \Longrightarrow (\exists \text{ computable } R)[\tag{9}$$

$$[A \supseteq B_0 \cap R] \,\&\, [(B_1 \cup \overline{C}) \cap R \text{ is not c.e.}]]]. \tag{10}$$

THEOREM 2.4. (i) $(\forall W)[W$ *is low* $\Longrightarrow \neg NL(W)]$.
(ii) $(\exists A)[NL(A)$ & A *is promptly simple and low*$_2$ & $\overline{A}$ *is semi-low*$_{1.5}$].

This refutes the appealing conjecture based on Maass *1983*, *1984*, *1985*, that if A and B are both promptly simple and $\mathscr{L}^*(A) \cong_{\mathit{eff}} \mathscr{L}^*(B)$ then $A \simeq B$ where $\mathscr{L}(A) = \{W : W \supseteq A\}$ and $\simeq$ denotes "is automorphic to." Choose B low and promptly simple and A as in (ii). Then by (ii) and Maass *1983* (see [12, p. 230]) $\mathscr{L}^*(A) \cong_{\mathit{eff}} \mathscr{L}^*(B)$ but by (i) $A \not\simeq B$.

To understand $NL(A)$ we assume that BLUE satisfies the hypotheses of $NL(A)$ in (8) and (9), and let R be the reply by RED. Let all sets be restricted to R. Let v_1 be $R - C$, v_2 be $C - (B_0 \cup B_1)$, v_3 be $B_0 - A$, v_4 be $B_0 \cap A$, v_5 be $B_1 - A$, and v_6 be $B_1 \cap A$, again interpreted dynamically. The second clause of (10) guarantees that $R - C$ is not c.e. , so there is a flow of infinitely many elements from state v_1 to v_2. When such an element x arrives in v_2, BLUE can wait an arbitrarily long time but must eventually put x either into B_1 (providing $x \in A$ already because of the second clause of (6)) or he can put x into B_0 (state v_3) from which RED must eventually move x into A (state v_4) because of the first clause of (10). (Note there is no flow from v_2 to v_5 only to v_6.)

Assume that A satisfies $NL(A)$. BLUE claims that A is not low, and indeed there is no computable partial function φ_e such that $\lim_s \varphi_e(x, s)$ is the characteristic function of $\{x : W_x \cap \overline{A}\} \neq \emptyset$. BLUE first defeats a *fixed* φ_e by playing B_0 and B_1 satisfying the hypotheses of $NL(A)$ in (5) and (6), and considers all the possible replies R_i, $i \in \omega$. He chooses a fresh element $x \in v_2^i$, namely in $R^i \searrow C$ and puts x in $Z_{e,i}$, a c.e. set whose index $f(e, i)$ is known a priori by the Kleene fixed point theorem. BLUE waits until $\varphi_e(f(e, i), s) \downarrow = 1$ and then puts x into B_0. Then BLUE waits for $\varphi_e(f(e, i), s) \downarrow = 0$ before repeating the procedure. If repeated infinitely often then $\lim_s \varphi_e(f(e, i), s) \uparrow$. If repeated finitely often and if $\lim_s \varphi_e(f(e, i), s) \downarrow$ then it gives the wrong answer.

§3. Automorphisms of $\mathscr{E}$. Automorphisms are useful for two reasons. First, if we are unable to exhibit an $\mathscr{E}$ definition of some property P (see $\overline{\mathbf{L}}_1$ in §4), then we may be able to produce an automorphism Φ of $\mathscr{E}$ mapping some A with property P to some B with $\neg P$, thereby proving P is undefinable in $\mathscr{E}$. The second use is in the spirit of Klein's *Erlanger*

Programm which is to classify some mathematical object such as a geometry in terms of the properties left invariant under its automorphisms. The first application of the automorphism method was to classify orbits of maximal sets following Klein's program.

