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PURE PROOF THEORY
AIMS, METHODS AND RESULTS

EXTENDED VERSION OF TALKS GIVEN AT OBERWOLFACH AND HAIFA

WOLFRAM POHLERS

Apologies. The purpose of the following talk is to give an overview of the present state of aims, methods and results in Pure Proof Theory. Shortage of time forces me to concentrate on my very personal views. This entails that I will emphasize the work which I know best, i.e., work that has been done in the triangle Stanford, Munich and Münster. I am of course well aware that there are as important results coming from outside this triangle and I apologize for not displaying these results as well.

Moreover the audience should be aware that in some points I have to oversimplify matters. Those who complain about that are invited to consult the original papers.

§1. Aims.

1.1. General. Proof theory started with Hilbert's Programme which aimed at a finitistic consistency proof for mathematics.

By Gödel's Theorems, however, we know that we can neither formalize all mathematics nor even prove the consistency of formalized fragments by finitistic means. Inspite of this fact I want to give some reasons why I consider proof theory in the style of Gentzen's work still as an important and exciting field of Mathematical Logic. I will not go into applications of Gentzen's cut-elimination technique to computer science problems—this may be considered as applied proof theory—but want to concentrate on metamathematical problems and results. In this sense I am talking about *Pure Proof Theory*.

Mathematicians are interested in structures. There is only one way to find the theorems of a structure. Start with an axiom system for the structure and deduce the theorems logically. These axiom systems are the objects of proof-theoretical research. Studying axiom systems there is a series of more or less obvious questions. For example:

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(i) *Is the axiom system consistent? The answer, given in a naive way, is of course "yes" since the axiom system has to be sound. However, we know from Gödel's 2nd Theorem that we are unable to prove that in an elementary way. So the questions arise*:
  (a) *Is there a—in any sense—"smallest" system which proves the consistency of the axiom system? What could be a good candidate for such a system?*
  (b) *Are there plausible reasons why the axiom system should be consistent? This essentially asks if there is a "constructive" justification for the axiom system.*

(ii) *We know by Gödel's 1st Theorem that for most structures there is no complete axiom system. In that case we may ask the following questions*:
  (a) *Is there a characterization of the theorems of the structure which are not provable from the axiom system?*

(iii) *If there is no complete axiom system for a structure there will be many different axiom systems—which often will extend each other. Then we may ask*:
  (a) *Is it possible to give a—at least rough—characterization of the "strength" of these axiom systems which helps in their comparison?*

  Also a very interesting closely connected problem is
  (b) *to find the "weakest" axiom system which proves a given theorem of the structure.*

  But this will mostly depend on the theorem and not so much on the intended structure. So this question seems to have a slightly different flavor.

(iv) *A very natural question (originally posed by Kreisel) to ask is*:
  (a) *Do we know more than the sheer truth of a sentence if we "know" the axiom system in which it is provable?*

In the following talk I will explain that all these problems—perhaps with the exception of a full solution of problem (iii) (b) which often requires a profound mathematical analysis of the particular theorem—can be solved in a quite uniform manner. This approach will moreover yield insight into the construction of the mathematical universe and I hope that at the end of the talk I will have convinced you that this is the real exciting part of the whole enterprise. What is especially attractive in these investigations is that they use methods from all areas of Mathematical Logic. We need subtle set-theoretical considerations together with results from recursion theory (both classical and generalized) to provide sufficiently large ordinal notations, the basics of model theory and last but not least proof theoretical means which are further developments of Gentzen's technique.

1.2. The standard model. Following the tradition of proof theory we regard structures which include the structure of natural numbers. There is a common agreement on what we mean by the first order standard model of the natural numbers. Of greater importance, however, are axiom systems which involve sets (although most of the results obtained will also boil down to first order). There is also a common agreement on what the standard model of Second Order Number Theory should be—namely the structure

$$\mathbb{N} := \big(\omega, \mathrm{Pow}(\omega), \in, \underline{0}, \underline{1}, \underline{+}, \dots\big).$$

But we do not really know $\mathrm{Pow}(\omega)$. Power sets have always to be regarded relative to a set universe. Unfortunately there is no commonly accepted standard universe of set theory. To remedy this situation we will stick to the smallest possible set universe—the constructible universe $\mathbf{L}$ and regard it as our standard model. This restricts us to axiom systems about $\mathbf{L}$. But practically this means no restriction because all axiom systems which we can handle so far have models in $\mathbf{L}$. It is—at least to me—completely unclear what will happen as soon as we advance to axiom systems which are so strong that $\mathbf{L}$ no longer suffices. But this is an exciting question of the future.

1.3. Partial models. The constructible universe is the union of its stages $\mathbf{L}_\alpha$ which are defined by

$$\begin{aligned} \mathbf{L}_0 &= \emptyset \\ \mathbf{L}_{\alpha+1} &= \mathrm{Def}(\mathbf{L}_\alpha) \\ \mathbf{L}_\lambda &= \bigcup\{\, \mathbf{L}_\xi \mid \xi < \lambda \,\} \text{ for } \lambda \in \mathrm{Lim}\,. \end{aligned}$$

Recall that $\mathrm{Def}(\mathbf{L}_\alpha)$ denotes the collection of sets which are definable from $\mathbf{L}_\alpha$ with parameters, i.e., we have $a \in \mathrm{Def}(\mathbf{L}_\alpha)$ if there are elements $a_1, \dots, a_n$ in $\mathbf{L}_\alpha$ and a formula $F(x, x_1, \dots, x_n)$ in the language of Set Theory such that $a = \{\, x \in \mathbf{L}_\alpha \mid \mathbf{L}_\alpha \models F[x, a_1, \dots, a_n] \,\}$.

If we want to tackle problem (i) for an axiom system Ax, which has a model in $\mathbf{L}$, the first thing to do is to look for the smallest model of Ax within $\mathbf{L}$. Hence we define

$$\|\mathsf{Ax}\|_\infty := \min\{\, \alpha \mid \mathbf{L}_\alpha \models \mathsf{Ax} \,\}.$$

But is this indeed already a first step in the direction of an answer to question (i) (a)? One may observe that, if α is a countable ordinal and $\prec$ is a well-ordering on the natural numbers representing α, then $\mathbf{L}_\alpha \models \mathsf{Ax}$ can be expressed within the system $\mathsf{PA} + \mathsf{TI}(\prec)$ which is Peano Arithmetic augmented by the scheme $\mathsf{TI}(\prec)$ of transfinite induction along $\prec$. By Löwenheim-Skolem and the Condensation Lemma for the constructible universe $\|\mathsf{Ax}\|_\infty$ is a countable ordinal but it is of course unlikely that $\mathsf{PA} + \mathsf{TI}(\prec)$ is in any sense the "least" system which proves the consistency of Ax. In general, the well-ordering $\prec$ will not even be definable in the

language of PA and it is obvious that we are still far away from a constructive justification of $\mathsf{PA}+\mathsf{TI}(\prec)$. To make any progress we need a constructive ordinal α, i.e., an ordinal which is less than ω_1^{CK}. In general, however, $\|\mathsf{Ax}\|_\infty$ is much bigger than ω_1^{CK}.

But for consistency we do not need a full model of Ax. It suffices to have a partial model, i.e., a model which satisfies only formulas of restricted complexity. Therefore we call $\mathbf{L}_\alpha$ an $\mathcal{F}$-model of Ax if and only if

$$\mathbf{L}_\alpha \models F \text{ for all } F \in \mathcal{F} \text{ such that } F \text{ follows from } \mathsf{Ax}$$

where $\mathcal{F}$ is a complexity class of formulas and define

$$\|\mathsf{Ax}\|_{\mathcal{F}} := \min\{\,\alpha \mid \mathbf{L}_\alpha \text{ is an } \mathcal{F}\text{-model of } \mathsf{Ax}\,\}.$$

Since most axiom systems consist of Π_3-sentences we have

$$\|\mathsf{Ax}\|_{\Pi_3} = \|\mathsf{Ax}\|_\infty \tag{1.1}$$

for all these systems. This shows that we have to step down to Π_2-models in order to make progress. As an interesting remark we observe that

$$\|\mathsf{Ax}\|_{\Pi_2} = \|\mathsf{Ax}\|_{\Sigma_1}. \tag{1.2}$$

In general, however, the ordinals $\|\mathsf{Ax}\|_{\Pi_2}$ are still much bigger than ω_1^{CK}. Therefore we introduce the complexity classes Π_n^κ and Σ_n^κ which contain all formulas of the shape

$$(\forall x_1 \in \mathbf{L}_\kappa)(\exists x_2 \in \mathbf{L}_\kappa)\dots(\mathsf{Q}_n x_n \in \mathbf{L}_\kappa)F(\vec{x})$$

or

$$(\exists x_1 \in \mathbf{L}_\kappa)(\forall x_2 \in \mathbf{L}_\kappa)\dots(\mathsf{Q}_n x_n \in \mathbf{L}_\kappa)F(\vec{x}),$$

respectively, where $(\forall x_1)(\exists x_2)\dots(\mathsf{Q}_n x_n)$ and $(\exists x_1)(\forall x_2)\dots(\mathsf{Q}_n x_n)$ are alternating strings of quantifiers and $F(\vec{x})$ is an $\mathbf{L}_\kappa$-Δ_0-formula, i.e., a formula in which all quantifiers are restricted to elements of $\mathbf{L}_\kappa$. Observation (1.2) extends to the complexity classes Π_2^κ and Σ_2^κ. Hence

$$\|\mathsf{Ax}\|_{\Pi_2^{\omega_1^{\mathrm{CK}}}} = \|\mathsf{Ax}\|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}} < \omega_1^{\mathrm{CK}}. \tag{1.3}$$

The computation of the ordinal $\|\mathsf{Ax}\|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}}$ is called an *ordinal analysis* of Ax.

In all analyzed cases we get by an ordinal analysis of Ax already a primitive recursive (even elementary) well-ordering $\prec$ representing $\|\mathsf{Ax}\|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}}$ and obtain by metamathematical and proof theoretical considerations

$$\mathsf{Ax} \vdash F \Longrightarrow \mathsf{PA}+\mathsf{TI}(\prec_{\mathrm{in}}) \vdash F \tag{1.4}$$

for all Π^0_∞ (i.e., arithmetical) formulas F, where $\prec_{\text{in}}$ denotes a proper initial segment of $\prec$. As shown, e.g., in [16], this can be refined to

$$\mathsf{PA} + \mathsf{TI}(\prec_{\text{in}}) \vdash F \Longrightarrow \mathsf{PRA} + \mathsf{PRWO}(\prec_{\text{in}}) \vdash F \tag{1.5}$$

for all Π^0_2-formulas F, where PRA is the system of Σ_1 induction (which is equivalent to Primitive Recursive Analysis when function symbols for all primitive recursive functions are present), and $\mathsf{PRWO}(\prec)$ stands for the scheme

$$(\forall \vec{z})(\exists y)\big[\neg f(\vec{z}, y+1) \prec f(\vec{z}, y)\big]$$

in which f varies over all primitive recursive functions, thus stating that there are no infinite descending primitive recursive sequences in $\prec$. This result uses again metamathematical considerations (e.g., formalized infinitary proof figures) and the key in its proof is Mints' continuous cut elimination procedure for infinitary proof trees.

