Homework 8

Exercises (1)–(4) deal with the *Cauchy integral formula.* A complex-valued differential form on a complex manifold M is a sum $\omega = \omega_1 + i\omega_2$, where ω_1 and ω_1 are (real) differential form. The wedge product extends in the obvious way, and

$$d\omega := d\omega_1 + i d\omega_2, \quad \int_M \omega := \int_M \omega_1 + i \int_M \omega_2.$$

Note that Stokes' Theorem is valid for complex-valued forms since it is valid separately for the real part and the complex part. On $\mathbb{C} = \mathbb{R}^2$, let $z = x + iy$ be the complex coordinate and $\overline{z} = x - iy$ be its complex conjugate. Then $dz = dx + idy$ and $d\overline{z} = dx - idy$.

(1) Consider the function $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$. Show that $\omega = f(z)dz$ is closed (i.e., $d\omega = 0$) if and only if $f(z)$, viewed as a map $f(x, y)$ from $\mathbb{R}^2$ to $\mathbb{R}^2$, satisfies the Cauchy-Riemann equation

$$\tfrac{\partial f}{\partial y} = i\tfrac{\partial f}{\partial x}.$$

(2) Suppose f is a holomorphic function on a domain $\Omega \subset \mathbb{C}$. Prove that if γ_0 and γ_1 are smoothly homotopic curves in Ω, then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

Two curves $\gamma_0, \gamma_1 : S^1 \to \Omega$ are said to be *smoothly homotopic* if there exists a smooth map $\Gamma : S^1 \times [0, 1] \to \Omega$ such that $\Gamma(\theta, t) = \gamma_t(\theta)$ for $t = 0, 1$.

(3) Let C be a circle of radius $R > 0$ around $a \in \mathbb{C}$. Then prove that

$$\int_C \frac{1}{z-a} dz = 2\pi i.$$

(4) Let $f(z)$ be a holomorphic function on Ω, let $C \subset \Omega$ be a circle of radius $R > 0$ around $a \in \Omega$, and let γ be a smooth closed curve which is homotopic to C inside $\Omega - \{a\}$. Then prove that

$$\int_\gamma \frac{f(z)}{z-a} dz = 2\pi i f(a).$$

(5) Using the version of Stokes' Theorem given in class, prove the classical Stokes' Theorem: Let S be a compact, oriented 2-manifold (i.e., a surface) with boundary in $\mathbb{R}^3$ and let $F = (F_1, F_2, F_3)$ be a smooth vector field defined on a neighborhood of S. Then:

$$\int_S \langle \operatorname{curl} F, n \rangle dA = \int_{\partial S} F_1 dx + F_2 dy + F_3 dz,$$

where n is the unit normal to S which is consistent with the orientation, $\langle \cdot \rangle$ is the standard inner product, dA is the pullback to S of

$$n_1 dydz + n_2 dzdx + n_3 dxdy,$$

and

$$\operatorname{curl} F = \left(\tfrac{\partial F_3}{\partial y} - \tfrac{\partial F_2}{\partial z}, \tfrac{\partial F_1}{\partial z} - \tfrac{\partial F_3}{\partial x}, \tfrac{\partial F_2}{\partial x} - \tfrac{\partial F_1}{\partial y} \right).$$

(6) Let M be a manifold. Consider the map

$$\wedge : H^k(M) \times H^l(M) \to H^{k+l}(M), \qquad ([\omega], [\eta]) \mapsto [\omega \wedge \eta].$$

Prove that $\wedge$ is well-defined on the level of cohomology. Hence $H^*(M) := \oplus_{i=0}^{\dim M} H^i(M)$ has an $\mathbb{R}$-algebra structure.

(7) Let T^2 be the 2-dimensional torus. Compute the map $\wedge : H^1(T^2) \times H^1(T^2) \to H^2(T^2)$, i.e., give a basis $\{\omega_1, \dots, \omega_k\}$ for $H^1(T^2)$ and compute $\omega_i \wedge \omega_j$ for all i, j. In particular, prove that $\wedge$ is surjective.

(8) Prove, using the previous exercise, that every smooth map $f : S^2 \to T^2$ has degree zero.