I. Kostant partition function and flow polytopes

\(K_{A_n}(a_1, \ldots, a_{n+1})\) number of ways of writing \((a_1, \ldots, a_{n+1})\) as an \(\mathbb{N}\)-combination of positive roots \(e_i - e_j, i < j\).

\(K_{A_3}(1, 0, 0, -1) = 4:\)

\[(1, 0, 0, -1) = e_1 - e_4\]
\[= (e_1 - e_3) + (e_3 - e_4)\]
\[= (e_1 - e_2) + (e_2 - e_4)\]
\[= (e_1 - e_2) + (e_2 - e_3) + (e_3 - e_4)\]

- Kostant (1958) used them to give formulas for weight multiplicities of irreducible representations of semisimple Lie algebras. Lusztig (1983) studied a \(q\)-analogue.

Combinatorial approach (Baldoni-Vergne 2001)

View ways counted in \(K_{A_n}(\cdot)\) as lattice points of a flow polytope \(F_G(a_1, \ldots, a_{n+1})\)

\[P_n := F_G(1, 0, \ldots, 0, -1)\]
\[\text{volume}(P_n) = K_G(1, 2, 3, \ldots) = C_1 C_2 \cdots C_{n-2}\]
\[C_i = \frac{1}{i+1} \binom{2i}{i}\]

volume equals \# lattice points similar to permutahedra

Zeilberger 1999 by identity related to Selberg integral

Results

- [1] Extend from Lie type \(A\) to Lie types \(B, C, D\)

- [2][3] Connection to space of diagonal harmonics \(DH_n\), shuffle conjecture
  new polytope \(Q_n := F_G(1, 1, \ldots, -n)\), \(n!\) vertices, \(\text{volume}(Q_n) = \frac{1}{\prod_{i=1}^n (2i-1)^{n-i}} C_1 \cdots C_{n-1}\)

- [4] \(F_G(1, 0, \ldots, -1)\) when \(G\) is planar \(F_G \equiv \text{order polytope}\) of poset from dual of \(G\)
  Corollary: certain Kostant partition functions count linear extensions of posets.

- [5] lattice/Ehrhart theory flow polytopes parallel to generalized permutahedra

References

II. Hook formulas for skew shapes

The irreducible representations of the symmetric group $S_n$ are indexed by partitions $\lambda$ of $n$.

The dimension $f_\lambda$ of the irreducible representation counts standard tableaux: fillings of the diagram of $\lambda$ with all entries $1, 2, \ldots, n$ increasing in rows and columns (Young 1900)

$$f^{(3,2)} = 5$$

$\lambda = (3, 2)$

$f_\lambda$ has a product formula: the hook-length formula (Frame-Robinson-Thrall 1954)

$$f_\lambda = \frac{n!}{\prod_{(i,j) \in \lambda} \text{hook}(i, j)}$$

Applications:

- Greene-Nijenhuis-Wilf 79
- Novelli-Pak-Stoyanovski 97

The dimension of irreducible calibrated representations of the Hecke algebra counts standard tableaux of skew shape: fillings of $\lambda/\mu$ with all entries $1, 2, \ldots, n$ increasing in rows and columns (Ram 2004)

$$f^{(3,2)/(1)} = 5$$

$\lambda/\mu = (3, 2)/(1)$

$f^{\lambda/\mu}$ has no product formula

Naruse announced formula for $f^{\lambda/\mu}$ as a positive sum of products using equivariant Schubert calculus.

$$f^{\lambda/\mu} = n! \sum_D \prod_{(i,j) \in D} \frac{1}{\text{hook}(i,j)}$$

Results

- two $q$-analogues of Naruse formula [1]
- elementary proof of Naruse formula using Gessel-Viennot theory [2]
- asymptotics for $f^{\lambda/\mu}$ [3]
- new family of skew shapes with product formulas for $f^{\lambda/\mu}$ [4]

References


†, ‡ images made using from D. Romik MacTableaux.
III. $q$-analogue of placements of non-attacking rooks