The systems $\mathsf{PRA} + \mathsf{TI}(\prec)$ are pretty small and we think that they incorporate already a good solution to the question posed in (i) (a). So, a satisfying result is obtained as soon as we can establish (1.4). We speak therefore of a *profound ordinal analysis* whenever we have a primitive recursive well-ordering $\prec$ of minimal order-type for which (1.4) holds.

Once we have a profound ordinal analysis we are also one step closer to an answer of question (i) (b). In the next step one has to prove the well-foundedness of $\prec$ within the framework of a constructively justified system. Such systems are usually based on intuitionistic logic. Examples are, among others, Heyting Arithmetic, Theories for Iterated Inductive Definitions based on intuitionistic logic, some Martin-Löf Type Theories and—perhaps most spectacular—Feferman's theories for Explicit Mathematics. As far as I know no direct proof theoretical reduction of $(\Sigma^1_2\text{-AC}) + (\text{BI})$ to Feferman's system T_0 has yet been obtained. Here $(\Sigma^1_2\text{-AC}) + (\text{BI})$ denotes the subsystems of Second Order Number Theory with the Σ^1_2-axiom of choice and classical Bar Induction which is equivalent to the quantifier scheme

$$(\forall X)F(X) \;\rightarrow\; F(G). \tag{BI}$$

The reduction is still by showing

$$T_0 \vdash \mathsf{TI}(< \|(\Sigma^1_2\text{-AC}) + (\text{BI})\|_{\Sigma^{\omega_1^{\text{CK}}}_1})$$

where $\mathsf{TI}(< \|(\Sigma^1_2\text{-AC}) + (\text{BI})\|_{\Sigma^{\omega_1^{\text{CK}}}_1})$ is a sloppy notation for the well-foundedness of all initial segments of a primitive recursive representation for

$$\|(\Sigma^1_2\text{-AC}) + (\text{BI})\|_{\Sigma^{\omega_1^{\text{CK}}}_1}$$

(cf. [8]).

We will soon see that an ordinal analysis of an axiom system Ax also gives a rough solution of question (ii) (a). Since, as soon as we have introduced the notion of truth complexity, it will be clear that no Π_2 sentence with truth complexity above $\|\mathsf{Ax}\|_{\Pi_2^{\omega_1^{\mathrm{CK}}}}$ can be derived from Ax.

We postpone the discussion in how far an ordinal analysis also answers question (iv) (a) until the last section.

1.4. Connections to the conventional approach. Those who have studied the proof theory of subsystems of Second Order Number Theory may now ask how this fits together. Let me roughly sketch the connection on the example of Gentzen's result. We may restate Gentzen's result as

$$\mathsf{PA} \vdash \mathsf{TI}(X, <\varepsilon_0) \quad \text{but} \quad \mathsf{PA} \nvdash \mathsf{TI}(X, \varepsilon_0). \tag{1.6}$$

Moreover we assume that the language of PA contains symbols for all primitive recursive functions and set parameters $X, Y, Z, X_1, \dots$. It is well known that cut free ω-logic is complete for $\mathbb{N}$. The most elegant way to prove this is—in my opinion—by Schütte's search trees (Quasideduktionsbäume). Let us denote by

$$\models^{\alpha} F$$

that there is a cut free proof tree of depth α in ω-logic with endformula F. Define

$$\mathrm{tc}(F) := \min\{\, \alpha \mid \models^{\alpha} F \,\} \tag{1.7}$$

and call $\mathrm{tc}(F)$ the *truth complexity* of F. This is justified by the fact that in the case that F is a sentence, $\models^{\alpha} F$ is just the truth definition for F. We call formulas $F(X, \dots)$ which may contain set parameters but must not contain free number variables Π^1_1-*sentences* and define

$$\mathbb{N} \models F(X_1, \dots, X_n) \;:\Longleftrightarrow\; \mathbb{N} \models (\forall X_1) \dots (\forall X_n) F(X_1, \dots, X_n).$$

For true Π^1_1-sentences F we have $\mathrm{tc}(F) < \omega_1^{\mathrm{CK}}$—the search tree à la Schütte is recursive—and obtain

$$\mathbb{N} \models F \iff \mathrm{tc}(F) < \omega_1^{\mathrm{CK}} \tag{1.8}$$

for Π^1_1-sentences F. A simple induction on the complexity of F shows

$$\mathrm{tc}(F) < \omega$$

for true arithmetical sentences F. But in general

$$\sup\{\, \mathrm{tc}(F) \mid F \text{ is a } \Pi^1_1\text{-sentence and } \mathbb{N} \models F \,\} = \omega_1^{\mathrm{CK}}.$$

We define

$$\|\mathsf{PA}\|_{\Pi^1_1} := \sup\{\, \mathrm{tc}(F) \mid F \in \Pi^1_1 \text{ and } \mathsf{PA} \vdash F \,\}. \tag{1.9}$$

Since the set of consequences of a decidable axiom system is recursively enumerable and the search tree for F can be recursively constructed from F we have

$$\|\mathsf{PA}\|_{\Pi^1_1} < \omega_1^{\mathrm{CK}}.$$

Usually we define the proof theoretical ordinal of PA as

$$\|\mathsf{PA}\| := \sup\{\, \mathrm{otyp}(\prec) \mid \prec \text{ is primitive recursive and } \mathsf{PA} \vdash \mathsf{TI}(X, \prec)\,\}.$$

The canonical proof of $\mathsf{TI}(X, \prec)$ in ω-logic gives

$$\mathrm{tc}\big(\mathsf{TI}(X, \prec)\big) \leq \mathrm{otyp}(\prec). \tag{1.10}$$

The converse inequality is the Boundedness Lemma stating

$$\mathrm{otyp}(\prec) \leq \mathrm{tc}\big(\mathsf{TI}(X, \prec)\big) \tag{1.11}$$

for arithmetically definable orderings $\prec$. So we have

$$\mathrm{otyp}(\prec) = \mathrm{tc}\big(\mathsf{TI}(X, \prec)\big) \tag{1.12}$$

and hence

$$\|\mathsf{PA}\|_{\Pi^1_1} = \sup\{\, \mathrm{otyp}(\prec) \mid \mathsf{PA} \vdash \mathsf{TI}(X, \prec)\,\} \tag{1.13}$$

by the Π^1_1-completeness of $\mathsf{TI}(X, \prec)$. This shows that $\|\mathsf{PA}\|_{\Pi^1_1}$ is the proof theoretical ordinal of PA. We may therefore restate Gentzen's result as

$$\|\mathsf{PA}\|_{\Pi^1_1} = \varepsilon_0. \tag{1.14}$$

As a word of warning, however, we emphasize that it is *not true* that $\mathsf{PA} \vdash F$ holds for all Π^1_1-sentences F with $\mathrm{tc}(F) < \|\mathsf{PA}\|_{\Pi^1_1}$. True arithmetical sentences which are independent from PA are counterexamples. These sentences are unprovable but have truth complexities below ω.

Let us go back to (1.8). There is a hidden existential quantifier in it which tells us that a Π^1_1-sentence holds if there is an infinite proof tree of depth $< \omega_1^{\mathrm{CK}}$. These proof trees live in $\mathbf{L}_{\omega_1^{\mathrm{CK}}}$. Thus (1.8) tells us that every Π^1_1-sentence is equivalent to a $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-sentence, the well known Hyperarithmetical Quantifier Theorem. This theorem links subsystems of Second Order Arithmetic to subsystems of set theory. If we regard PA as a subsystem of set theory we get

$$\|\mathsf{PA}\|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}} = \|\mathsf{PA}\|_{\Pi^1_1} = \|\mathsf{PA}\|. \tag{1.15}$$

Equation (1.15) shows how Π^1_1- and $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-norms are linked. The $\Pi_2^{\omega_1^{\mathrm{CK}}}$-norm makes no sense for PA. But similar observations hold for subsystems

of Second Order Arithmetic stronger than PA in which also the $\Pi^{1\text{-CK}}_2$-norm becomes meaningful.[1]

§2. Methods.

2.1. A complete infinitary calculus. Though $\|\mathsf{Ax}\|_\infty$ is known for most axioms systems by results of Generalized Recursion Theory or Set Theory it is not at all obvious how to compute $\|\mathsf{Ax}\|_{\mathcal{F}}$. Let me roughly outline the general approach.

We introduce an infinitary system which is complete for $\mathbf{L}$. This turns out to be much easier than it was for ω-logic. First we define the L-language by choosing a constant L_α for every ordinal α and then define L-terms and L-sentences by the following clauses:

(2.1) *Every* L_α *is a term of stage* α.

If $F(x, y_1, \ldots, y_n)$ *is a formula of Set Theory which contains only the indicated variables free and* s *and* $(t_1, \ldots, t_n)$ *are terms of stages* $< \alpha$ *then* $F(s, t_1, \ldots, t_n)^{\mathsf{L}_\alpha}$ *is an* L*-sentence and*

$$\{\, x \in \mathsf{L}_\alpha \mid F(x, t_1, \ldots, t_n)^{\mathsf{L}_\alpha} \,\}$$

is a term of stage α.

We denote the stage of an L-term t by $\mathrm{stg}(t)$.

Interpreting L_α by $\mathbf{L}_\alpha$ and continuing this in the obvious way we obtain the semantics for the L-language.

Observe that there are only bounded quantifiers in L-sentences. For technical reasons we regard $s = t$ as an abbreviation for

$$(\forall x \in s)[x \in t] \wedge (\forall x \in t)[x \in s].$$

We say that sentences of the form

$$r \in t \quad \text{and} \quad (\exists x \in t)F$$

have $\vee$-type and dually that sentences of the form

$$r \notin t \quad \text{and} \quad (\forall x \in t)F$$

[1] To be honest, the true reasons for switching to subsystems of set theory were the technical difficulties we encountered in the proof-theoretical analysis of stronger impredicative subsystems of Second Order Number Theory. Pursuing the methods of then known predicative proof theory we tried to work within the ramified analytical hierarchy. However, the coding machinery needed there turned out to be unfeasible. As soon as we switched to the constructible hierarchy everything worked canonically. Then it was of course natural also to look at subsystems of ZFC. It should be mentioned that it was G. Jäger, then my student, who was the first to study the role of the constructible universe in proof theory systematically (cf. [9]).

