A Brief Summary of My Researches

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My research interest is to prove various conjectures from physics by using mathematical techniques with simple and clean ideas. Localization techniques in various settings, combining with various subjects of mathematics, have played central roles in my researches. See my ICM lecture [38] for more detailed discussions. My first favorite work is on the applications of modular invariance, the infinite dimensional symmetry in Kac-Moody algebras and mathematical physics, to geometry and topology. From modular invariance we derived several quite general rigidity, divisibility and vanishing results about certain topological invariants of a manifold. The second most interesting work of mine is to use explicit expressions of heat kernels on Lie groups to study the geometry and topology of moduli spaces of flat connections on Riemann surfaces, this proves certain conjectural formulas of Witten and gives several new results including general vanishing formulas of intersection numbers. The third most interesting work of mine is the understanding of mirror symmetry and counting curves on projective manifolds. My recent work, joint with Lian and Yau, solves some very general conjectures about the relationship between the numbers of algebraic curves in projective manifolds and hypergeometric series.

I have also solved some long-standing open problem about algebraic surfaces by using Weil-Peterson metric and Quillen metric; proved certain localization formulas for non-compact group action; proved a Shafarevich conjecture for odd dimensional Calabi-Yau manifolds; and derived various topological and geometrical results on foliated manifolds. See the following for details. The references are listed in the bibliography of my resume.

1 Elliptic Genus

1.1 $\hat{A}$-vanishing theorem for loop spaces

This is a loop space analogue of the Atiyah-Hirzebruch $\hat{A}$-vanishing theorem for group actions and the loop space $\hat{A}$-genus, or the Witten genus. An analogue of the Lawson-Yau's vanishing theorem for non-abelian group action is also derived.The proof involves index theory and certain subtle properties of the Jacobi theta-functions. See [3]

1.2 General vanishing theorems associated to loop groups

These theorems give us new obstructions for group actions on manifolds. See [3]. Their proofs uses the modular invariance of the characters of Kac-Moody algebras in a substantial way.

1.3 General rigidity theorem associated to loop groups

This theorem generalizes the famous Witten rigidity conjectures as proved by Taubes, Bott-Taubes, Hirzebruch, Krichever, Landweber-Stong and Ochanine. Here is the first time that Kac-Weyl character formula came into geometry and topology. See [3].

1.4 General miraculous cancellation formula

By using modular forms I generalized a 12-dimensional formula of Alveraz-Gaume and Witten, which they called the miraculous cancellation formula, to arbitrary dimensions and general vector bundles. Joint with Zhang, I found relations between elliptic genus and other geometric invariants, such as holonomy, the APS eta-invariants and the Rochlin invariants. See [4].

1.5 Mod 2 elliptic genus

I have studied mod 2 elliptic genera and answered a question raised by Witten about mod 2 partition functions as mod 2 modular forms in quantum field theory. See [1].

1.6 Elliptic cohomology

In [7] I described an approach to the geometric construction of elliptic cohomology by using the K-group of infinite dimensional vector bundles. A Riemann-Roch type theorem for such K-group was proved.

1.7 Family rigidity and vanishing theorems

Recently in joint works with Xiaonan Ma and Weiping Zhang, we proved several rigidity and vanishing theorems for the family indices of elliptic operators. Applications of these results to fundamental groups are expected. See papers [26]-[30]

1.8 Elliptic genus and foliation

Joint with Ma and Zhang, we proved certain general rigidity and vanishing theorems of elliptic genus for foliated manifolds. To proceed, we constructed new elliptic operators associated to the foliation. See [33].

The proofs of the above results can be considered as combinations of index theory with modular invariance.

1.9 Elliptic genus and vertex operator algebras

Recently with Dong and Ma we are able to prove a rigidity theorem for vertex operator algebra bundles [37]. This strongly indicates that the geometric construction of elliptic cohomlogy is related vertex operator algebras.

2 Heat Kernel and Moduli Spaces

2.1 Intersection numbers on moduli spaces

By using explicit formulas of heat kernels on Lie groups, the Reidemeister torsion and symplectic geometry, I derived very general formulas for the intersection numbers on the moduli spaces of flat connections over a Riemann surface. As consequences several conjectural formulas derived by Witten by using path-integral method are proved. New vanishing formulas for the intersection numbers are derived by using the new method. In principle these formulas contain all the information needed for the Verlinde formula. Such

moduli spaces have been studied from many different point of views in the past 50 years. People have used geometric invariant theory, gauge theory and loop groups. My results are for general compact semi-simple Lie groups. Our method also generalizes to the cases when the moduli spaces are singular, as well as when the Riemann surface has several boundary components. See [14] and [15].

2.2 Vanishing theorems on moduli spaces

As another consequence our method gives several very general new vanishing theorems about the characteristic numbers of the moduli spaces, which actually follows from the delta function property of heat kernels. Some partial results in this direction for $G = SU(n)$ were previously obtained by Atiyah-Bott and Witten in their well-known papers. See [14], [15].