To answer a question of Martin and Lachlan, Soare *1974* produced a new method for constructing automorphisms of $\mathcal{E}$ and used it to prove that any two maximal sets are automorphic. The method begins by choosing an appropriate skeleton for the c.e. sets (i.e., one member $U_{g(e)}$ from each class W_e^*) and then, by a fairly complicated construction, building an automorphism of $\mathcal{E}$ which is *effective* in the sense that there is a computable function $h(e)$ such that $\Phi(U_e) =^* W_{h(e)}$. An automorphism Φ is Δ_3^0 if there is a Δ_3^0 permutation h of ω such that $\Phi(W_e) = W_{h(e)}$ for all $e \in \omega$.

Recently, Harrington and Soare [7], and simultaneously Cholak [2], building on some conversations with Harrington, combined the essence of the effective automorphism method with the tree method of Lachlan *1975* to produce a powerful new method for constructing Δ_3^0-automorphisms, and used it to prove, for example, the following.

THEOREM 3.1 (Harrington-Soare [7], Cholak [2]). *For every noncomputable c.e. set A there is a c.e. set B which is high (i.e., $deg(B') = \mathbf{0}''$) such that A is Δ_3^0-automorphic to B.*

Theorem 3.1 asserts that every nontrivial orbit contains a high set. This has some interesting corollaries for noninvariant classes as we shall see in §4. Before considering this, let us consider the relation of Theorem 3.1 to Theorem 2.2. Theorem 3.1 implies that $Q(A)$ which prevents A from being complete cannot be extended to cause A to be low or even nonhigh. It says if we are willing to extend the target set from the complete degree to the high degrees, then the automorphism builder (BLUE) wins. On the other hand, if we insist on mapping A to a complete set, what stronger hypotheses must we place on A?

THEOREM 3.2 (Harrington-Soare [7]). *If A is any c.e. set which is prompt (i.e., of promptly simple degree) or even if A is* almost prompt *then A is automorphic to a complete set.*

Cholak, Downey, and Stob [1] proved this result under the stronger hypothesis "A is a promptly simple *set*," and Harrington and Soare extended this with the much weaker hypothesis "of promptly simple degree." They then realized that the essence of the hypothesis is a much weaker promptness property still, which they named "almost promptly simple," and which is based on the following notion of n-c.e. (also called n-r.e.) sets.

In related work Downey and Stob have proved that the class of sets known as HHM sets are all automorphic to complete sets. We discuss the relationship between HHM and almost prompt in [7, §12]. Also Harrington proved that there is no "fat" orbit, i.e., one containing a set in every nonzero degree.

DEFINITION 3.3. (i) A set $X \leq_T K$ is *n-c.e.* if $X = \lim_s X_s$ for some computable sequence $\{X_s\}_{s\in\omega}$ such that for all x, $X_0(x) = 0$ and

$$\mathrm{card}\{s : X_s(x) \neq X_{s+1}(x)\} \leq n.$$

For example, the only 0-c.e. set is $\emptyset$, the 1-c.e. sets are the usual c.e. sets, and the 2-c.e. sets are the d.c.e. sets (also called d.r.e.).

(ii) Such a sequence $\{X_s\}_{s\in\omega}$ is called an *n-c.e. presentation* of X.

It is well-known and easy to show [12, Exercise III.3.8., p. 38] that for $n > 0$, X is n-c.e. iff

$$X = (W_{e_1} - W_{e_2}) \bigcup (W_{e_3} - W_{e_4}) \bigcup \dots \bigcup W_{e_{2k+1}}, \tag{11}$$

or

$$X = (W_{e_1} - W_{e_2}) \bigcup (W_{e_3} - W_{e_4}) \bigcup \dots \bigcup (W_{e_{2k+1}} - W_{e_{2k+2}}), \tag{12}$$

according as $n = 2k + 1$ is odd or $n = 2k + 2$ is even.