have $\wedge$-type. To every L-sentence we associate an infinite set $\mathrm{it}(F)$ of L-sentences, the *infinity type of* F, such that for L-sentences of $\vee$-type we have

$$\mathbf{L} \models F \quad \text{if and only if} \quad \mathbf{L} \models \bigvee \mathrm{it}(F)$$

and dually

$$\mathbf{L} \models F \quad \text{if and only if} \quad \mathbf{L} \models \bigwedge \mathrm{it}(F)$$

for sentences of $\wedge$-type. We define

$$\mathrm{it}(t \in \mathsf{L}_\alpha) := \{\, s = t \mid \mathrm{stg}(s) < \alpha \,\}$$
$$\mathrm{it}(t \in \{\, x \in \mathsf{L}_\alpha \mid F(x)^{\mathsf{L}_\alpha} \,\}) := \{\, s = t \wedge F(s)^{\mathsf{L}_\alpha} \mid \mathrm{stg}(s) < \alpha \,\}$$

and

$$\mathrm{it}\big((\exists x \in \mathsf{L}_\alpha) G(x)\big) := \{\, G(s) \mid \mathrm{stg}(s) < \alpha \,\}$$
$$\mathrm{it}\big((\exists x \in \{\, y \in \mathsf{L}_\alpha \mid F(x)^{\mathsf{L}_\alpha} \,\}) G(x)\big) := \{\, F(s)^{\mathsf{L}_\alpha} \wedge G(s) \mid \mathrm{stg}(s) < \alpha \,\}$$

for L-sentences of $\vee$-type. If F is a sentence of $\wedge$-type then $\neg F$ is of $\vee$-type and we may put

$$\mathrm{it}(F) := \{\, \neg G \mid G \in \mathrm{it}(\neg F) \,\}$$

for sentences of $\wedge$-type. It can be seen that the sentences in $\mathrm{it}(F)$ are of lower complexity than F; i.e., we have

$$\mathrm{rk}(G) < \mathrm{rk}(F) \quad \text{for } G \in \mathrm{it}(F)$$

if $\mathrm{rk}(F)$ denotes the complexity of the sentence F.[2]

We obtain the infinitary system by just two rules

($\vee$) *If F is a sentence of $\vee$-type and $\models^{\alpha_0} G$ with $\alpha_0 < \alpha$ holds for some $G \in \mathrm{it}(F)$ then $\models^{\alpha} F$.*

and

($\wedge$) *If F is a sentence of $\wedge$-type and $\models^{\alpha_G} G$ with $\alpha_G < \alpha$ holds for all $G \in \mathrm{it}(F)$ then $\models^{\alpha} F$.*

By induction on $\mathrm{rk}(F)$ we obtain easily

(2.2) $$\mathbf{L} \models F \quad \text{if and only if} \quad (\exists \alpha) \models^{\alpha} F.$$

In analogy to the situation in the previous section we define

$$\mathrm{tc}(F) := \min\{\, \alpha \mid \models^{\alpha} F \,\}.$$

[2]It is then obvious that for true sentences F we have $\mathrm{tc}(F) \leq \mathrm{rk}(F)$.

The direction from right to left in (2.2) sharpens to[3]

$$\mathrm{tc}(F) \leq \alpha \Longrightarrow \mathbf{L}_\alpha \models F. \tag{2.3}$$

Equation (2.3) shows that it suffices to compute $\mathrm{tc}(F)$ for all $F \in \mathcal{F}$ which are provable from Ax in order to get $\|\mathsf{Ax}\|_{\mathcal{F}}$. There are of course rough estimates for $\mathrm{tc}(F)$ such as $\mathrm{rk}(F)$. It is also obvious that the infinite proof tree for a $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-sentence F is $< \omega_1^{\mathrm{CK}}$ branching which yields $\mathrm{tc}(F) < \omega_1^{\mathrm{CK}}$. But this gives us only $\mathbf{L}_{\omega_1^{\mathrm{CK}}}$ as a $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-model for Ax. And this is a triviality because $\mathbf{L}_{\omega_1^{\mathrm{CK}}} \models F$ is obvious for true $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-sentences F.

2.2. Ordinals and collapsing functions. So we have to do something more subtle. We have not yet used the fact that there is a proof for F from the axioms in Ax. That means that there are finitely many axioms $\{A_1, \ldots, A_n\} \subseteq \mathsf{Ax}$ and a cut free derivation

$$\vdash A_1 \wedge \cdots \wedge A_n \to F \tag{2.4}$$

in pure logic. It is not hard to see that this derivation translates into

$$\models^k A_1^* \wedge \cdots \wedge A_n^* \to F^* \tag{2.5}$$

for some finite k where A_i^* and F^* are suitable translations into L-formulas. (E.g., $F^* := F^{\mathsf{L}_\alpha}$ for $\alpha = \|\mathsf{Ax}\|_\infty$.)

Then we compute upper bounds for $\mathrm{tc}(A^*)$ for all $A \in \mathsf{Ax}$.[4]

This gives us

$$\models^{\alpha_i} A_i^* \tag{2.6}$$

for $i = 1, \ldots, n$, but then we are stuck since there is no way to glue (2.5) and (2.6) together. What is lacking is a cut rule. If we add a cut rule and denote (still tentatively) by $\vdash^\alpha_\rho F$ that there is an infinite proof tree of depth $\leq \alpha$ whose cut sentences all have complexities $< \rho$ then we get from (2.5) and (2.6)

$$\mathsf{Ax} \vdash F \Longrightarrow \vdash^\alpha_\rho F^* \tag{2.7}$$

with α as an upper bound for the truth complexities of all axioms and ρ a strict bound for the complexities of all axioms.[5]

[3]Though true, this is not so easy to prove. But it becomes trivial if we require in $\vee$-clauses that its witness $G \in \mathrm{it}(F)$ is controlled by α. I.e., inspecting the definition of $\mathrm{it}(F)$ we see that for every $G \in \mathrm{it}(F)$ there is a characteristic L-term s for which we require $\mathrm{stg}(s) < \alpha$. We sloppily denote that by $G < \alpha$.

[4]Usually this is not too difficult because the truth of axioms should be easily recognizable. Otherwise they are not good axioms.

[5]Because of $\mathrm{tc}(F) \leq \mathrm{rk}(F)$ the ordinals α and ρ in (2.7) are in general not too distant.

What we need now is a function, say η such that

$$\vdash^{\alpha}_{\rho} F \Longrightarrow \models^{\eta(\rho,\alpha)} F.$$

This situation is well-known to proof theorists. For predicative axiom systems the Veblen function (cf. Section 3.3 below) φ does that. For impredicative axiom systems, however, we encounter an additional difficulty. Even if F is a $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-sentence we cannot be sure that $\vdash^{\alpha}_{\rho} F$ is represented by a $< \omega_1^{\mathrm{CK}}$ branching tree because there might be cuts of higher complexities. Therefore we have to look for a function η which also collapses large ordinals below ω_1^{CK}. The best way to visualize the situation is to think of two functions: First a function ϕ, which is quite similar to the Veblen function, which gives

$$\vdash^{\alpha}_{\rho} F \Longrightarrow \vdash^{\phi(\rho,\alpha)}_{0} F \tag{2.8}$$

and then a collapsing function, say Ψ, which collapses $\phi(\rho,\alpha)$ below ω_1^{CK} and for which we have[6]

$$\vdash^{\beta}_{0} F \Longrightarrow \models^{\Psi(\beta)} F. \tag{2.9}$$

The development of such *collapsing functions* is one of the central problems in impredicative proof theory. It is at this point where we need Set Theory and Generalized Recursion Theory. There will be no time to go into details but I want to explain the principles on a simple example of a partial collapsing function

$$\Psi\colon \mathsf{On} \to_{\mathrm{p}} \omega_1^{\mathrm{CK}}.$$

The collapsing functions have to meet certain requirements which put them close to Mostowski collapses. So we have to "thin out" the ordinals. We define sets $B(\alpha) \subseteq \mathsf{On}$ by the following clauses:

$$\begin{aligned}
&\{0, \omega_1\} \subseteq B(\alpha)\\
&B(\alpha) \textit{ is closed under } +,\ \varphi\\
&\xi \in B(\alpha) \cap \alpha \Longrightarrow \Psi(\xi) \in B(\alpha)\\
&\Psi(\alpha) := \min\{\, \xi \mid \xi \notin B(\alpha)\,\}
\end{aligned} \tag{2.10}$$

where ω_1 denotes the first uncountable ordinal. This definition is by recursion on α. It is obvious that $B(\alpha)$ is a countable set. Hence $\Psi(\alpha) < \omega_1$ even for $\alpha > \omega_1$. So Ψ collapses ordinals above ω_1 into countable ordinals. Above ω_1 the set $B(\alpha)$ only contains ordinals below the first fixed-point,

[6] To tell the truth the collapsing function is already needed in the cut-elimination procedure (a faint hint is given below). So the above function ϕ will already need Ψ. But we didn't want to go too much into technical details.

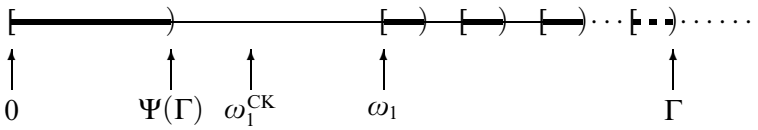

FIGURE 1. The set $B(\Gamma)$.

say Γ, of the Veblen function above ω_1. (A kind of uncountable Γ_0.) Hence $B(\alpha) \subseteq B(\Gamma)$ for all α and it turns out that

$$B(\Gamma) \cap \omega_1 = \Psi(\Gamma).$$

We see from the definition of $B(\Gamma)$ that every ordinal in $B(\Gamma)$ can be represented by a term built up from the symbols 0, ω_1, $+$, φ, Ψ. This representation can be made unique and thus provides an ordinal notation system which includes the segment of ordinals below $\Psi(\Gamma)$.

The relation $< \restriction B(\alpha)$ turns out to be primitive recursive. Hence $\Psi(\Gamma) < \omega_1^{CK}$ and Ψ is a function which collapses ordinals above ω_1 into ordinals below ω_1^{CK}. This is, however, not really what we wanted. But after having computed $\Psi(\Gamma)$ it is not too hard to see that we may replace ω_1 by (almost) any ordinal from the interval $\big(\Psi(\Gamma), \omega_1\big]$—and thus especially also by ω_1^{CK} without changing the segment in $B(\Gamma)$. Thus Ψ may be regarded as a partial function which collapses ordinals above ω_1^{CK} into ordinals below ω_1^{CK}.

This example elucidates the role of cardinals in the development of ordinal notation systems. If we start with

$$\{0, \omega_1^{CK}\} \subseteq B(\alpha)$$

instead of the first line of (2.10) and want to show $(\forall \xi)[\Psi(\xi) < \omega_1^{CK}]$ we have to examine the complexity of all the involved sets and functions. By starting with ω_1 instead, we get $(\forall \xi)[\Psi(\xi) < \omega_1]$ by a simple cardinality argument. So, the *regular* ordinal ω_1 may be regarded as an idealization of the *recursively regular* ordinal ω_1^{CK} which is used to avoid the additional examination of the complexity of the involved sets and functions. It is true that in the end we have to show that the resulting term-notation system is recursive in order to get $(\forall \xi)[\Psi(\xi) < \omega_1^{CK}]$. But this is much easier.[7]

This simple example expands to all stronger notation systems. So for instance the system for the analysis of $(\Sigma_2^1\text{-AC}) + (\text{BI})$ (which on the set

[7]This strategy resembles that in Physics where one abstracts from side effects in order to study the main effect. It has long be conjectured—and recently carried through by Rathjen for simpler and more recently independently by Rathjen and Schlüter for the strongest notation systems—that the collapsing functions can also be developed directly on the basis of recursively regular ordinals instead of regular ones. This has the advantage that the development can be carried through in (a slight extension of) the analyzed axiom system itself but the development becomes *much* more involved.

theoretical side corresponds to the system KPi axiomatizing an admissible union of admissible universes) needs the first weakly inaccessible cardinal in its development, the notation system for the subsystem KPM, axiomatizing a recursively Mahlo universe, needs the first Mahlo cardinal, and the notation system for Π_3-reflection needs the first weakly compact cardinal. The analysis of (Π^1_2-CA), as announced by Rathjen, seems to use cardinals above supercompactness.