2.3 Compact Lie groups and finite groups

Our heat kernel method gives a systemic and very simple way to find explicit formulas for the numbers of solutions of equations in finite groups. The same technique applies to the derivation of the push-forward measures for commutator maps between compact Lie groups. See [25].

This heat kernel method can be viewed as a natural extension of the heat kernel proof of the Atiyah-Singer index formula, as proposed by Mckean-Singer. It turns out that Witten's nonabelian localization formula can also be seen naturally from the heat kernel point view.

3 Mirror Principle

3.1 Euler data and the Candelas formula

In my joint works with Lian and Yau, we introduced the general notion of Euler data. These are sequences of equivariant cohomology classes in the linear sigma models, the simplest compactifications of the moduli spaces of holomorphic maps from curves into certain projective manifolds with symmetry. We study in detail the algebraic and geometric properties of the Euler data and their properties under mirror transformations. A key technique involved is the Atiyah-Bott equivariant localization formula. Many examples

of Euler data arise from the stable compactifications of the above mentioned moduli spaces. We have a conceptual understanding of the mirror symmetry to compute the characteristic classes on the stable map moduli in terms of hypergeometric series. As one of the corollaries we gave a complete proof of the mirror conjecture which relates the counting series of rational curves in a Calabi-Yau quintic manifold to the hypergeometric series of its mirror, as proposed by Candelas and his collaborators.

3.2 General mirror principle

Our method works for computing general characteristic classes on the moduli spaces of stable maps into projective balloon manifolds, which include toric manifolds and homogeneous manifolds, and many of their submanifolds. So far almost all of the formulas as conjectured by string theorists for counting rational curves can be derived from mirror principle.

3.3 Local mirror symmetry

Particularly interesting is our proof of the local mirror symmetry, which was called geometric engineering by physicists. Seiberg-Witten curves always show up in our computations, as predicted by string theory.

3.4 Counting higher genus curves

Now we are in the process of generalizing our methods to count higher genus curves. Most part of our theory works well for higher genus curves. Some new ideas are needed to prove certain general conjectures from string theory.

See [17], [18], [19], [20], [23] and [31] for the details of the above works. Note that we solved these problems with the classical local to global principles in geometry, as developed in the sixties by Atiyah, Bott and Singer.

4 Other Works

4.1 Equivariant cohomology

In [13] I derived a localization formula which relates different fixed point components and used it to study the moduli spaces of flat connections on

a Riemann surface, the Verlinde formula and non-abelian localizations in symplectic geometry.

In [9] I studied equivariant cohomology for non-compact holomorphic or meromorphic group actions on complex manifolds. Several localization and the Duistermaat-Heckman type formulas were obtained.

In [32], we use the Atiyah-Bott-Singer fixed point formula to derive the generating series of the Hirzebruch χ_y-genus for the Hilbert schemes of any complex surface. We are now trying to compute the elliptic genera of Hilbert schemes with this method which will prove a conjecture of string theorists.

4.2 Geometric height inequalities

By using Yau's Schwarz lemma and the Weil-Peterson geometry of the moduli spaces of curves, I obtained quite general geometric height inequalities, the simplest of which implies the functional field analogue of the Mordell conjecture. One of the results was used to solve a conjecture of Beauville about the minimal number of singular fibers for stable fibration over CP^1, and another long-standing conjecture about the strict Chern number inequality of general Kodaira surfaces, which, combining with a result of Yau, implies that the Kodaira surfaces can not be covered by a ball. See [8], [16].

4.3 Foliation

This is joint work with Zhang. By using adiabatic limit method and index theory, we studied foliations and obtained some vanishing theorems for the characteristic numbers of foliations. Particularly interesting is that we derived the Bott connection as the adiabatic limit of a Riemannian connection. See [12].

4.4 Shafarevich conjecture on Calabi-Yau

Joint with Todorov, Yau and Zuo we prove the Shafarevich conjecture about the finiteness of certain families of odd dimensional Calabi-Yau manifolds. See [21].

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  Department of Mathematics, MIT, Cambridge, Massachusetts.

  C. L. E. Moore Instructor, June 1993-July 1996

  Department of Mathematics, Harvard University, Cambridge, Massachusetts .

  Research and Teaching Assistant, September 1988-June 1993.

- **SERVICE**

  Communications in Analysis and Geometry, Editor-in-Chief.

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  * Guggenheim Fellowship, 2002.

* ICM Invited Speaker, 2002, Mathematical Physics.

* ICCM Invited Speaker, 2001, Plenary Speaker.

* ICCM Award, 1998.

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* Jian-Qing Zhong Award for the Best Young Mathematicians in China, 1988.

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- **TALKS**

  * Invited Speaker at ICM 2002; at the Shanghai International Algebraic Geometry Conference 2002; at ICCM 2001; at the Wisconsin Orbifold String Theory Conference 2001; at the Southwest Topology Conference 2001.

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  * Invited Speaker in Geometry, Topology, Gauge Theory, Infinite Dimensional Lie Algebras and Mathematical Physics seminars of University of California at Irvine, at Santa Cruz, at Davis, at Berkeley, at Los Angeles and at San Diego; at Cornell University, Harvard University, Columbia University, Rutgers University, Stanford University, Yale

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