Electrical Circuits and Random Walks: Challenge Problems

LA Math Circle, High School II[*]

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1 Effective Resistance

The most efficient way of calculating electrical properties of a circuit is actually not just setting up an equation for each vertex and solving the system. Instead, we try to simplify the problem, modifying the graph in a way that doesn't change our final answer.

Consider a graph interpreted as an electrical circuit, with a source vertex and a ground vertex. Imagine that the entire circuit between these two vertices (except for the battery and the ground) is stuffed into an obfuscating box, so that all we see is the battery attached to two vertices, each attached to this box containing an unknown circuit:
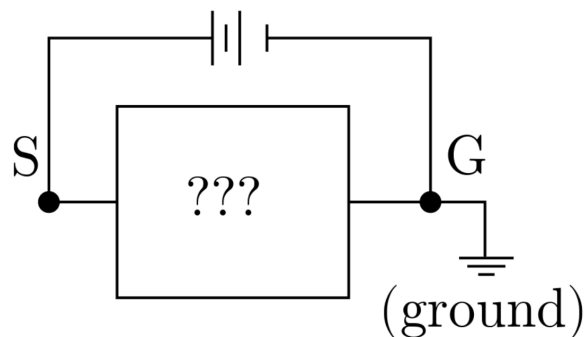
What information do we know about this box? We can still measure the current flowing out of the source, $\sum_{x \in N(S)} i_{Sx}$, or the amount of current flowing into the ground, $\sum_{x \in N(G)} i_{xG}$. Physically, these quantities should be equal, as they both describe the amount of current flowing into the battery on one side and out the other, and if they are not equal, charge would be building up in the battery or being depleted over time. However, we did not assume that currents coming in and out of the battery are equal in our graph-theoretic model, so we should still check that this is true.

Problem 1 Using Kirchoff's Law (which we only assumed holds at vertices other than S and G), prove that $\sum_{x \in N(S)} i_{Sx} = \sum_{x \in N(G)} i_{xG}$.

If we call this current $i_S = \sum_{x \in N(S)} i_{Sx}$, we can now pretend that the box contains just one big resistor, with current i_A flowing through it. This gives us enough information to calculate its resistance. Ohm's Law tells us that its resistance would have to be given by $\frac{V(S) - V(G)}{i_S}$, which we will call R_{eff}, the *effective* resistance.
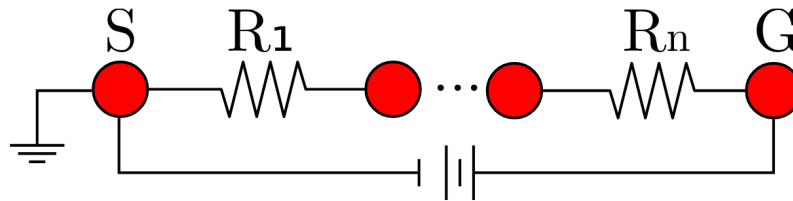

Problem 2 Using Ohm's Law and Kirchoff's Law, calculate the effective resistance R_{eff} of these two circuits:

(a) Resistors in *parallel*
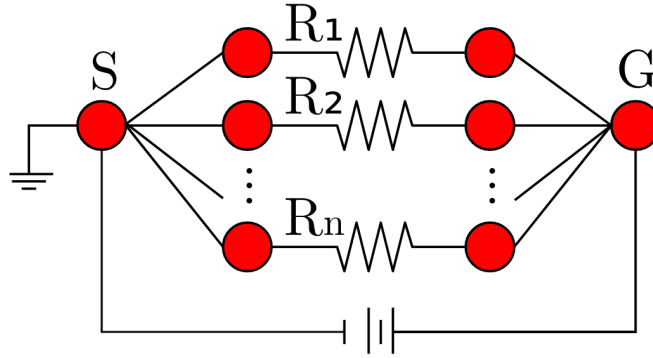
(b) Resistors in *series*

Now that we know how to calculate the effective resistance of certain circuits, we can use these effective resistances to simplify other complicated circuits. Whenever we have resistors in parallel or series, that is, a portion of the circuit resembling one of these diagrams, we can replace this portion with a single resistor, with resistance determined by the effective resistance of the piece of circuit it's replacing.