We want, however, to emphasize that all *these large cardinals are only* needed as *idealizations*. As soon as we have developed the notation system $\mathbb{B}$ we may regard it as a formally defined primitive recursive notation system which then of course needs to be proved well-founded. So, those who don't like ordinals may take the primitive recursively defined term system $\mathbb{B}$, show its well-foundedness, preferably within a constructively justified system, and then use it to analyze the axiom system in question.

However, we believe that the large cardinals used for the notation are themselves good indicators for the strength of the analyzed axiom system. It is true that the closure properties of those cardinals are in some sense—which would lead too far to explain here—reflected in the axiom systems.

2.3. The semi-formal system. Let us assume that we succeeded in constructing a notation system $\mathbb{B}$ together with appropriate collapsing functions Ψ_κ (in general just one collapsing function is not sufficient but we need a collapsing function Ψ_κ with $\Psi_\kappa(\alpha) < \kappa$ for every recursive number class which we assume to begin with the admissible ordinal κ. We stick, however, to the notation Ψ if we want to denote $\Psi_{\omega_1^{\mathrm{CK}}}$).

In order to make (2.8) and (2.9) precise we have to define a semi-formal system

$$\vdash^\alpha_\rho F$$

suitably. What are the requirements it has to meet? The ordinal α has to be collapsible, i.e., an ordinal of $\mathbb{B}$. Therefore we must only assign ordinals from $\mathbb{B}$. But $\mathrm{otyp}(\mathbb{B}) < \omega_1^{\mathrm{CK}}$ which means that there are no real admissible (recursively Mahlo, Π_3-reflecting, ...) ordinals in $\mathbb{B}$. Therefore we have to add defining rules for the (symbols for) admissible ordinals in $\mathbb{B}$ to the semi-formal system, e.g., the rules Π^κ_2-Ref below. It has turned out to be very convenient to use a Tait-calculus (some people call it also a *one sided sequent calculus*) in the definition of the semi-formal system. We are therefore going to define the semi-formal provability relation

$$\vdash^\alpha_\rho \Delta \tag{2.11}$$

for finite sentence-sets Δ. Since we want to collapse the depth of the derivations in the semi-formal system we introduce a relation

$$\alpha \ll \beta \;:\Longleftrightarrow\; \alpha < \beta \text{ and } \Psi(\alpha) < \Psi(\beta).$$

If we assign ordinals to the derivations which grow in the sense of $\ll$ instead of $<$ then we get

$$(\vdash^{\alpha}_{\rho} \Delta \Longrightarrow \vdash^{\alpha'}_{\rho'} \Delta') \Longrightarrow (\vdash^{\Psi(\alpha)}_{\rho} \Delta \Longrightarrow \vdash^{\Psi(\alpha')}_{\rho'} \Delta').$$

So why do we not assign directly $\Psi(\alpha)$ instead of α? The reason lies in the infinitely many premises of $\wedge$-inferences. There it may happen that there are not enough $\alpha_G \ll \alpha$. Here we introduce a relation

$$\langle\, \alpha_G \mid G \in \mathrm{it}(F) \,\rangle \ll \alpha$$

which we won't explain in detail (although it is a very important point). Thus we define the semi-formal provability relation (2.11) by the following clauses:

($\vee$) *If F is a sentence of $\vee$-type and $\vdash^{\alpha_0}_{\rho} \Delta$, G with α_0, $G \ll \alpha$ holds for some $G \in \mathrm{it}(F)$ then $\vdash^{\alpha}_{\rho} \Delta$, F*

($\wedge$) *Let F be a sentence of $\wedge$-type such that $\vdash^{\alpha_G}_{\rho} \Delta$, G holds for every $G \in \mathrm{it}(F)$ and $\langle\, \alpha_G \mid G \in \mathrm{it}(F) \,\rangle \ll \alpha$ then $\vdash^{\alpha}_{\rho} \Delta$, F*

(Π^{κ}_2-Ref) *If $\vdash^{\alpha_0}_{\rho} \Delta$, $(\forall x \in \mathsf{L}_\kappa)(\exists y \in \mathsf{L}_\kappa) F(x,y)$ for a $\Delta_0^{\mathsf{L}_\kappa}$-sentence $F(x,y)$ and κ, $\alpha_0 \ll \alpha$ then*

$$\vdash^{\alpha}_{\rho} \Delta,\ (\exists z \in \mathsf{L}_\kappa)(\forall x \in z)(\exists y \in z) F(x,y)$$

(cut) *If $\vdash^{\alpha_1}_{\rho} \Delta$, F, $\vdash^{\alpha_2}_{\rho} \Delta$, $\neg F$, α_1, $\alpha_2 \ll \alpha$ and $\mathrm{rk}(F) < \rho$ then $\vdash^{\alpha}_{\rho} \Delta$*

where I have suppressed some technical details. The notation $G \ll \alpha$ in clause ($\vee$) should be read analogously to $G < \alpha$ in footnote 3 on page 168. The semi-formal system is tailored so that we have the Collapsing Property

$$\vdash^{\alpha}_{\kappa} \Delta \text{ for a finite set } \Delta \text{ of } \Sigma_1^{\mathsf{L}_\kappa}\text{-sentences} \Longrightarrow \vdash^{\Psi_\kappa(\alpha)}_{\kappa} \Delta \tag{2.12}$$

and

$$\vdash^{\alpha}_{0} \{F_1, \ldots, F_n\} \text{ for } \alpha < \omega_1^{\mathrm{CK}} \Longrightarrow \models^{\alpha} F_1 \vee \cdots \vee F_n \tag{2.13}$$

for $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-sentences $F_1, \ldots, F_n$.

Since we have $\vdash^{\alpha}_{\rho} F^*$ for all formulas provable from Ax the problem of ordinal analysis for Ax will be solved by cut elimination for the semi-formal system. But notice that we have not only to get rid of all cuts but also of the additional reflection rules (Π^{κ}_2-Ref). The technique we use here is called *local predicativity* first developed by myself for the analysis of axiom systems for iterated inductive definitions then extended by Jäger and myself to systems for set theory. An essential simplification in both technical and conceptual

respect is due to Buchholz (operator-controlled derivations cf. [3]) and most recently Rathjen has extended the method considerably.

I don't want to go too much into details. One of the key ideas is a formalization of (2.3) which leads to the following Boundedness Property

$$\vdash^{\alpha}_{\rho} \Delta,\ F^{\mathsf{L}_\kappa} \text{ and } \alpha \leq \kappa \Longrightarrow \vdash^{\alpha}_{\rho} \Delta,\ F^{\mathsf{L}_\alpha}. \tag{2.14}$$

Combining the Boundedness and the Collapsing Properties we see how to eliminate $(\Pi^\kappa_2\text{-Ref})$-inferences in derivations of Σ^κ_1-sentences. For a set Δ, F^{L_κ} of $\Sigma^{\mathsf{L}_\kappa}_1$-sentences we may in (2.14) replace α by $\Psi_\kappa(\alpha) < \kappa$ using the Collapsing Property. A $(\Pi^\kappa_2\text{-Ref})$-inference of the form

$$\vdash^{\alpha_0}_{\kappa} \Delta,\ (\forall x \in \mathsf{L}_\kappa)(\exists y \in \mathsf{L}_\kappa)F(x,y) \Longrightarrow \vdash^{\alpha}_{\kappa} \Delta,\ (\exists z \in \mathsf{L}_\kappa)(\forall x \in z)(\exists y \in z)F(x,y)$$

can thus be turned into an $(\vee)$-inference

$$\vdash^{\Psi_\kappa(\alpha_0)}_{\kappa} \Delta,\ (\forall x \in \mathsf{L}_{\Psi_\kappa(\alpha_0)})(\exists y \in \mathsf{L}_{\Psi_\kappa(\alpha_0)})F(x,y) \Longrightarrow \vdash^{\Psi_\kappa(\alpha)}_{\kappa} \Delta,\ (\exists z \in \mathsf{L}_\kappa)(\forall x \in z)(\exists y \in z)F(x,y)$$

with $\mathsf{L}_{\Psi_\kappa(\alpha_0)}$ as witness.

The interplay between the Collapsing and Boundedness Properties allows also to remove all cuts of complexities $\geq \kappa$ in proofs of $\Sigma^{\mathsf{L}_\kappa}_1$-sentences. The crucial case is a cut with a cut sentence $(\exists z \in \mathsf{L}_\kappa)F^z$ which is derived by an instance of the Π^κ_2-reflection rule (Π^κ_2)-Ref. Then we have the situation

$$\dfrac{\dfrac{\vdash^{\alpha_0}_{\kappa} \Delta, F^{\mathsf{L}_\kappa}}{\vdash^{\alpha}_{\kappa} \Delta, (\exists z \in \mathsf{L}_\kappa)F^z} \quad \dfrac{\langle\, \vdash^{\beta_s}_{\kappa} \Delta, \neg F^s \mid \mathrm{stg}(s) < \kappa \,\rangle}{\vdash^{\beta}_{\kappa} \Delta, (\forall z \in \mathsf{L}_\kappa)\neg F^z}}{\vdash^{\gamma}_{\kappa+1} \Delta}$$

for a set Δ of $\Sigma^{\mathsf{L}_\kappa}_1$-sentences. First we may replace α_0 by $\Psi_\kappa(\alpha_0) < \kappa$ using the Collapsing Property and then reduce the derivation to

$$\dfrac{\vdash^{\Psi_\kappa(\alpha_0)}_{\kappa} \Delta, F^{\mathsf{L}_{\Psi_\kappa(\alpha_0)}} \quad \vdash^{\beta_{\Psi_\kappa(\alpha_0)}}_{\kappa} \Delta, \neg F^{\mathsf{L}_{\Psi_\kappa(\alpha_0)}}}{\vdash^{\alpha\#\beta}_{\kappa} \Delta}\ .$$

Of course we have to ensure that $\Psi_\kappa(\alpha_0)$ and $\beta_{\Psi_\kappa(\alpha_0)}$ are both collapsibly less than $\alpha\#\beta$ which depends essentially on the features of the relation

$$\langle\, \alpha_G \mid G \in \mathrm{it}(F) \,\rangle \ll \alpha.$$

Once we are down to a cut rank of ω_1^{CK} we may use the familiar cut elimination for predicative systems to obtain a cut- (and reflection-) free derivation.

Summing up we can state that an ordinal analysis of an axiom system Ax is obtained by the following steps:

- *Embedding of the formal proof* $\mathsf{Ax} \vdash F$ *into the semi-formal system to obtain* $\vdash^{\alpha}_{\rho} F^*$.