DEFINITION 3.4. For $n = 0$ let $X^0_0 = \emptyset$. For $n > 0$ and $e = \langle e_1, e_2, \dots, e_n\rangle$ define

$$X^n_e = (W_{e_1} - W_{e_2}) \bigcup \dots, \tag{13}$$

as in (11) or (12) according as n is odd or even. We say that $\langle n, e\rangle$ is an *n-c.e. index* for X^n_e. Let

$$X^n_{e,s} = (W_{e_1,s} - W_{e_2,s}) \bigcup \dots. \tag{14}$$

DEFINITION 3.5. Let A be a c.e. set and let $\{A_s\}_{s\in\omega}$ be a computable enumeration of A. We say A is *almost prompt*, abbreviated *a.p.*, if there is a nondecreasing computable function $p(s)$ such that for all n and e,

$$X^n_e = \overline{A} \Longrightarrow (\exists x)(\exists s)[x \in X^n_{e,s} \;\&\; x \in A_{p(s)}]. \tag{15}$$

(ii) We say A is *very tardy* if A is *not* almost prompt, namely if for every nondecreasing computable function $p(s)$, the negation of (15) holds. In this case, if a fixed n works uniformly for *all* such functions p then we say A is *n-tardy*. Note that for the case $n = 2$ this is equivalent to our definition in §3 (3) of being 2-tardy which is a very important special case, as we shall see.

The results here on almost prompt sets and in §2 on 2-tardy sets stress the very important, but previously often hidden, connection between dynamic properties on one hand, and definable properties and automorphisms on the other.

Here is another unexpected connection. In Theorem 3.2 we put extra hypotheses on the set A so that it could be mapped to a complete set. Now we ask what hypotheses are necessary on a set D so that it can be computably coded into some set B in *every* nontrivial orbit. The answer involves the 2-tardy sets of §1 in an unexpected way. The *orbit* of A is the class $[A]$ of all sets B automorphic to A, written $A \simeq B$. By a *nontrivial* orbit we mean the orbit of a *noncomputable* c.e. set A.

DEFINITION 3.6. (i) We say X can be *coded into the orbit of* A, denoted by $X \leq_T [A]$, if $X \leq_T B$ for some $B \in [A]$.

(ii) We say X is *codable* if X can be coded in every nontrivial orbit, namely if $X \leq_T [A]$ for every $A >_T \emptyset$.

THEOREM 3.7 (Harrington-Soare, Coding Theorem [4]). *If D is 2-tardy (say if $Q(D)$ holds) then D is codable.*

COROLLARY 3.8. *A set X is codable iff $X \leq_T D$ for some D satisfying $Q(D)$ iff $X \leq_T D$ for some 2-tardy D.*

PROOF. If $X \leq_T D$ and D is 2-tardy, then $X \leq_T D \leq_T [A]$ for every $A >_T \emptyset$ by Theorem 3.7. If $X \leq_T [A]$ for every $A >_T \emptyset$ then $X \leq_T C$ for some C and D such that $C \in [D]$ and $Q(D)$, hence D is 2-tardy. Hence, $Q(C)$ because Q is $\mathcal{E}$-definable. ⊣

Note that $Q(D)$ implies that D is a major subset and hence D is high. Thus, Theorem 3.7 is a very strong generalization of Theorem 3.1 because by Theorem 3.7 if A is noncomputable and $Q(D)$ holds, then there exists $B \in [A]$ such that $D \leq_T B$ so D and B are both high. Thus, $Q(D)$, and its associated property of D being 2-tardy, were originally introduced just to force the enumeration of elements into D to be sufficiently *slow* so that D would have to be incomplete. Now we see that it also forces a sufficiently slow enumeration so that the machinery building an automorphism $\Phi(A) = B$ has time to code into B the fact that x enters D. This connection of the *speed* of elements entering D to its *codability* into orbits is so interesting that we explore it further for a moment.