Problem 3 Consider a circuit including the star-shaped piece on the left:
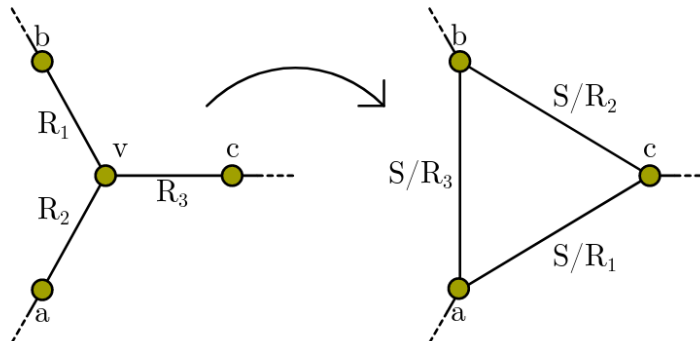
Show that replacing that piece with the triangle-shaped piece on the right does not change the effective resistance of the circuit if $S = R_1 R_2 + R_1 R_3 + R_2 R_3$.

Problem 4 Suppose that we make a circuit by connecting the 2^n vertices of an n-dimensional cube with 1Ω resistors at the edges of the cube. If we place the ground and source at opposite vertices of the cube, what is the effective resistance of this circuit?

2   Alternate Random Walk Problem and Interpretations of Current

Let H be a graph, with designated source vertex S, ground vertex G and edge weights $w_{xy} \in (0, +\infty)$. Let's consider the same type of random walk as before, except we start at S, and instead of stopping if we ever reach S, we keep going, picking a neighbor to walk to according to edge weights like with the vertices that aren't S and G. We still stop if while walking randomly we reach G. Now define a function $u : V(H) \to [0, +\infty)$ by letting $u(x)$ be the average number of times we walk through vertex x before reaching G.

Let's now make this random walk problem into a circuit, once again letting the conductance of an edge be its weight ($C_{xy} = w_{xy}$ or $R_{xy} = \frac{1}{w_{xy}}$). Recall the definition $C_x = \sum_{y \in N(x)} C_{xy}$. Then ground G so that $V(G) = 0$ and place a voltage difference of $\frac{u(S)}{C_S}$ between G and S such that $V(S) = \frac{u(S)}{C_S}$.

Problem 5 Show that in this circuit, for any vertex x, $V(x) = \frac{u(x)}{C_x}$.

Problem 6 In this type of random walk problem, current also takes on an interesting meaning. Show that if $\{x, y\}$ is an edge of our graph, i_{xy} is the average number of times the walker walks along this edge from x to y minus the number of times they walk along the edge from y to x.

2.1   Understanding i_S

Let's say we modify this problem, so that the voltage at S is 1. This will simply scale down all voltages by a factor of $\frac{u(S)}{C_S}$. Then we can find another algebraic expression for the current i_S coming out of the source, which will help us explain the meaning of i_S in random walk problems:

Problem 7 Prove using Ohm's Law and Kirchoff's Law that

$$i_S = C_S \left(v(S) - \sum_{y \in N(S)} v(y) \frac{C_{Sy}}{C_S} \right)$$

This may not look helpful at first, but we at least have a random walk interpretation for each of these terms.

Problem 8 Show that the expression

$$\sum_{y \in N(S)} v(y) \frac{C_{Sy}}{C_S}$$

appearing in our equation for i_S above is the probability that a walker starting at S returns to S before reaching G. Deduce that the probability of reaching G before returning to S is $\frac{1}{R_{\text{eff}} C_S}$. We call this probability p_{esc}, the probability of escape.

3 Escaping to Infinity: Pólya's Random Walk Problem

Suppose you place a random walker at the origin of a d-dimensional integer lattice $\mathbb{Z}^d$, and let them wander. What is the probability that they eventually return to the origin?

In general, this is a hard problem, but we can start simple by considering the one-dimensional case. The walker sits at 0 on a number line and steps to the left or the right with equal probability, until they eventually return to 0, or wander forever without returning. We'd like to use the tools we've developed so far, but these have all assumed our graphs were finite, but this one isn't.

We can, however, answer a related problem with finite graphs. Given the same random walk, on the same number line graph, what is the probability that the walker returns to the origin before wandering out to distance r from the origin? This is almost a question we can answer with our previous framework, as we can build a finite graph out of the finitely many vertices which are distance at most r from the origin. We start a random walk from the origin, and see whether we get back to the origin first, or to one of the distance r vertices first. However, in our previous problems, we only asked about the relative probabilities of arriving at *two* vertices, the source and the ground. We can put this problem in those terms by gluing (or in the case of an electrical circuit, it'd be more like soldering) all the vertices at distance r together into one vertex called G, grounding this one, and placing the source S at the origin.