This step works best for *balanced* systems (as Kreisel called them). A system is *unbalanced* if it lacks full foundation. Typically you find a sentence F in an unbalanced system Ax such that $\mathsf{Ax} \nvdash \mathsf{TI}(F, \varepsilon_0)$ though Ax may have a proof theoretical ordinal much bigger than ε_0. Examples for unbalanced systems are subsystems of Second Order Number Theory with restricted comprehension but without the scheme

$$(\mathbf{BI}) \qquad (\forall X)F(X) \to F(G)$$

of (classical) Bar Induction or subsystems of set theory with restricted foundation scheme. Since full foundation comes for free in **L** we need additional considerations for unbalanced systems. A frequently used technique (cf., e.g., [17]) is to embed the axiom system into an intermediate system which allows partial cut elimination. Embedding the reduced derivations of the intermediate system into the semi-formal system then produces lower cut complexities and thus often yields lower upper bounds for truth complexities.

Having embedded the axiom system directly or via an intermediate system, the next step consists in

- *cut elimination for the semi-formal system to obtain* $\vdash^{\eta(\alpha,\rho)}_{0} F^*$.

Then we use (2.13) to obtain upper bounds for $\|\mathsf{Ax}\|_{\Sigma^{\omega}_1}^{\mathrm{CK}}$.

The usual technique to prove that an obtained upper bound α is precise is to show

$$\mathsf{Ax} \vdash \mathsf{TI}(< \alpha).$$

The above sketch makes already clear that an ordinal analysis of Ax needs, besides transfinite induction, only means which are formalizable within PA (including the semi-formal system whose proof trees have to be formalized by recursive trees). Therefore we regularly obtain a profound ordinal analysis of Ax by formalizing the whole procedure in $\mathsf{PA} + \mathsf{TI}(< \alpha)$ (where the latter stands for transfinite induction along all initial segments of order type $< \alpha$ of the well-ordering which is obtained from an arithmetization of the notation system $\mathbb{B}$).

§3. Results.

3.1. General. Before we list some of the axiom systems which have been successfully analyzed we want to discuss the results of an ordinal analysis in a more general setting. A glance at Figure 2 helps here. It shows the typical

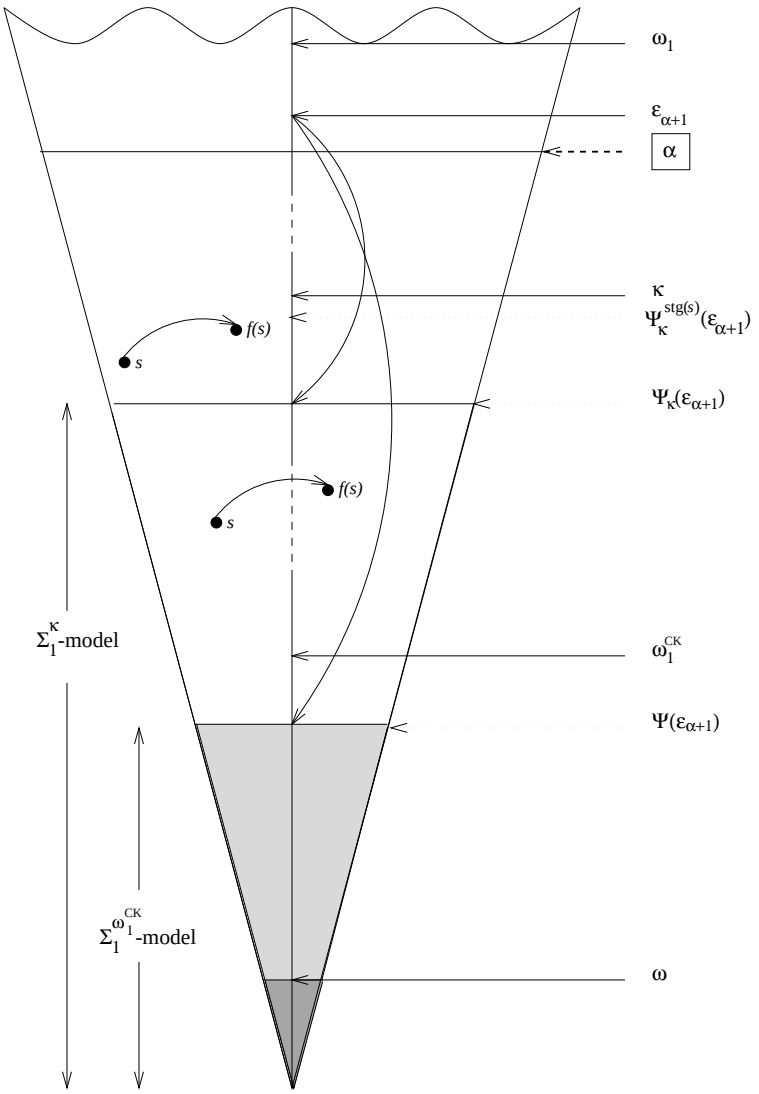

FIGURE 2. Spectrum of an axiom system with ∞-norm α.

"spectrum" of a balanced impredicative axiom system with $\|\mathsf{Ax}\|_\infty = \alpha$. You recognize the ordinal α at which we have a full model of Ax. The ordinal α is typically an admissible ordinal or the limit of admissibles. Since we have full induction, there is a virtual well-ordering $<_p$ (see Section 3.3 below) of ω for which Ax proves $\mathsf{TI}(<_p\restriction\eta)$ for every $\eta < \varepsilon_{\alpha+1}$ by the usual technique of Gentzen. This explains the increase to $\varepsilon_{\alpha+1}$ in the figure. For

each set $\{F_1, \ldots, F_n\}$ of $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-sentences we obtain

$$\vdash_0^\beta \{F_1, \ldots, F_n\} \Longrightarrow \models^\beta F_1 \vee \cdots \vee F_n. \tag{3.1}$$

Therefore we have

$$\mathrm{tc}(F) < \Psi(\varepsilon_{\alpha+1})$$

for every $\Sigma_1^{\omega_1^{\mathrm{CK}}}$-sentence provable from Ax. Hence

$$\|\mathsf{Ax}\|_{\Sigma_1^{\omega_1^{\mathrm{CK}}}} \leq \Psi(\varepsilon_{\alpha+1}). \tag{3.2}$$

The result displayed in (3.1) depends on the fact that $\Psi(\varepsilon_{\alpha+1})$ is a segment of the notation system $\mathbb{B}$. This is no longer true for ordinals bigger than $\Psi(\varepsilon_{\alpha+1})$. So what are we to do if we want to extend (3.1) to Σ_1^κ-sentences? To explain that roughly we have to say a bit more about the notation system $\mathbb{B}$. Let's forget for a moment that we developed the notation system on the basis of cardinals and just talk about admissible ordinals. The parameters used in the development of $\mathbb{B}$ are 0, $+$, possibly a function like the Veblen function φ and—and that is really important—a function which enumerates a segment of the admissible ordinals. Therefore every ordinal in $\mathbb{B}$ can be denoted by a Skolem-term which is built up only from these constants. This term notation can be made unique and it turns out that the induced order relation on the notation terms is primitive recursive. As soon as we want to denote segments of the ordinals above ω_1^{CK} we need more parameters. If we allow all ordinals $\leq \sigma$ as parameters in the construction of $\mathbb{B}$ then this modified $\mathbb{B}^\sigma$ contains a segment to which σ belongs. This defines new collapsing functions, say Ψ_κ^σ, for $\kappa \geq \sigma$. If we choose $\sigma = \kappa^-$ (the biggest admissible ordinal below κ) then we obtain

$$\Psi_\kappa^{\kappa^-}(\varepsilon_{\alpha+1}) = \Psi_\kappa(\varepsilon_{\alpha+1}) \tag{3.3}$$

but observe that $\mathbb{B}^{\kappa^-} \cap \Psi_\kappa(\varepsilon_{\alpha+1})$ is a segment of the ordinals while $\mathbb{B} \cap \Psi_\kappa(\varepsilon_{\alpha+1})$ contains gaps. It is self-evident that $\mathbb{B}^\sigma$ is no longer primitive recursive but only primitive recursive in $\{\, \xi \mid \xi \leq \sigma \,\}$. In that sense we may generalize (3.1) to

$$\vdash_{\kappa^-}^\beta \Delta \text{ and } \Delta \subseteq \Sigma_1^\kappa \Longrightarrow \models^\beta \bigvee \Delta \tag{3.4}$$

where β is now supposed to be an ordinal in $\mathbb{B}^{\kappa^-}$. So, considering (3.3), we can generalize (3.2) to

$$\|\mathsf{Ax}\|_{\Sigma_1^\kappa} \leq \Psi_\kappa(\varepsilon_{\alpha+1})$$

as displayed in Figure 2. Observe that the Σ_1-model (which coincides with the Π_2-model) is the Σ_1^α-model and is thus obtained at $\Psi_\alpha(\varepsilon_{\alpha+1})$.

But there is also a more dynamical aspect in our results. We have

$$\|\mathsf{Ax}\|_{\Sigma_1^\kappa} = \|\mathsf{Ax}\|_{\Pi_2^\kappa} = \Psi_\kappa(\varepsilon_{\alpha+1})$$

for all admissibles $\kappa \leq \alpha$. This shows that the ordinals $\Psi_\kappa(\varepsilon_{\alpha+1})$ are closed under all κ-recursive functions f which are provably total in Ax. This is an example for a closure property which is reflected downwards. The regular cardinal K is closed under any function. Its recursive counterpart, the admissible ordinal κ, is closed under all $\mathbf{L}_\kappa$-recursive functions, while its collapse $\Psi_\kappa(\varepsilon_{\alpha+1})$ (relative to the axiom system Ax) is closed under all $\mathbf{L}_\kappa$-recursive functions whose totality is provable in Ax.

But what happens if Ax proves the Π^κ_2 sentence $(\forall x \in \mathsf{L}_\kappa)(\exists y \in \mathsf{L}_\kappa)G(x, y)$ and we choose any $s \in \mathbf{L}_\kappa$ with $\mathrm{stg}(s) \geq \Psi_\kappa(\varepsilon_{\alpha+1})$? Then there is a $t \in \mathbf{L}_\kappa$ with $G(s, t)$. Can we say anything about the stage of t? Here we may use $\{\, \xi \mid \xi \leq \mathrm{stg}(s) \,\}$ as parameters in the development of $\mathbb{B}^{\mathrm{stg}(s)}$ and thus obtain by (3.4)

$$\models^{\Psi_\kappa^{\mathrm{stg}(s)}(\varepsilon_{\alpha+1})} (\exists y) G(s, y)$$

which finally yields

$$\mathbf{L}_\kappa \models G(s, t) \text{ for some } t \in \mathbf{L}_{\Psi_\kappa^{\mathrm{stg}(s)}(\varepsilon_{\alpha+1})}.$$

This shows that every κ-recursive function f which is provably total in Ax is bounded by the function $\lambda\xi.\Psi_\kappa^\xi(\varepsilon_{\alpha+1})$.

This result would also make sense below ω since it would give a characterization of the provably recursive functions of Ax. Such a characterization can already be obtained by a profound ordinal analysis using (1.4) and (1.5) and the fact that the provably recursive functions of $\mathsf{PRA} + \mathsf{PRWO}(\prec)$ are the $\prec$-descendent recursive functions (cf. [16]). But a more direct characterization, avoiding all the metamathematical deviations, would be desirable.