COROLLARY 3.9. *If S is a promptly simple set (or even of promptly simple degree) then S is not codable.* ⊣

PROOF. If S is promptly simple (or even of prompt, i.e., of promptly simple degree) and $S \leq_T C$ then C is also prompt [12, Corollary XIII.1.9, p.

287], thus not tardy, thus not 2-tardy, thus $\neg Q(C)$. Hence, by Corollary 3.8 S is not codable. $\dashv$

Thus, codable sets can be high by Theorem 3.1, while noncodable sets can be low (choose a low promptly simple set in Corollary 3.9). Therefore, one of our main conclusions is that *the question of whether a set X can be coded* into an arbitrary orbit $[A]$ *depends more on the speed of enumeration of X (prompt or tardy) than on its information content (high or low).*

The fact that K is not codable has more to do with the fact that K is prompt (i.e., of promptly simple degree) than that K has complete information content. For example, to show that $K \leq_T B$ for some $B \in [A]$ we must do very rapid coding, but if $Q(A)$ holds then $Q(B)$ holds for every $B \simeq A$ (because Q is $\mathscr{E}$-definable although "2-tardy" is not). Thus, B is 2-tardy, hence tardy, and hence incomplete.

To code D into $[A]$, the orbit of a given noncomputable set A, we must construct $B \simeq A$ and a computable functional Ψ such that $D = \Psi(B)$. We must define the use function $\psi(n)$ to be a convenient element y not yet in B such that if n enters D, then we can gradually move y into B. The property of D being 2-tardy, as described after (3), will imply that there is a computable function g (played by BLUE) such that if n wants to enter D, it must first *declare* that intention at some stage s and then wait until some stage $t \geq p(s)$ before doing so. Since the automorphism machinery imposes considerable delay in putting $\psi(n)$ into B after first starting the process, BLUE arranges that when n declares its intention at stage s, BLUE starts $\psi(n)$ toward B immediately and makes $p(s)$ so large that $\psi(x)$ has arrived in B by stage $p(s)$ before n has arrived in D.

The entire automorphism construction is more complicated and is played on a tree T of nodes. Thus, our actual coding procedure is a bit more complicated as it is performed repeatedly for several nodes $\alpha \in T$. Perhaps, $\psi(n)$ begins in the region R_γ (defined in [7]) with witness $y_\gamma < \psi(n)$. Now in its journey toward D, element n passes through a series of "gates" G_α for $\alpha \in T$ with witnesses y_α, each time undergoing a delay as above with G_α in place of D. (In reality these sets G_α are simply different names for the set D, at least for $\alpha \subset f$, where f is the true path through T.) After each successful entry y_α into B, the use function is redefined to some number above y_β where $\beta = \alpha^-$, the predecessor of α and $\psi(n)$ passes from region R_α to region R_β. Best of all, to complete the proof we do not need to know anything about the machinery for generating Δ^0_3-automorphisms of $\mathscr{E}$. Rather, in [7, §7] we develop the full automorphism machinery and a coding theorem, which can be applied here without further proof. Further details can be found in [4].

§**4. Invariance.** A property of c.e. sets or class $\mathscr{C} \subseteq \mathscr{E}$ is *invariant* if it is invariant under $\mathrm{Aut}(\mathscr{E})$, and $\mathscr{E}$-*definable* if there is a first order property in the language $L(\subset)$ which defines it over $\mathscr{E}$. A class $\mathbf{C}$ of c.e. degrees is *invariant* if it is the class of degrees of sets in some class $\mathcal{C} \subseteq \mathscr{E}$ which is invariant (e.g. if $\mathcal{C}$ is $\mathcal{E}$-definable). For $\mathbf{R}$ the c.e. degrees and $\mathbf{C} \subset \mathbf{R}$ define

$$\begin{aligned}
\mathbf{H}_n &= \{\mathbf{a} \in \mathbf{R} : \mathbf{a}^{(n)} = \mathbf{0}^{(n+1)}\},\\
\mathbf{L}_n &= \{\mathbf{a} \in \mathbf{R} : \mathbf{a}^{(n)} = \mathbf{0}^{(n)}\},\\
\mathbf{L}_0 &= \{\mathbf{0}\},\ \mathbf{H}_0 = \{\mathbf{0}'\},\ \text{ and } \overline{\mathbf{C}} = \mathbf{R} - \mathbf{C}.
\end{aligned}$$

The degrees in $\mathbf{H}_n$ ($\mathbf{L}_n$) are called *high*$_n$ (*low*$_n$) and the high$_1$ (low$_1$) degrees are called *high* (*low*).