Problem 9 For a natural number r, let H_r be the graph consisting of vertices $v_{-r+1}, v_{-r+2}, \ldots, v_{r-2}, v_{r-1}$ and G. We let $S = v_0$, and join v_i and v_{i+1} with an edge for each number $-r+1 \leq i < r-1$, and then join G to both v_{-r+1} and v_{r-1}. This is the same graph we get from taking all vertices of the integer lattice at distance at most r from the origin, and gluing together all the vertices at distance r into one vertex G.

Viewing H_r as a circuit with source S and vertex G and each edge having resistance 1Ω, what is its effective resistance R_{eff}? Given this, what is the chance p_r of reaching G before returning to the origin S?

Problem 10 When we do a random walk starting from the origin in $\mathbb{Z}$, in order to walk off to infinity without returning to the origin, we need to hit something at distance r before returning to the origin for any choice of r. Thus the probability of not returning to the origin in $\mathbb{Z}$ is at most what it is in H_r for any r, that is, p_r. From this information, can you determine the probability of never returning to the origin while walking in $\mathbb{Z}$?

4 More Dimensions

To tackle the 2-dimensional case, we need two more tricks in order to compute effective resistances, which we can then use to compute escape probabilities of pieces of the 2-dimensional lattice.

4.1 Identifying Vertices Together

Problem 11 Let x, y be two vertices of a circuit that are not connected by an edge, such that $V(x) = V(y)$. Show that if we add an edge/resistor with resistance R_{xy} between them, then the flow of current, and thus all voltages and the effective resistance of the whole circuit, is not changed. Start by showing that in this new modified circuit where there is a resistor of resistance R_{xy} between them, the current i_{xy} is 0, and then deduce that all other currents must be the same as in the original circuit.

When we set $R_{xy} = 0$, this is basically the same thing as fusing the vertices x and y together, as current is allowed to flow freely between them. Thus, based on what we've just proven, fusing together two vertices of the same voltage does not change the effective resistance of a circuit.

Problem 12 Use this technique to more quickly solve for the effective resistance of the n-dimensional cube, gluing together vertices that by symmetry must have the same voltage.

4.2 Rayleigh's Theorem

Theorem 4.1. *If we increase the resistance of a particular edge, then this only increases the effective resistance of a circuit. For this purpose a lack of an edge at all between two vertices is equivalent to connecting them with a resistor with infinite resistance, or 0 conductance, so deleting an edge always increases the resistance of that edge, increasing the effective resistance of the circuit.*

Similarly, decreasing the resistance of a particular edge, or adding a new edge, or even gluing two vertices together, will only decrease the effective resistance.

We will not prove this circuit here, but in terms of electricity, it makes some intuitive sense. However, in the case of random walks, it tells us something more profound: deleting edges, by increasing the effective resistance, only decreases the escape probability, while adding edges or gluing together only increases the escape probability.

Problem 13 Now we can think about the 2-dimensional integer lattice $\mathbb{Z}^2$. For every natural number r, let V_r be the collection of vertices v such that the shortest path through the graph from v to the origin $(0,0)$ is of length r. Then, we create a circuit/graph H_r by looking at the vertex set $V_0 \cup V_1 \cup \cdots \cup V_r$, the set of all vertices at (path-length) distance at most r from the origin. Then we glue/solder all of V_r together, to make one vertex r, ground this vertex, and make the origin the source.

(a) Using symmetries, addition of edges, and even gluing of vertices, show that H_r can be simplified into a circuit H_r' that looks like this:
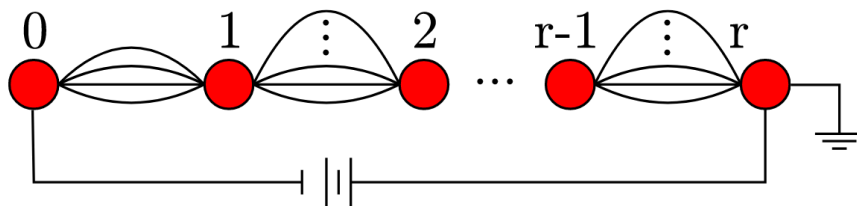
Does this process raise or lower the effective resistance?

(b) Find out how many edges should be between each pair of vertices i and $i+1$ in H_r'.

(c) Using the parallel and series laws for effective resistance, compute the effective resistance, and thus escape probability for H_r', and from this, get an upper or lower bound on the escape probability p_r of H_r.

(d) Given this information about the escape probabilities p_r, can you find the probability of returning to the origin in a random walk on the lattice $\mathbb{Z}^2$?