So the question arises if it is not possible to find a partial collapsing function

$$F : \mathsf{On} \to_{\mathrm{p}} \omega$$

and extend the above results to collapses below ω. Of course we cannot expect to find a Σ^0_1- or Π^0_2-model below ω—the least such model is always $\mathbf{L}_\omega$—but, as mentioned a minute ago, the dynamical aspect of majorizing provably recursive function makes sense also below ω.

The development of such a collapsing function, however, turned out to be quite difficult. It was just recently that Weiermann discovered the right way to define collapsing-functions

$$\lambda\alpha.F_\alpha(n) : \mathsf{On} \to_{\mathrm{p}} \omega$$

in terms of a new approach to subrecursive hierarchies. Generalizing the techniques of local predicativity he could show

$$\mathsf{Ax} \vdash (\forall x \in \omega)(\exists y \in \omega)G(x, y) \Longrightarrow (\forall n)(\exists m \in F_\beta(n))G(n, m)$$

where $\beta := \|\mathsf{Ax}\|^{\mathrm{CK}}_{\Sigma^1_{\omega_1}}$ and thus obtained upper bounds for the growth of the provably recursive functions of the axiom system Ax in terms of the subrecursive hierarchy F_β (cf. [27], [2]). This hierarchy corresponds closely to the familiar Hardy and so called "fast growing" subrecursive hierarchies (cf. [4]).

These results incorporate some kind of answer to question (iv) (a). If we have an ordinal analysis of an axiom system we know more of the provable Π^κ_2- and Σ^κ_1-sentences of Ax than their sheer truth. We know the minimal models of Σ^κ_1- and Π^κ_2-sentences in the constructible hierarchy and we can describe functions majorizing the Skolem-functions of Π^κ_2-sentences (including $\kappa = 0$).

3.2. Some examples. There is not enough space to list all the axioms systems which are analyzed today. We content ourselves to give some landmarks as displayed in Figure 3. It shows the proof-theoretical ordinals of some familiar axiom systems together with their "cardinal origin". The leftmost cone is supposed to represent the set theoretical universe **V**. The ordinals shown there are the cardinals used in the development of the notation system. I_0 stands for the first weakly inaccessible cardinal, M_0 for the first weak Mahlo cardinal and K_0 for the first weakly compact cardinal.

The cones in the middle and on the right hand show **L** on different scales. In the middle we display the **L**-cone below the first uncountable ordinal ω_1. The ordinals shown there are the "recursive" counterparts of the ordinals on the left. So ι_0 stands for the first recursively inaccessible ordinal, μ_0 for the first recursively Mahlo ordinal and κ_0 for the first Π_3-reflecting ordinal. These ordinals are the $\|\mathsf{Ax}\|_\infty$-norms of the axiom systems shown between the middle and the right cone. On the right hand we see **L** below ω_1^{CK} and the "proof theoretical" collapses $\|\mathsf{Ax}\|^{\mathrm{CK}}_{\Sigma^1_{\omega_1}}$ of the ordinals $\|\mathsf{Ax}\|_\infty$ in the middle relative to the axiom systems shown.

The weakest axiom system in the figure is PA having proof theoretical ordinal ε_0. On the side of subsystems of set theory PA corresponds to Kripke-Platek Set Theory with foundation restricted to Π^0_∞-formulas.

Full Kripke-Platek Set Theory comprises the axioms for pairing, union, Δ_0-separation and Δ_0-collection together with extensionality and the scheme of foundation. If we add also an infinity axiom we obtain the system $\mathsf{KP}\omega$ which represents one of the simplest impredicative axiom systems. Alternatively, $\mathsf{KP}\omega$ can be axiomatized by a scheme of Π_2-reflection. It is as strong as the axiom system ID_1 which axiomatizes the existence of sets which can be inductively defined by arithmetical positive operators. It is not hard to see that on the side of subsystems of Second Order Arithmetic it corresponds to Π^1_1-comprehension

$$(\Pi^1_1\text{-CA}) \qquad (\exists X)(\forall x)[x \in X \leftrightarrow F(x, X)]$$

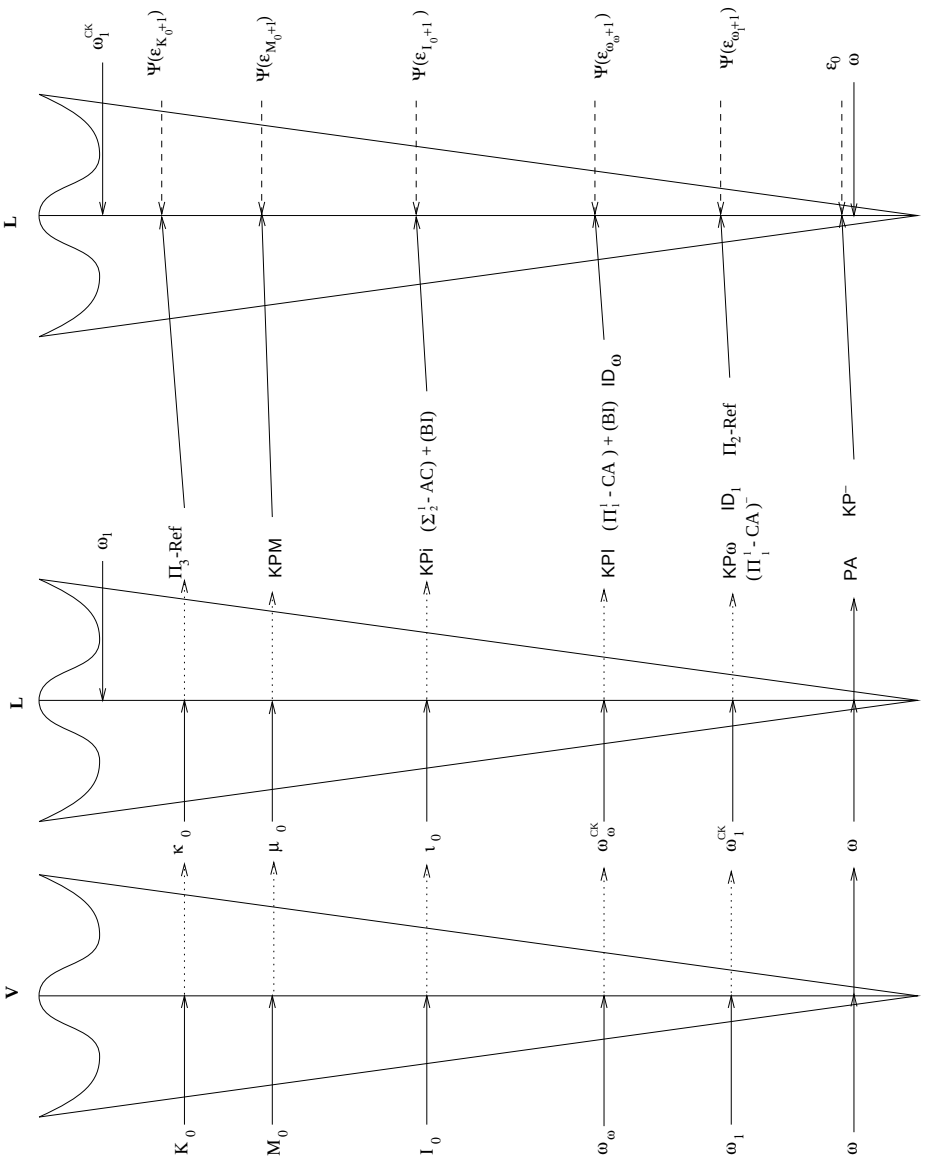

FIGURE 3. Some proof theoretic ordinals and their origins.

where $F(X, x)$ is a Π^1_1-formula which must not contain set parameters besides X.

Allowing additional set parameters in $F(X, x)$ means to allow to iterate (Π^1_1-CA) arbitrarily (finitely) often. This system is unbalanced but becomes balanced by adding the scheme (BI) of classical Bar Induction. On the side of inductive definitions this system corresponds to ID_ω and on the

side of subsystems of Set Theory to the system KPI which axiomatizes that the universe is a limit of admissible universes.

If we require that the universe is not only a limit of admissible universes but itself also admissible we obtain the system KPi which on the side of subsystems of Second Order Number Theory corresponds to $(\Delta^1_2\text{-CA})+(\text{BI})$ which is of the same strength as $(\Sigma^1_2\text{-AC})+(\text{BI})$. Its proof theoretical ordinal is that of Feferman's system T_0 of Explicit Mathematics.

Adding the Mahlo-Scheme

$$(\forall x)(\exists y)F(x,y) \to (\exists z \in \text{Ad})(\forall x \in z)(\exists y \in z)F(x,y),$$

where Ad denotes the class of admissible ordinals, gives the axiom system KPM. On the side of subsystems of Second Order Arithmetic this systems corresponds to $(\Delta^1_2\text{-CA}) + (\text{BI})$ together with a reflection scheme which tells that every Π^1_3-sentence is reflected in some β-model of $(\Delta^1_2\text{-CA})$.

The strongest system (whose analysis is known to me) is the system of Π_3-reflection with the scheme

$$(\Pi_3\text{-Ref}) \quad (\forall x)(\exists y)(\forall z)F(x,y,z) \to (\exists u)(\forall x \in u)(\exists y \in u)(\forall z \in u)F(x,y,z)$$

for Δ_0-formulas $F(x,y,z)$. This system has been analyzed by Rathjen ([20]) and it is quite clear how this can be generalized to systems for Π_n-reflection. Most recently he succeeded in analyzing $(\Pi^1_2\text{-CA})$ using cardinals above supercompactness on the left hand side (of the figure).

3.3. Concluding remarks. In the previous section we talked about the "proof theoretical" collapses of admissible ordinals but wisely equipped that notion with the addendum "relative to an axiom system Ax". This relativization is justified by the fact that while the admissible ordinal κ is characterized by being closed under all κ-recursive functions, its proof theoretical collapse relative to Ax is characterized by being closed under all κ-recursive functions whose totality is provable in Ax. One may wonder whether there is also an absolute notion of proof theoretical collapse. We do not know the final answer but want to indicate in what direction one should search.