A set $A \in \mathscr{E}$ is *maximal* if A is maximal in the inclusion ordering, i.e., $\neg(\exists W)[A \subset^* W \subset^* \omega]$, and a coinfinite A is *atomless* if A has no maximal superset. Martin *1966b* showed that the degrees of maximal sets are exactly $\mathbf{H}_1$, see [12, p. 217]. Lachlan *1968a* and Shoenfield *1976* showed that the degrees of atomless sets are exactly the nonlow$_2$ c.e. degrees $\overline{\mathbf{L}}_2$, see [12, p. 231]. Thus, $\mathbf{H}_1$ and $\overline{\mathbf{L}}_2$ are invariant. For the trivial jump classes corresponding to $n = 0$, $\mathbf{L}_0$ $\overline{\mathbf{L}}_0$, and $\mathbf{H}_0$ are invariant, while $\overline{\mathbf{H}}_0$ is noninvariant by Theorem 3.2. These three results and his work at the time on projective determinacy led Martin to make the following conjecture.

Conjecture 4.1 (Martin's Invariance Conjecture). *Among all the jump classes* $\mathbf{H}_n$ *and* $\mathbf{L}_n$ *for* $n > 0$, *and their complements* $\overline{\mathbf{H}}_n$ *and* $\overline{\mathbf{L}}_n$, *the invariant classes are exactly* $\mathbf{H}_{2n-1}$ *and* $\overline{\mathbf{L}}_{2n}$.

As first stated, the conjecture stated that these were the only invariant classes among *all* nontrivial classes of degrees, but this was soon refuted by Lerman and Soare *1980* who showed that the d-simple sets form an $\mathscr{E}$-definable class which splits $\mathbf{L}_1$. Therefore, the conjecture was modified to be restricted to just the jump classes and their complements. It was also modified to exclude the case of $n = 0$ because these classes tend to be pathological. The alternation of every odd $\mathbf{H}_n$ and every even $\overline{\mathbf{L}}_n$ was inspired by the behavior of projective determinacy. The following immediate corollary of Theorem 3.1 confirms the Invariance Conjecture prediction for the *downward* closed jump classes for $n > 0$.

Corollary 4.2. *For all* $n > 0$ *the* downward *closed jump classes of c.e. degrees* $\mathbf{L}_n$ *and* $\overline{\mathbf{H}}_n$ *are noninvariant.*

For the *upward* closed classes $\mathbf{H}_n$ and $\overline{\mathbf{L}}_n$, $n > 0$, after the discovery of invariance of $\mathbf{H}_1$ and $\overline{\mathbf{L}}_2$, attention has been focused on $\overline{\mathbf{L}}_1$ because of the important role played by the low c.e. sets. Researchers had tried unsuccessfully to find a property for the definability player (i.e., the RED player) which would define the class of degrees $\overline{\mathbf{L}}_1$ analogously as the property "atomless" defines $\overline{\mathbf{L}}_2$. The property $NL(A)$ in §2.3 almost succeeded, but not quite. After years of unsuccessful efforts by several researchers, Harrington and Soare discovered that $\overline{\mathbf{L}}_1$ is *non*invariant, thereby giving further evidence for Martin's conjecture. However, when they started to write down the proof it became complicated because infinitely many automorphisms had to be constructed simultaneously. They thought through the problem again and found the following fairly natural property for the automorphism player (the BLUE player) called *locally low* so as to break the problem into two separate parts, and to give much more insight into the structure of $\mathscr{E}$ than the original proof.