In complexity theory computing the permanent of a (0-1) matrix is hard ($\#P$-complete) (Valiant 79)

$$\text{perm}(A) = \sum_{w \in S_n} A_{w_1} \cdots A_{w_n}$$

For an $n \times n$ 0-1 matrix $A$, $\text{perm}(A) = \# \text{ placements of } n \text{ non-attacking rooks on support of } A.$

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{pmatrix}
\]

some nice cases:

$$\text{perm}(A) = \prod_i (\lambda_i - i + 1)$$

\[d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n - i)!\]

Results

**new $q$-analogue of rook placements** [1]: for $S \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$

$$M_q(S) := \# \text{ invertible matrices } A \text{ entries in finite field } \mathbb{F}_q \text{ support in } S \subseteq \text{GL}_n(\mathbb{F}_q)$$

- **Theorem**: enumerative $q$-analogue of rook placements:
  \[\text{i.e. } \lim_{q \rightarrow 1} M_q(S) = \# \text{ rook placements in } S\]

- not always polynomial in $q$, can be Polynomial On Residue Classes (Stembridge 98)

- polynomial in $q$ when support is diagram of partition (Haglund 97)
  and when support is a skew diagram [2]

- **Theorem**: polynomial in $q$ when zeros are on inversions of any permutation [4]
  proved using coding theory (settling conjecture from [2])

- not known if there are non PORC examples

References

IV. Colored factorizations in $S_n$ and $\text{GL}_n(\mathbb{F}_q)$

Structure constants of the group algebra of the symmetric group count factorizations

$$F_{\lambda,\mu} = \# \{(\pi_1, \pi_2) \mid \pi_1, \pi_2 \in S_n, \text{ cycle type } \pi_1 = \lambda, \text{ cycle type } \pi_2 = \mu, \pi_1 \pi_2 = (1 \ldots n)\}$$

In the generating function of $F_{\lambda/\mu}$ do a change of basis like $x^r \rightarrow x(x-1) \cdots (x-r+1)$ and obtain new coefficients $C_{\alpha,\beta}$.

$C_{\alpha,\mu}$ count factorizations with colored cycles, usually they have nicer formulas.

**Results** (symmetric group $S_n$)

- two factors [1]

$$F_{\lambda,\mu} \quad \xrightarrow{\text{change of basis}} \quad C_{\alpha,\beta} = \frac{n(n - \ell(\alpha))!(n - \ell(\beta))!}{(n + 1 - \ell(\alpha) - \ell(\beta))!}$$

- $k$ factors (Jackson 88, bijective proof [2])

$$F_{\lambda^{(1)}, \ldots, \lambda^{(k)}} \quad \xrightarrow{\text{change of basis}} \quad C_{\alpha^{(1)}, \ldots, \alpha^{(k)}} = n!^{k-1} \cdot \frac{S^{n-1}_{\ell_1-1, \ldots, \ell_k-1}}{\prod_{i=1}^{k} \ell_i} \cdot \prod_{i=1}^{k} (n-\ell_i)$$

where $\ell_i = \ell(\alpha^{(i)})$, $S_{r_1, \ldots, r_k}^n = \# \{(S_1, \ldots, S_n) \mid S_i \subseteq [k], r_j \text{ sets } S_i \text{ contain } j\}$.

In $\text{GL}_n(\mathbb{F}_q)$ (Lewis-Reiner-Stanton 2013; Huang-Lewis-Reiner 2015):

- analogue of long cycle $(1, 2, \ldots, n)$ $\rightarrow$ Singer cycle

- analogue of number of cycles of $\pi$ $\rightarrow$ fixed space dimension of matrix

- number of factorizations Singer cycle in $\text{GL}_n(\mathbb{F}_q)$ into $n$ reflections is $(q^n - 1)^n$

**Results** [3]

$$F_{r,s}(n, q) = \# \{(A, B) \mid A, B \in \text{GL}_n(\mathbb{F}_q), A \cdot B = \text{fixed Singer cycle}, \text{ dimension fixed space } A, B = r, s\}$$

$$F_{r,s}(n, q) \quad \xrightarrow{\text{change of basis}} \quad C_{r,s}(n, q) = \frac{(q^{n-r-s}q^{r-s}q^s+1)(n-r-1)!(n-s-1)!}{(q-1)(n-1)!}$$

**References**