The situation is simple as long as we stay within predicative proof theory. If we regard an initial segment $\mathcal{A}$ of the set theoretical universe there are two ordinals which in some sense determine the behavior of $\mathcal{A}$. These are

$$o(\mathcal{A}) := \min\{\,\alpha \in \mathsf{On} \mid \alpha \notin \mathcal{A}\,\}$$

and the ordinals pinned down by sentences of the $\mathcal{A}$-fragment of $\mathcal{L}_{\infty,\omega}$ (cf. [1]). The notion of pinning down ordinals is a very model theoretic one and should be replaced by more proof theoretic terms. We say that an ordinal α is *provably pinned down* by $\mathcal{A}$ if there is a well-order $\prec$ of

order-type α and a sentence $\mathrm{Wf}(\prec)$ of $\mathcal{L}_{\infty,\omega}$ belonging to $\mathcal{A}$ expressing the well-foundedness of $\prec$ and an infinitary proof of $\mathrm{Wf}(\prec)$ belonging to $\mathcal{A}$. Let

$$h(\mathcal{A}) := \min\{\, \alpha \in \mathsf{On} \mid \alpha \text{ is not provably pinned down by } \mathcal{A} \,\}.$$

Then we have in general

$$o(\mathcal{A}) \leq h(\mathcal{A}).$$

Typically we have $\mathcal{A} = \mathbf{L}_\alpha$ for some countable α, which entails $o(\mathbf{L}_\alpha) = \alpha$, and obtain that an ordinal β is provably pinned down by $\mathbf{L}_\alpha$ if there is a well-ordering $\prec$ of the natural numbers of order type β such that we have $\vdash^{\xi}_{\rho} \mathrm{Fu}(\prec)^{\mathbf{L}_\eta}$ for $\xi, \rho, \eta < \alpha$ where $\mathrm{Fu}(\prec)$ stands for the sentence

$$(\forall u)\{(\forall x \in \omega)[(\forall y \in \omega)(y \prec x \to y \in u) \to x \in u] \to (\forall x \in \omega)[x \in u]\}.$$

It is then more or less obvious that it$\big(\mathrm{Fu}(\prec)^{\mathsf{L}_\eta}\big)$ and the proof tree $\vdash^{\xi}_{\rho} \mathrm{Fu}(\prec)^{\mathbf{L}_\eta}$ (which, in the predicative case, needs neither the collapsible less relation $\ll$ nor the additional reflection rules) belong to $\mathbf{L}_\alpha$. We call an ordinal α *autonomous* if and only if α is provably pinned down by $\mathbf{L}_\alpha$, i.e., if

$$\alpha = o(\mathbf{L}_\alpha) < h(\mathbf{L}_\alpha)$$

(this notion is essentially due to Feferman). If

$$\alpha = h(\mathbf{L}_\alpha)$$

we call α *autonomously closed*.

It is a well-known result of predicative proof theory (e.g., cf. [14]) that

$$\vdash^{\xi}_{\rho} \mathrm{Fu}(\prec)^{\mathsf{L}_\eta} \;\Rightarrow\; \mathrm{otyp}(\prec) \leq \varphi_\rho(\xi) \tag{3.5}$$

where φ_ρ is the hierarchy of Veblen functions, i.e., $\varphi_0 = \lambda\zeta.\omega^\zeta$ and φ_ρ enumerates the common fixed-points of all functions φ_η for $\eta < \rho$ if $\rho > 0$. We put

$$\mathrm{Cr}(\alpha) := \{\, \zeta \mid \varphi_\alpha(\zeta) = \zeta \,\}$$

and call the ordinals in $\mathrm{Cr}(\alpha)$ α-*critical*. The ordinals in $\mathsf{SC} := \{\, \alpha \mid \alpha \in \mathrm{Cr}(\alpha) \,\}$ are *strongly critical*. For $\alpha \in \mathsf{SC}$ and $\xi, \eta < \alpha$ we obtain $\varphi_\xi(\eta) < \varphi_\alpha(0) = \alpha$. It follows then from (3.5) that ω and the ordinals in SC are autonomously closed ordinals and, by a result of Feferman and Schütte, no other ordinal is autonomously closed. This shows that starting from ω we may construct a universe by provably pinning down an ordinal bigger than ω—a good candidate is is $\zeta_1 := \varphi_1(0)$ (cf. [14])—then constructing the universe up to that ordinal and, iterating this procedure, arriving at all stages $< \Gamma_0 := \min \mathsf{SC} = \sup_{i<\omega} \zeta_i$. This shows that $\mathbf{L}_\eta$ is for all $\eta < \Gamma_0$ accessible from below while $\mathbf{L}_{\Gamma_0}$ is not. In that sense $\mathbf{L}_{\Gamma_0}$ is the *Predicative Universe* and Γ_0 is the bounding ordinal for predicativity.

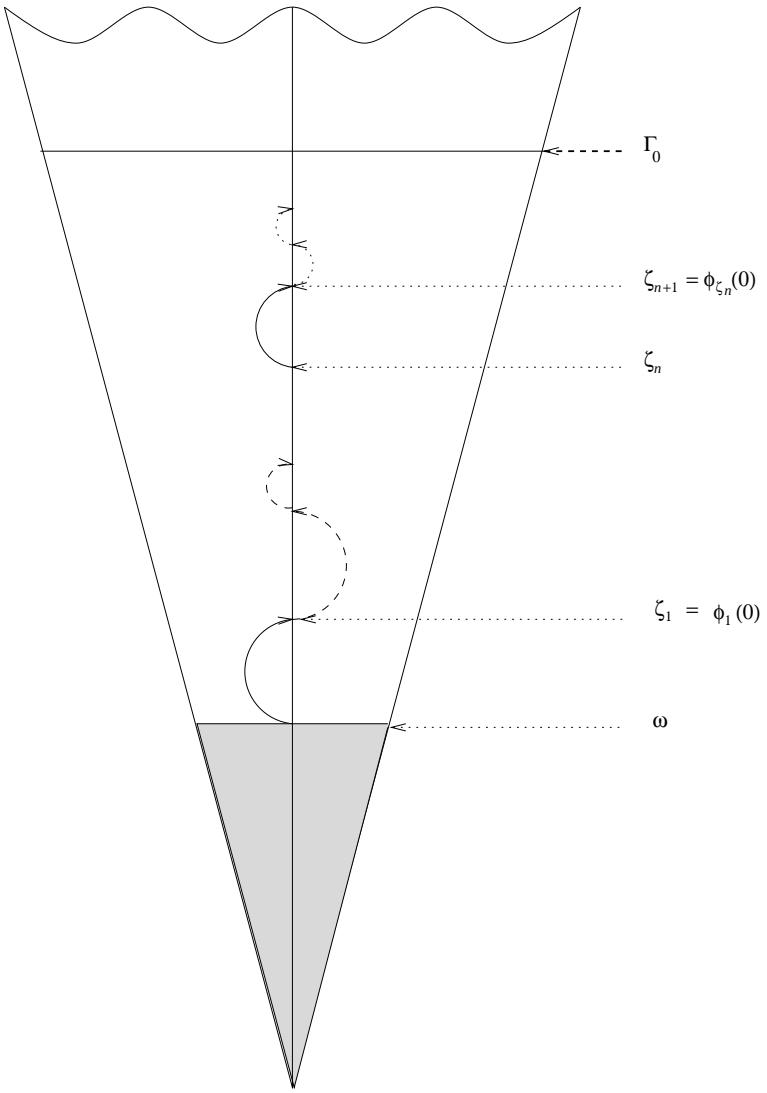

FIGURE 4. The Predicative Universe.

The situation becomes dramatically more involved in the impredicative case. As we have just seen there is no way to step beyond Γ_0 from below. Therefore we have to do something from above. To illustrate the situation we define Γ_0 alternatively. We construct the α-Skolem-hull $\mathrm{Cl}_\alpha(\beta)$ of an ordinal β by the following clauses:

(C0) *Put all members of* $\beta \cup \{0\}$ *into* $\mathrm{Cl}_\alpha(\beta)$;

(C1) *Close* $\mathrm{Cl}_\alpha(\beta)$ *under ordinal addition*;

(C2) *For $\xi, \eta \in \mathrm{Cl}_\alpha(\beta)$ and $\xi < \alpha$ put $\theta_\xi(\eta) \in \mathrm{Cl}_\alpha(\beta)$;*

(C3) *Let θ_α enumerate the class*

$$\mathrm{In}(\alpha) := \{\, \eta \in \mathsf{On} \mid \eta \notin \mathrm{Cl}_\alpha(\eta) \,\}.$$

For obvious reasons we call $\mathrm{In}(\alpha)$ the class of α-inaccessible ordinals. (These notions are due to Feferman.) It is not too hard to see that $\mathrm{Cl}_{\omega_1}(\beta) = \mathrm{Cl}_\alpha(\beta)$ for all $\alpha \geq \omega_1$, that $\mathrm{Cl}_{\omega_1}(0)$ contains exactly a segment of ordertype Γ_0 and that $\theta_\xi(\eta) = \varphi_\xi(\eta)$. If we want to step beyond Γ_0 we need to add a single point, e.g., we replace clause (C0) by

(C0') $\beta \cup \{0, \omega_1\} \subseteq \mathrm{Cl}_\alpha(\beta)$.

It is then still true that

$$\mathrm{In}(\alpha) := \{\, \eta \in \mathsf{On} \mid \eta \notin \mathrm{Cl}_\alpha(\eta) \,\} \cup \{\omega_1\}$$

is closed and unbounded (club) in all regular cardinals. Hence θ_α operates within the infinite number classes, i.e., $\eta < \omega_1$ implies $\theta_\alpha(\eta) < \omega_1$. Therefore $\lambda\alpha.\theta_\alpha(0)$ is again a collapsing function. We have obviously $\mathrm{Cl}_\alpha(\gamma) \subseteq \mathrm{Cl}_\beta(\gamma)$ for $\alpha \leq \beta$ and $\mathrm{Cl}_\alpha(\gamma) \subseteq \mathrm{Cl}_{\omega_2}(\gamma)$ for all ordinals α which shows that the hierarchy $\mathrm{Cl}_\alpha(0)$ collapses at some $\alpha \leq \omega_2$.

The set $\mathrm{Cl}_\alpha(\beta) \cap \omega_1$ is an ordinal and we have especially $\mathrm{Cl}_\alpha(0) \cap \omega_1 = \theta_\alpha(0)$. It follows that $\Gamma_0 = \mathrm{Cl}_{\omega_1}(0) \cap \omega_1 = \theta_{\omega_1}(0)$ and $\Psi(\varepsilon_{\omega_1+1}) = \mathrm{Cl}_{\varepsilon_{\omega_1+1}}(0) \cap \omega_1 = \theta_{\varepsilon_{\omega_1+1}}(0)$. The equation $\Gamma_0 = \theta_{\omega_1}(0)$ shows that Γ_0 already notationally looks like the collapse of ω_1 and it may well be regarded as the proof theoretic collapse of ω_1 in an absolute sense because it is closed under every function f whose totality has an infinite proof belonging to $\mathbf{L}_{\Gamma_0}$.

The ordinals in $\mathrm{Cl}_{\omega_2}(0)$ have notations by Skolem terms using only the symbols 0, ω_1, $+$ and θ and these terms can be made unique. The segment $\mathrm{Cl}_{\omega_2}(0) \cap \omega_2 =: \theta_{\omega_2}(0)$ is thus the largest segment of the ordinals which can be represented by those Skolem terms.

Again we can see that ω_1 can be replaced by practically any ordinal in the interval $\left(\theta_{\omega_2}(0), \omega_1\right]$ (having some quite elementary closure properties) without changing the segment in $\mathrm{Cl}_{\omega_2}(0)$. This shows that ω_1 serves only as a parameter for an ordinal which is safely outside the eventually obtained segment in $\mathrm{Cl}_{\omega_2}(0)$. We can prove that $< \restriction \mathrm{Cl}_{\omega_2}(0)$ is a primitive recursive relation and may thus replace ω_1 by ω_1^{CK}. Therefore we use ω_1 in the sense of ω_1^{CK} below.

The impredicativity of the procedure is—at least prima facie—reflected by the fact that the definition of the segment in $\mathrm{Cl}_{\omega_2}(0)$ depends on the parameter ω_1 which, relative to the set $\mathrm{Cl}_{\omega_2}(0)$, represents the eventually obtained segment.