THEOREM 4.3 (Harrington-Soare [6]). (i) *Every c.e. set A which is locally low is automorphic to a low set B.*

(ii) *There is a c.e. set $D \in low_2$–low_1 such that every c.e. $A \leq_T D$ is locally low.*

COROLLARY 4.4. $\overline{\mathbf{L}}_1$ *is not an invariant class of c.e. degrees.*

PROOF. Suppose it is invariant. Choose D as in (ii). Now every $A \equiv_T D$ is locally low and hence by (i) A is automorphic to a low set B. ⊣

To understand the new property of locally low, we turn first to a stronger lowness related property which was used by Soare *1982* to prove that $\mathscr{L}^*(A) \cong^{\mathit{eff}} \mathscr{E}^*$. (A weaker property was used by Maass *1983* to get a necessary and sufficient condition.) The property, called *semi-low*, refers to the complement $\overline{A}$ of a c.e. set A and asserts [12, Exer. 4.7] that there is a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that for all e,

$$(\exists^\infty s)[W_{e,s} - A_s \neq \emptyset] \Longrightarrow W_e - A \neq \emptyset. \tag{16}$$

We can think of this as a guarantee that if at infinitely many stages we see a promise that some $x \in W_{e,s} - A_s$, then in the limit infinitely many (but not necessarily all such x) promises will be kept, in the sense that x will indeed rest in $W_e - A$.

The usefulness of this condition in building an automorphism Φ mapping A to B is clear because without it the RED player may play U_0 such that $U_0 \backslash A$ is infinite, causing BLUE to make $\Phi(U_0)\backslash B$ infinite also, but then RED plays $U_0 - A = \emptyset$ and BLUE , who does not control B, cannot avoid $\Phi(U_0) - B$ infinite.

We cannot construct D to have the properties in Theorem 4.3 and *also* have $\overline{D}$ semi-low. Indeed, our method is to make it nonlow by making $\overline{D}$ *non*semi-low. However, with the tree method of Lachlan we do not really need the full power of (16), but merely a local version which preserves the *structural* properties but not the *information content* properties.

DEFINITION 4.5. A low_2 c.e. set A is *locally low* if there is a computable enumeration $\{A_s\}_{s\in\omega}$ of A such that,

(1) $(\forall i)(\forall s)[g(i,s) \leq g(i,s+1)]$,
and a uniformly computable canonically indexed sequence of nonempty finite sets $\{F_e\}_{e\in\omega}$ (i.e., $F_n = D_{h(n)}$ for some computable h) such that

$$(\forall e \in \omega)(\exists i_e \in F_e)[$$

(2) $\lim_s \ g(i_e, s) = \infty$ and

(3) $(\exists^\infty s)[(W_{e,s} - A_s) \upharpoonright g(i_e, s) \neq \emptyset] \Longrightarrow W_e - A \neq \emptyset]]$,
where $Y \upharpoonright x$ denotes the restriction of the set Y to elements $z < x$.

Suppose A and B are locally low. The tree giving the automorphism construction of $\Phi(A) = B$ will now have extra branching beyond [7] to guess at the correct $i_e \in F_e$. Those $g(j,s)$ for $j < i_e$ may not satisfy (2) and may not give enough room to build the automorphism, but their action will be finite; those $j > i_e$ may satisfy (2) but not (3), so their promises will not be kept, but the automorphism method will ignore their action. For the true $g(i_e, s)$ it will appear that the world is exactly as in the original case of semi-low with (16). Building a set D which is low_2, locally low, and $\overline{D}$ is not semi-low is similar to the construction of a low_2 nonsemi-low set and is naturally done on a tree because the requirement for the latter produces an infinite sequence of Σ_2 or Π_2 outcomes which the nodes on the tree guess at, and which naturally tends to produce the tree-like locally low property.

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