The ordinals in $\mathrm{Cl}_\alpha(\beta)$ are, roughly speaking, defined by a kind of back and forth procedure. As soon as the closing properties of $+$ are exhausted we produce more ordinals below ω_1 by the collapsing functions $\theta_\xi(\eta)$ (for $\eta < \omega_1$)—this is, for example, the way how $\Gamma_0 = \theta_{\omega_1}(0)$ is put into the set—and then use these ordinals to produce bigger ordinals above ω_1, etc., ... Crucial for an ordinal $\alpha \geq \omega_1$ are therefore the ordinals below ω_1 which are used in its definition. Call these ordinals the ω_1-parts of α, denoted by $P_{\omega_1}(\alpha)$. Then we have that $P_{\omega_1}(\alpha) = \{\alpha\}$ for $\alpha < \omega_1$ and $P_{\omega_1}(\omega_1) = \emptyset$. For additively decomposable ordinals $\alpha =_{\mathrm{NF}} \alpha_1 + \cdots + \alpha_n$ in Cantor normal-form (which is the meaning of $=_{\mathrm{NF}}$) we get

$$P_{\omega_1}(\alpha) := \bigcup \{\, P_{\omega_1}(\alpha_i) \mid 1 \leq i \leq n \,\}$$

and for additively indecomposable ordinals of the form $\alpha = \theta_\xi(\eta) > \omega_1$ we put $P_{\omega_1}(\alpha) := P_{\omega_1}(\xi) \cup P_{\omega_1}(\eta)$.

Now we want to transfer the construction of the predicative universe to the impredicative situation. We know by the result of Feferman and Schütte that we need new parameters. The obvious candidate for such a parameter is ω_1^{CK}. But adding ω_1^{CK} as the set theoretical ordinal would make the universe uncollapsible which is not what we intend. So we try to add ω_1^{CK} as a point without spoiling the collapsibility of the universe. This can be done by introducing the relation

$$\alpha <_p \beta \;:\Longleftrightarrow\; \begin{array}{l} \alpha < \beta \text{ and all ordinals in } P_{\omega_1}(\alpha) \\ \text{can be provably pinned down.} \end{array}$$

We say that an ordinal α can be *virtually pinned down* if a proof of $\mathrm{Wf}(<_p \restriction \alpha)$ belongs to the semi-formal system as introduced in Section 2.3. The term "virtual" is justified because $\mathrm{Wf}(<_p \restriction \alpha)$ does not secure α as a real ordinal but only virtually, i.e., relative to $<_p$, in the event that $\omega_1 \leq \alpha$. Then we have by definition that ω_1 can be virtually pinned down.

Relative to the relation $<_p$ we have—vaguely speaking—the same possibilities to put ordinals above ω_1 into $\mathrm{Cl}_{\omega_2}(0)$ as we had in the construction of $\mathrm{Cl}_{\omega_1}(0)$ before we added the additional parameter ω_1. (This is because we have no possibility to collapse ordinals $\geq \omega_2$ into the set.) This vague analogy can be made precise by showing that all ordinals in $\mathrm{Cl}_{\omega_2}(0) \setminus \omega_1$ can be virtually pinned down. This is obtained by proving that the virtually pinned down ordinals are closed under $+$ and θ in the same way as it is done for the provably pinned down ordinals in the predicative case. The crucial point is then the *Condensation Property* telling that if $\alpha \geq \omega_1$ can be virtually pinned down then its collapse below ω_1 (which regularly is the ordinal $\theta_\alpha(0)$) can be provably pinned down. So all ordinals in $\mathrm{Cl}_{\omega_2}(0) \cap \omega_1$ can be provably pinned down—and are thus real ordinals—and we obtain a universe as shown in Figure 5. Observe that the ordinal $\theta_{\omega_2}(\omega_1 + 1)$ shown there is

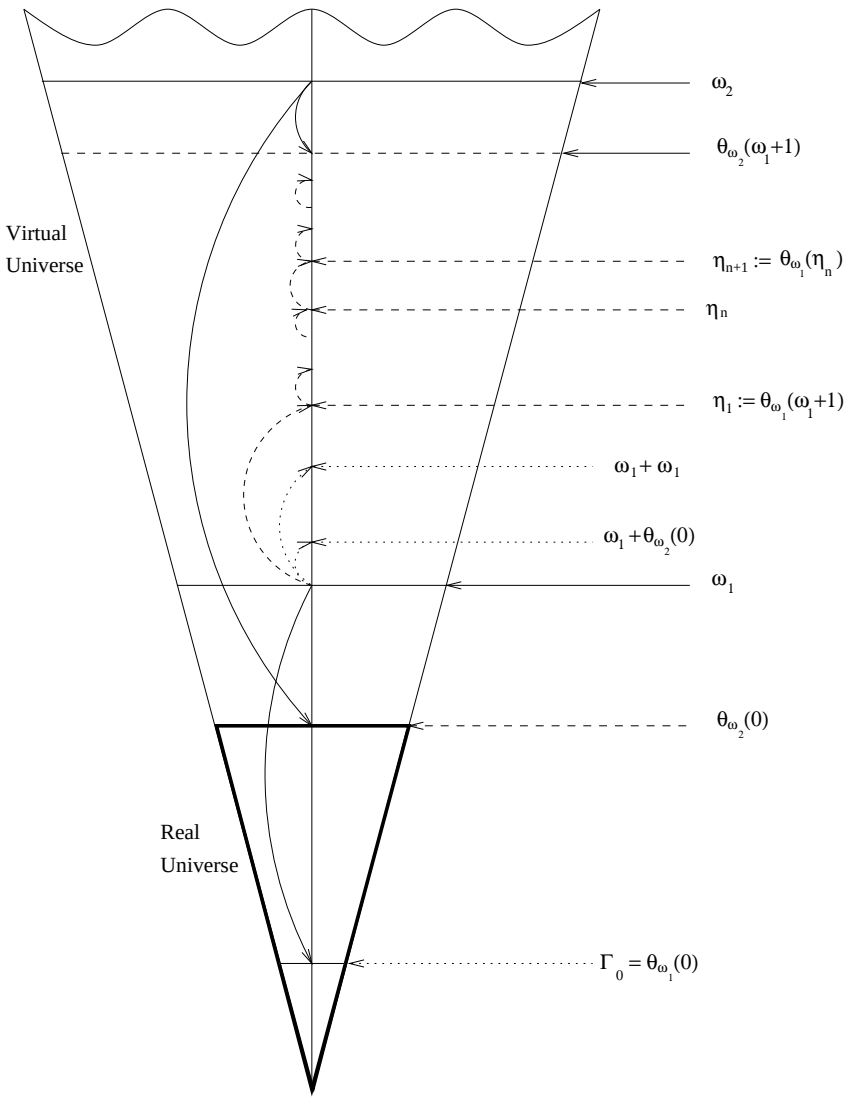

FIGURE 5. Extending the Predicative Universe.

the ordinal Γ in Figure 1. The ordinals η_i in the figure form a fundamental sequence for $\theta_{\omega_2}(\omega_1 + 1)$, i.e., we have

$$\theta_{\omega_2}(\omega_1 + 1) = \sup_{i \in \omega} \eta_i$$

(which shows that the function $i \mapsto \eta_i$ cannot belong to the universe). There are good reasons to regard $\theta_{\omega_2}(0)$ as the proof theoretic collapse of ω_2 in an absolute sense.

This example of a construction of an impredicative universe is admittedly far less canonical than that of the predicative universe. Instead of working within the constructible universe we have to work in the semi-formal system which in turn depends on the development of the ordinal notations (and especially on the collapsing functions provided by the notation system). Also the relation $<_p$ relies on the notation of the ordinals. We are still far from knowing the canonical construction. There are hints—already faintly visible in the above example—that developing the ordinal notations on the basis of admissible ordinals instead of cardinals will play a certain role in a canonical construction. The interplay between the construction of the sets $\mathrm{Cl}_\alpha(\beta)$ and the "provable sets" of the universe will probably be of crucial importance. But until today we do not really know what it means that a set is provable. Though obvious for sets in the real part of the universe it is still mysterious for the sets in the virtual part. This is because they are not transitive in the classical sense. The missing transitivity will probably have to be replaced by certain uniformity constraints but it is—at least to me—still unclear how this has to be done. (A variant of the classical question "What is a set?"). There seem to be some connections to "E-recursion" theory. Work in these directions has been done by A. Schlüter [21], [22] and M. Rathjen [19].

Moreover, the example shows only the lowest degree of impredicativity. In it we stressed much more the "back" direction (i.e., collapsing ordinals from above) and not so much the "forth" direction (i.e., defining larger ordinals from below). A higher degree of impredicativity is obtained as soon as we allow a function with virtual domain enumerating the admissible ordinals in the universe (thus stressing also the "forth" direction). The weakest axiom system which requires such notations is Feferman's theory $\mathsf{ID}_{\prec^*}$ which allows us to iterate inductive definitions along the accessible part (a virtual thing) of a primitive recursive order relation $\prec$. An even higher dimension of impredicativity is needed in the analysis of axiom systems like KPi and KPM let alone Π_3-Ref or even Π^1_2-comprehension.

At any rate we believe that exploring the "inner construction" of the (provable parts of the) mathematical universe should be the main focus of pure proof theory. H. Friedman, S. Simpson and their followers in their program of *Reverse Mathematics* have shown that practically all everyday mathematics can be formalized within quite weak systems nearly never exceeding Π^1_1-comprehension. So the interest in stronger systems is not so much their mathematical power but the principles which are implicitly stated in their axioms. They become explicit in their ordinal analyses and may thus be set in correspondence to constructively more perspicuous principles.

On the other hand, one should not forget that ordinal analysis of a sufficiently strong axiom system Ax provides us with a primitive recursively definable order relation $\prec$ on the natural numbers such that for every $n \in \mathrm{field}(\prec)$

there is an ordinal $\alpha_n < \|\mathsf{Ax}\|^{\mathrm{CK}}_{\Sigma^1_1(\omega)}$ with

$$\mathsf{Ax} \vdash (\exists f)[f : \text{field}(\prec \restriction n) \to \alpha_n \text{ and } f \text{ is order preserving}]$$

but there is no ordinal α for which we have

$$\mathsf{Ax} \vdash (\exists f)[f : \text{field}(\prec) \to \alpha \text{ and } f \text{ is order preserving}].$$

This can be viewed as a combinatorial principle—which for many systems can be turned into a form of the Hydra game—independent from Ax. These principles have miniaturizations of the form that there is a Π^0_2-sentence $(\forall x \in \omega)(\exists y \in \omega)F(x, y)$ such that

$$\mathsf{Ax} \vdash (\exists y \in \omega)F(\underline{n}, y)$$

for all $n \in \omega$ but

$$\mathsf{Ax} \nvdash (\forall x \in \omega)(\exists y \in \omega)F(x, y)$$

(which essentially means that there are recursive functions whose totality is unprovable in Ax). Thus ordinal analysis may help to detect strong combinatorial principles which possibly might also be of interest outside Proof Theory or even Mathematical Logic.

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A more comprehensive list of references can be found in [14].

INSTITUTE FOR MATHEMATICAL LOGIC AND FOUNDATIONAL RESEARCH
UNIVERSITY OF MÜNSTER
GERMANY

E-mail: pohlers@math.uni-muenster.de